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Existence of solution for a fractional differential equation via a new type of (ψ, F)-contraction in b-metric spaces

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Abstract

In this paper, we further develop the notion of cyclic (α, β)-admissible mappings introduced in ([14], S. Chandok, K. Tas, A. H. Ansari, *Some fixed point results for TAC-type contractive mappings,* J. Function spaces, 2016, Article ID 1907676, 1–6) and (ψ, F)-contraction mappings introduced in ([34], D. Wardowski, *Solving existence problems via F-contractions,* Proceedings of the American Mathematical Society, 146 (4), (2018), 1585–1598), in the framework of b-metric spaces. To achieve this, we introduce the notion of $(\alpha, \beta) - S$-admissible mappings and a new class of generalized (ψ, F)-contraction types and establish a common fixed point and fixed point results for these classes of mappings in the framework of complete b-metric spaces. As an application, we establish the existence and uniqueness of the solutions to differential equations in the framework of fractional derivatives involving Mittag-Leffler kernels via the fixed point technique. The results obtained in this work provide extension as well as substantial generalization and improvement of the fixed point results obtained in [14, 34, 35] and several well-known results on fixed point theory and its applications.

Keywords: Fixed point, $(\alpha, \beta) - S$-admissible mappings, Generalized (ψ, F)-contraction, b-metric space, Differential equation
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1 Introduction and Preliminaries

The theory of fixed point plays an important role in nonlinear functional analysis and is known to be very useful in establishing the existence and uniqueness theorems for nonlinear differential and integral equations. Banach [10] in 1922 proved the well celebrated Banach contraction principle in the frame work of metric spaces. The importance of the Banach contraction principle cannot be over emphasized in the study of fixed point theory and its applications. Due to its importance and fruitful applications, many authors have generalized this result by considering classes of

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nonlinear mappings which are more general than contraction mappings and in other classical and important spaces (see [1, 28] and the references therein). For example, Berinde [11, 12] introduced and studied a class of contractive mappings, which is defined as follows:

Definition 1.1. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a generalized almost contraction if there exist $\delta \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x, y \in X$.

Furthermore, in 2008, Suzuki [31] introduced a class of mappings satisfying condition (C), known as Suzuki-type generalized nonexpansive mapping and he proved some fixed point theorems for this class of mappings.

Definition 1.2. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to satisfy condition (C) if for all $x, y \in X$,

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

Theorem 1.3. Let (X, d) be a compact metric space and $T : X \to X$ be a mapping satisfying

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) < d(x, y),$$

for all $x, y \in X$. Then T has a unique fixed point.

In 2012, Wardowski [35] introduced the notion of F-contractions, which is defined as follows:

Definition 1.4. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that for all $x, y \in X$;

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{1.1}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

(F_1) F is strictly increasing;
(F_2) for all sequences $\{\alpha_n\} \subseteq \mathbb{R}^+, \lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F_3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

He established the following result:

Theorem 1.5. Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for each $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to x^*.

Remark 1.6. [35] If we suppose that $F(t) = \ln t$, an F-contraction mapping becomes the Banach contraction mapping.

In [25], Piri et al. used the continuity condition instead of condition (F_3) and proved the following result:

Theorem 1.7. Let X be a complete metric space and $T : X \to X$ be a selfmap of X. Assume that there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$,

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is continuous strictly increasing and $\inf F = -\infty$. Then T has a unique fixed point $z \in X$, and for every $x \in X$, the sequence $\{T^n x\}$ converges to z.

In 2013, Secelean established the following result

Lemma 1.8. [30] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be an increasing mapping and $\{\alpha_n\}$ be a sequence of positive integers. Then the following assertion hold:

1. if $\lim_{n\to\infty} F(\alpha_n) = -\infty$ then $\lim_{n\to\infty} \alpha_n = 0$;
2. if $\inf F = -\infty$ and $\lim_{n\to\infty} \alpha_n = 0$ then $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Furthermore, the author in [30] replaced the condition (F_2) in the definition of F-contraction with the following condition.

(F_*) $\inf F = -\infty$ or, also by

(F_{**}) there exists a sequence $\{\alpha_n\}$ of positive real numbers such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$. In the same year, Turinici in [33] observed that the condition (F_2) in the definition of F-contraction can be replaced with

$(F_2^{'})$ $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Then, the implication is as follows

$(F_2^{''})$ $\lim_{n\to\infty} F(\alpha_n) = -\infty \Rightarrow \alpha_n \to 0$, where can be derived from (F_1).

Motivated by the work of Turinici [33], Wardowski [34] introduced a modified F-contraction called (ψ, F)-contraction in the setting of a metric space. He gave the following definition. Let (X, d) be a metric space a mapping $T : X \to X$ is called (ψ, F)-contraction if there are $\psi : [0,\infty) \to [0,\infty)$ and $F : [0,\infty) \to \mathbb{R}$ such that

1. F satisfies (F_1) and $(F_2^{'})$;
2. $\liminf_{s\to t^+} \psi(s) > 0$ for all $t \geq 0$;
3. $\psi(d(x, y)) + F(d(Tx, Ty)) \leq F(d(x, y))$ for all $x, y \in X$ such that $Tx \neq Ty$.

In 2016, Chandok et al. [14] introduced a new type of contractive mappings using the notion of cyclic admissible mappings in the framework of metric spaces.

Definition 1.9. [14] Let $T : X \to X$ be a mapping and let $\alpha, \beta : X \to \mathbb{R}^+$ be two functions. Then T is called a cyclic (α, β)-admissible mapping, if

1. $\alpha(x) \geq 1$ for some $x \in X$ implies that $\beta(Tx) \geq 1$,
2. $\beta(x) \geq 1$ for some $x \in X$ implies that $\alpha(Tx) \geq 1$.

Definition 1.10. [14] Let (X, d) be a metric space and let $\alpha, \beta : X \to [0,\infty)$ be two mappings. We say that T is a TAC-contractive mapping, if for all $x, y \in X$,

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(d(Tx, Ty)) \leq f(\psi(d(x, y)), \phi(d(x, y))),$$

where ψ is a continuous and nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$, ϕ is continuous with $\lim_{n\to\infty} \phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$ and $f : [0,\infty)^2 \to \mathbb{R}$ is continuous, $f(a, t) \leq a$ and $f(a, t) = a \Rightarrow a = 0$ or $t = 0$ for all $s, t \in [0,\infty)$.

Theorem 1.11. [14] Let (X, d) be a complete metric space and let $T : X \to X$ be a cyclic (α, β)-admissible mapping. Suppose that T is a TAC contraction mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1, \beta(x_0) \geq 1$ and either of the following conditions hold:

1. T is continuous,
2. if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq 1$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq 1$.

In addition, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in F(T)$ (where $F(T)$ denotes the set of fixed points of T), then T has a unique fixed point.

One of the most interesting generalizations of metric spaces is the concept of b-metric spaces (to be defined in Section 2) introduced by Czerwik in [15]. He proved the Banach contraction principle in this setting with the fact that d need not to be continuous. Thereafter, several results have been extended from metric spaces to b-metric spaces. In addition, a lot of results have been published on the fixed point theory of various classes of single-valued and multi-valued operators in the frame work of b-metric spaces (see [8, 13, 15, 27, 36] and the references therein).

Definition 1.12. [15] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is called a b-metric if for all $x, y, z \in X$, the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d). It is clear that, the class of b-metric spaces is larger than that of metric spaces. If $s = 1$, a b-metric become a metric.

Example 1.13. [8] Let $X = \mathbb{R}$ and $d(x, y) = |x - y|^2$ for all $x, y \in X$. It is easy to see that (x, d) is a b-metric space with coefficient $s = 2$, but (X, d) is not a metric space.

Definition 1.14. [13] Let (X, d) be a b-metric space. A sequence $\{x_n\}$ in X is said to be

1. b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.
2. b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Definition 1.15. [13] Let (X, d) be a b-metric space. Then X is said to be complete if every b-Cauchy sequence in X is b-convergent.

Yamaod and Sintunawarat [36] introduced the notion of (α, β)-(ψ, φ)-contraction mapping in the frame work of b-metric spaces as follows:

Definition 1.16. Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $\alpha, \beta : X \to [0, \infty)$ be two given mappings. We say that $T : X \to X$ is an (α, β)-(ψ, φ)-contraction mapping if the following conditions holds: for all $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$ implies that

$$\psi(s^3 d(Tx, Ty)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)),$$

where $M_s(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}$ and $\psi, \varphi : [0, \infty) \to [0, \infty)$ are alternating distance functions.

Theorem 1.17. Let (X, d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ an (α, β)-(ψ, φ)-contraction mapping. Suppose that one of the following conditions holds:

1. there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$,
2. there exists $y_0 \in X$ such that $\alpha(y_0) \geq 1$,

and the following holds:

1. T is continuous,
2. T is cyclic (α, β)-admissible.

Then T has a fixed point.

Recently, Babu et al. [8] generalized the result of Chandok et al. [14] by introducing a generalized TAC-contractive mapping in the frame work of b-metric spaces.

Definition 1.18. Let (X, d) be a b-metric space, $\alpha, \beta : X \to [0, \infty)$ be two given mappings and T be a self map on X. The mapping T is said to be generalized TAC-contrative map in b-metric spaces, if for all $x, y \in X$,

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(s^3 d(Tx, Ty)) \leq f(\psi(M_s(x, y)), \phi(M_s(x, y))),$$

where $M_s(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}$, ψ is an alternating distance function, ϕ is continuous with $\lim_{n\to\infty} \phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$ and $f : [0, \infty)^2 \to \mathbb{R}$ is continuous with $f(a, t) \leq a$ and $f(a, t) = a \Rightarrow a = 0$ or $t = 0$ for all $s, t \in [0, \infty)$.

Theorem 1.19. Let (X, d) be a complete b-metric space with coefficient $s \geq 1$. Let $T : X \to X$ be a generalized TAC-contraction mapping. Suppose the following conditions hold:

1. T is a cyclic (α, β)-admissible mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

3. T is continuous,
4. if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq 1$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq 1$.

Then T has a fixed point.

Definition 1.20. [17] Let X be a nonempty set and $S, T : X \to X$ be any two mappings.

1. A point $x \in X$ is called:
   (a) coincidence point of S and T if $Sx = Tx$,
   (b) common fixed point of S and T if $x = Sx = Tx$.
2. If $y = Sx = Tx$ for some $x \in X$, then y is called the point of coincidence of S and T.
3. A pair (S, T) is said to be:
   (a) commuting if $TSx = STx$ for all $x \in X$,
   (b) weakly compatible if they commute at their coincidence points, that is $STx = TSx$, whenever $Sx = Tx$.

We denote by $\mathcal{F}$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ and $\psi : [0, \infty) \to [0, \infty)$ which satisfy conditions
(F_1^*) F satisfies (F_1) and $(F_2^{'})$;
(F_2^*) F is continuous on $(0, \infty)$;
(F_3^*) $\liminf_{s \to t^+} \psi(s) > 0$ for all $t \geq 0$.

Furthermore, some interesting generalization of the F-contraction has been established by different authors in the literature. To mention a few, Acar [3] considered a fixed-point problem for mappings involving a rational type and almost type contraction on complete metric spaces. He established some fixed point results in this direction. Also, Acar [4] introduced a rational type F-contraction for multivalued integral type mapping on a complete metric space. Using Wardowski's technique, he established the existence of a fixed point of the multivalued integral type mapping provided that the mapping is continuous. For other related work on the generalization of F-contraction, the reader should see [5, 7, 21, 22] and the references therein.

To the best of our knowledge the results obtain in this work is new in this area of research. Motivated by the works of Wardowski [34, 35], Chandok [14] and the research work in this direction. The purpose of this work is to further develop the notion of cyclic (α, β)-admissible mappings and (ψ, F)-contraction in the framework of b-metric spaces. To do this, we introduce the notion of $(\alpha, \beta) - S$-admissible mappings, generalized (ψ, F)-contraction type I, generalized (ψ, F)-contraction type II, generalized (ψ, F)-contraction type III and generalized (ψ, F)-contraction type IV and establish common fixed point and fixed point results for these classes of mappings in the framework of complete b-metric spaces. Finally, we apply our fixed point result in establishing the existence and uniqueness of a fractional differential equation. The results obtained in this work provide extension as well as substantial generalization and improvement of the fixed point results obtained in [14, 35, 34] and several well-known results on fixed point theory and its applications.

2 Main Result

In this section, we introduce the notion of $(\alpha, \beta) - S$-admissible mappings, generalized (ψ, F)-contraction type I, generalized (ψ, F)-contraction type II, generalized (ψ, F)-contraction type III and generalized (ψ, F)-contraction type IV and establish common fixed point and fixed point results for these classes of mappings in the framework of complete b-metric spaces.

Definition 2.1. Let X be a nonempty set and $s \geq 1$. Let $S, T : X \to X$ and $\alpha, \beta : X \to [0, \infty)$ be given mappings. The mapping T is said to be cyclic $(\alpha, \beta) - S$-admissible mapping, if

1. $\alpha(Sx) \geq s^3$ for some $x \in X$ implies that $\beta(Tx) \geq s^3$,
2. $\beta(Sx) \geq s^3$ for some $x \in X$ implies that $\alpha(Tx) \geq s^3$.

Remark 2.2.
1. Clearly, if $s = 1$ and $Sx = x$, Definition 2.1 reduces to Definition 1.9.
2. Clearly, if $s = 1$, then Definition 2.1 reduces to
   (a) $\alpha(Sx) \geq 1$ for some $x \in X$ implies that $\beta(Tx) \geq 1$,
   (b) $\beta(Sx) \geq 1$ for some $x \in X$ implies that $\alpha(Tx) \geq 1$,

   which also generalizes Definition 1.9.

Lemma 2.3. Suppose (X,d) is a b-metric space with $s \geq 1$. Let $\{Sx_n\}$ be a sequence in X such that $d(Sx_n, Sx_{n+1}) \to 0$ as $n \to \infty$. If $\{Sx_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{Sm_k\}$ and $\{Sn_k\}$ with $m_k > n_k > k$ satisfying $d(Sx_{m_k}, Sx_{n_k}) \geq \epsilon$ and $d(Sx_{m_k}, Sx_{n_{k-1}}) < \epsilon$ such that

1. $\epsilon \leq \liminf_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) \leq \limsup_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) \leq s\epsilon$;
2. $\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq \limsup_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq s^3\epsilon$;
3. $\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d(Sx_{m_k}, Sx_{n_{k+1}}) \leq \limsup_{k\to\infty} d(Sx_{m_k}, Sx_{n_{k+1}}) \leq s^2\epsilon$;
4. $\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq \limsup_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq s^3\epsilon$.

Proof . Suppose $\{Sx_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ satisfying

$$d(Sx_{m_k}, Sx_{n_k}) \geq \epsilon \quad \text{and} \quad d(Sx_{m_k}, Sx_{n_{k-1}}) < \epsilon. \tag{2.1}$$

We choose m_k, the least positive integer satisfying (2.1). We now prove (1). Using (2.1)

$$\begin{aligned}\epsilon \leq d(Sx_{m_k}, Sx_{n_k}) &\leq sd(Sx_{m_k}, Sx_{n_{k-1}}) + sd(Sx_{n_{k-1}}, Sx_{n_k}) \\ &< s\epsilon + sd(Sx_{n_{k-1}}, Sx_{n_k}),\end{aligned} \tag{2.2}$$

clearly, using our hypothesis, we have that and thus

$$\epsilon \leq \liminf_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) \leq \limsup_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) \leq s\epsilon. \tag{2.3}$$

We now prove (2). Now observe that

$$\begin{aligned}d(Sx_{m_k}, Sx_{n_k}) &\leq sd(Sx_{m_k}, Sx_{m_{k+1}}) + sd(Sx_{m_{k+1}}, Sx_{n_k}) \\ &\leq sd(Sx_{m_k}, Sx_{m_{k+1}}) + s^2d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) + s^2d(Sx_{n_{k+1}}, Sx_{n_k}))\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) &\leq sd(Sx_{m_{k+1}}, Sx_{m_k}) + sd(Sx_{m_k}, Sx_{n_{k+1}}) \\ &\leq sd(Sx_{m_{k+1}}, Sx_{m_k}) + s^2d(Sx_{m_k}, Sx_{n_k}) + s^2d(Sx_{n_k}, Sx_{n_{k+1}}).\end{aligned} \tag{2.5}$$

Using our hypothesis, (2.4) and (2.5), we have that

$$\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq \limsup_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq s^3\epsilon. \tag{2.6}$$

We now prove (3). Note that,

$$d(Sx_{m_k}, Sx_{n_k}) \leq sd(Sx_{m_k}, Sx_{n_{k+1}}) + sd(Sx_{n_{k+1}}, Sx_{n_k}) \tag{2.7}$$

and

$$d(Sx_{m_k}, Sx_{n_{k+1}}) \leq sd(Sx_{m_k}, Sx_{n_k}) + sd(Sx_{n_k}, Sx_{n_{k+1}}). \tag{2.8}$$

Using our hypothesis, (2.8) and (2.7), we have that

$$\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d(Sx_{m_k}, Sx_{n_{k+1}}) \leq \limsup_{k\to\infty} d(Sx_{m_k}, Sx_{n_{k+1}}) \leq s^2\epsilon. \tag{2.9}$$

We now prove (4). Now observe that

$$\epsilon \leq d(Sx_{m_k}, Sx_{n_k}) \leq sd(Sx_{m_k}, Sx_{m_{k+1}}) + s^2d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) + s^2d(Sx_{n_{k+1}}, Sx_{n_k}) \tag{2.10}$$

and

$$d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq sd(Sx_{m_{k+1}}, Sx_{n_k}) + sd(Sx_{n_k}, Sx_{n_{k+1}}), \tag{2.11}$$

thus, using our hypothesis, (2.10), (2.11) and (3), we have

$$\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq \limsup_{k\to\infty} d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \leq s^3\epsilon. \tag{2.12}$$

$\square$

Lemma 2.4. Let X be a nonempty set and $T : X \to X$ be a cyclic $(\alpha_s, \beta_s) - S$-admissible mapping. Suppose that there exists $Sx_0 \in X$ such that $\alpha(Sx_0) \geq s^3$ and $\beta(Sx_0) \geq s^3$. Define the sequence $Sx_{n+1} = Tx_n$, then $\alpha(Sx_m) \geq s^3$ implies that $\beta(Sx_n) \geq s^3$ and $\beta(Sx_m) \geq s^3$ implies that $\alpha(Sx_n) \geq s^3$, for all $n, m \in \mathbb{N} \cup \{0\}$ with $m < n$.

Proof . Using the fact that T is a cyclic $(\alpha_s, \beta_s) - S$-admissible mapping and our hypothesis, we have that there exists $Sx_0 \in X$ such that

$$\alpha(Sx_0) \geq s^3 \Rightarrow \beta(Tx_0) = \beta(Sx_1) \geq s^3$$

and

$$\beta(Sx_0) \geq s^3 \Rightarrow \alpha(Tx_0) = \alpha(Sx_1) \geq s^3.$$

Continuing this way, we obtain that

$$\alpha(Sx_n) \geq s^3 \Rightarrow \beta(Tx_n) = \beta(Sx_{n+1}) \geq s^3$$

and

$$\beta(Sx_n) \geq s^3 \Rightarrow \alpha(Tx_n) = \alpha(Sx_{n+1}) \geq s^3$$

for all $n \in \mathbb{N} \cup \{0\}$. Using similar approach, we obtain that

$$\alpha(Sx_m) \geq s^3 \Rightarrow \beta(Tx_m) = \beta(Sx_{m+1}) \geq s^3$$

and

$$\beta(Sx_m) \geq s^3 \Rightarrow \alpha(Tx_m) = \alpha(Sx_{m+1}) \geq s^3,$$

for all $m \in \mathbb{N} \cup \{0\}$. In addition, since

$$\alpha(Sx_m) \geq s^3 \Rightarrow \beta(Tx_m) = \beta(Sx_{m+1}) \geq s^3 \Rightarrow \alpha(Sx_{m+2}) \geq s^3 \cdots$$

with $m < n$, we deduce that

$$\alpha(Sx_m) \geq s^3 \Rightarrow \beta(Sx_n) \geq s^3.$$

Using similar approach, we have that

$$\beta(Sx_m) \geq s^3 \Rightarrow \alpha(Sx_n) \geq s^3.$$

□

Definition 2.5. Let (X, d) be a b-metric space with $s \geq 1$, $\alpha, \beta : X \to (0, \infty)$ be two functions, $\psi : [0, \infty) \to [0, \infty)$ and S, T be a self map on X. The mapping T is said to be generalized (ψ, F)-contraction type I if $F \in \mathcal{F}$ and $L \geq 0$ such that for all $x, y \in X$

$$\begin{aligned} &d(Tx, Ty) > 0 \\ &\Rightarrow \psi(d(Sx, Sy)) + F(\alpha(Sx)\beta(Sy)d(Tx, Ty)) \leq F(d(Sx, Sy)) + LN(x, y), \end{aligned} \tag{2.13}$$

where $N(x, y) = \min\{d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\}$

Remark 2.6. We note that

1. if $\alpha(Sx) = \beta(Sy) = 1$ and $L = 0$, we obtain

$$d(Tx, Ty) > 0 \Rightarrow \psi(d(Sx, Sy)) + F(d(Tx, Ty)) \leq F(d(Sx, Sy)), \tag{2.14}$$

which is a generalization of (ψ, F)-contraction.

2. if $Sx = x, \alpha(x) = \beta(y) = 1$ and $L = 0$, we obtain (ψ, F)-contraction mappings.

Definition 2.7. Let (X,d) be a b-metric space with $s \geq 1$, Let (X,d) be a b-metric space with $s \geq 1$, $\alpha,\beta : X \to (0,\infty)$ be two functions, $\psi : [0,\infty) \to [0,\infty)$ and S,T be a self map on X. The mapping T is said to be generalized (ψ,F)-contraction type II if $F \in \mathcal{F}$ and $L \geq 0$ such that for all $x,y \in X$ $d(Tx,Ty) > 0$ and $\alpha(Sx)\beta(Sy) \geq s^3$, implies that

$$\psi(d(Sx,Sy)) + F(s^5 d(Tx,Ty)) \leq F(d(Sx,Sy)) + LN(x,y), \tag{2.15}$$

where $N(x,y) = \min\{d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx)\}$

Definition 2.8. Let (X,d) be a b-metric space with $s \geq 1$, Let (X,d) be a b-metric space with $s \geq 1$, $\alpha,\beta : X \to (0,\infty)$ be two functions, $\psi : [0,\infty) \to [0,\infty)$ and T be a self map on X. The mapping T is said to be generalized (ψ,F)-contraction type III if $F \in \mathcal{F}$ and $L \geq 0$ such that for all $x,y \in X$, $d(Tx,Ty) > 0$ implies that

$$\psi(d(x,y)) + F(\alpha(x)\beta(y)d(Tx,Ty)) \leq F(d(x,y)) + LN(x,y), \tag{2.16}$$

where $N(x,y) = \min\{d(x,y), d(y,Ty), d(x,Ty), d(y,Tx)\}$.

Definition 2.9. Let (X,d) be a b-metric space with $s \geq 1$, Let (X,d) be a b-metric space with $s \geq 1$, $\alpha,\beta : X \to (0,\infty)$ be two functions, $\psi : [0,\infty) \to [0,\infty)$ and T be a self map on X. The mapping T is said to be generalized (ψ,F)-contraction type IV if $F \in \mathcal{F}$, such that for all $x,y \in X$, $d(Tx,Ty) > 0$ implies

$$\psi(d(x,y)) + F(s^5 d(Tx,Ty)) \leq F(d(x,y)). \tag{2.17}$$

Theorem 2.10. Let (X,d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a generalized (ψ,F)-contraction type I mapping. Suppose the following conditions hold:

1. T is a cyclic $(\alpha_s,\beta_s) - S$-admissible mapping,
2. there exists $Sx_0 \in X$ such that $\alpha(Sx_0) \geq s^3$ and $\beta(Sx_0) \geq s^3$,
3. $T(X) \subseteq S(X)$,
4. $T(X)$ is complete in $S(X)$,
5. if for any sequence $\{Sx_n\}$ in X with $\beta(Sx_n) \geq s^3$, for all $n \geq 0$ and $Sx_n \to Sx$ as $n \to \infty$, then $\beta(Sx) \geq s^3$.

Then the pair (T,S) has a coincidence point in X. In addition, if the pair (T,S) is weakly compatible, then the pair (T,S) has a common fixed point.

Proof . Since, $T(X) \subseteq S(X)$, we can define a sequence $\{Sx_n\}$ by $Sx_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If we suppose that $Sx_{n+1} = Sx_n = Tx_n$, we obtain that x_n is the coincidence point of S and T. Now, suppose that $Sx_{n+1} \neq Sx_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is a cyclic $(\alpha_s,\beta_s) - S$-admissible mapping and $\alpha(Sx_0) \geq s^3$, we have $\beta(Sx_1) = \beta(Tx_0) \geq s^3$ and this implies that $\alpha(Sx_2) = \alpha(Tx_1) \geq s^3$, continuing the process, we have

$$\alpha(Sx_{2k}) \geq s^3 \quad \text{and} \quad \beta(Sx_{2k+1}) \geq s^3 \quad \forall \quad k \in \mathbb{N} \cup \{0\}. \tag{2.18}$$

Using similar argument, we have that

$$\beta(Sx_{2k}) \geq s^3 \quad \text{and} \quad \alpha(Sx_{2k+1}) \geq s^3 \quad \forall \quad k \in \mathbb{N} \cup \{0\}. \tag{2.19}$$

It follows from (2.18) and (2.19) that $\alpha(Sx_n) \geq s^3$ and $\beta(Sx_n) \geq s^3$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\alpha(Sx_n)\beta(Sx_{n+1}) \geq s^3$, we obtain from (2.13)

$$\begin{aligned}
\psi(d(Sx_n,Sx_{n+1})) + F(d(Sx_{n+1},Sx_{n+2})) =& \psi(d(Sx_n,Sx_{n+1})) + F(d(Tx_n,Tx_{n+1})) \\
\leq& \psi(d(Sx_n,Sx_{n+1})) + F(\alpha(Sx_n)\beta(Sx_{n+1})d(Tx_n,Tx_{n+1})) \qquad (2.20)\\
\leq& F(d(Sx_n,Sx_{n+1})) + L\min\{d(Sx_n,Sx_{n+1}), d(Sx_{n+1},Sx_{n+2}), d(Sx_n,x_{n+2}), \\
& d(Sx_{n+1},Sx_{n+1})\} \\
=& F(d(Sx_n,Sx_{n+1})),
\end{aligned}$$

which implies that

$$F(d(Sx_{n+1},Sx_{n+2})) \leq F(d(Sx_n,Sx_{n+1})) - \psi(d(Sx_n,Sx_{n+1})).$$

Using similar approach, it is easy to see that

$$F(d(Sx_n, Sx_{n+1})) \le F(d(Sx_{n-1}, Sx_n)) - \psi(d(Sx_{n-1}, Sx_n)).$$

Using the properties of ψ, then, there exists $c > 0$ and $n_0 \in \mathbb{N}$ such that $\psi(d(x_n, x_{n+1})) > c$ for all $n > n_0$. We obtain the following inequalities inductively

$$\begin{aligned} F(d(Sx_n, Sx_{n+1})) \le & F(d(Sx_0, Sx_1)) - (\psi(d(Sx_0, Sx_1)) + \cdots + \psi(d(Sx_{n_0-1}, Sx_{n_0}))) \\ & - (\psi(d(Sx_{n_0}, Sx_{n_0+1})) + \cdots + \psi(d(Sx_{n-1}, Sx_n))) \\ \le & F(d(Sx_0, Sx_1)) - (n - n_0)c. \end{aligned} \tag{2.21}$$

Since $F \in \mathcal{F}$, taking limit as $n \to \infty$ in (2.21) and using Lemma 1.8 we have

$$\lim_{n\to\infty} F(d(Sx_n, Sx_{n+1})) = -\infty \Leftrightarrow \lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = 0. \tag{2.22}$$

In what follows, we now show that $\{Sx_n\}$ is a b-Cauchy sequence. Suppose that $\{Sx_n\}$ is not a b-Cauchy sequence, then by Lemma 2.3, there exists an $\epsilon > 0$ and sequences of positive integers $\{Sx_{n_k}\}$ and $\{Sx_{m_k}\}$ with $n_k > m_k \ge k$ such that $d(Sx_{m_k}, Sx_{n_k}) \ge \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(Sx_{m_k}, Sx_{n_k}) \ge \epsilon, d(Sx_{m_k}, Sx_{n_{k-1}}) < \epsilon$ and $(1) - (4)$ of Lemma 2.3 hold. Since $\alpha(Sx_0) \ge s^3$ and $\beta(Sx_0) \ge s^3$, using Lemma 2.4, we obtain that $\alpha(Sx_{m_k})\beta(Sx_{n_k}) \ge s^3$ and we can choose $n_0 \in \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} \psi(d(Sx_{m_k+1}, Sx_{n_k+1})) + F(d(Sx_{m_k+1}, Sx_{n_k+1})) \le & \psi(d(Sx_{m_k+1}, Sx_{n_k+1})) + F(\alpha(Sx_{m_k})\beta(Sx_{n_k})d(Tx_{m_k}, Tx_{n_k})) \\ \le & F(d(Sx_{m_k}, Sx_{n_k})) + L \min\{d(Sx_{m_k}, Sx_{m_k+1}), d(Sx_{n_k}, Sx_{n_k+1}), \\ & dS(x_{m_k}, Sx_{n_k+1}), d(Sx_{n_k}, Sx_{m_k+1})\}. \end{aligned} \tag{2.23}$$

Since $F \in \mathcal{F}$, using Lemma 2.3, and (2.22), we have that

$$\begin{aligned} \liminf_{k\to\infty} \psi(d(Sx_{m_k}, Sx_{n_k})) + F(s\epsilon) &= \liminf_{k\to\infty} \psi(d(Sx_{m_k}, Sx_{n_k})) + F(s\epsilon) \\ &= \liminf_{k\to\infty} \psi(d(Sx_{m_k}, Sx_{n_k})) + F(s^3 \frac{\epsilon}{s^2}) \\ &= \liminf_{k\to\infty} [\psi(d(Sx_{m_k}, Sx_{n_k}) + F(\alpha(Sx_{m_k})\beta(Sx_{n_k})d(Tx_{m_k}, Tx_{n_k}))] \\ &\le F(\liminf_{k\to\infty} d(Sx_{m_k}, Sx_{n_k})) \\ &\le F(s\epsilon), \end{aligned}$$

where $0 < \liminf_{d(Sx_n, Sx)\to 0^+} \psi(d(Sx_{m_k}, Sx)) = \mu$. That is

$$\mu + F(s\epsilon) \le F(s\epsilon)$$

which is a contradiction. We therefore have that $\{Sx_n\}$ is b-Cauchy. Since $T(X)$ is complete in $S(X)$, there exists $x \in X$ such that $\lim_{n\to\infty} Sx_n = Sx$. More so, using the condition that $\beta(x_n) \ge s^3$ for all $n \in \mathbb{N} \cup \{0\}$, we obtain that $\beta(x) \ge s^3$. As such, we have that

$$\begin{aligned} \psi(d(Sx_n, Sx)) + F(d(Sx_{n+1}, Tx)) &\le \psi(d(Sx_n, Sx)) + F(\alpha(Sx_n)\beta(Sx)d(Tx_n, Tx)) \\ &\le F(d(Sx_n, Sx)) + L \min\{d(Sx_n, Sx_{n+1}), d(Sx, Tx), d(Sx_n, Tx), d(Sx, Tx_n)\}. \end{aligned}$$

Using the fact that $0 < \liminf_{d(Sx_n, Sx)\to 0^+} \psi(d(Sx_{m_k}, Sx)) = \mu$, $F \in \mathcal{F}$ and Lemma 1.8, we have that

$$\lim_{n\to\infty} F(d(Sx_{n+1}, Tx)) = -\infty$$

and so

$$\lim_{n\to\infty} d(Sx_{n+1}, Tx) = 0.$$

Now, observe that

$$d(Sx, Tx) = \lim_{n\to\infty} d(Sx_{n+1}, Tx) = 0.$$

Clearly, we have that

$$d(Sx, Tx) = 0 \Rightarrow Sx = Tx.$$

Hence, x is the coincidence point for the pair (T, S). Now,suppose that $y = Tx = Sx$, using condition () we have that

$$Ty = T(Sx) = S(Tx) = Sy.$$

It is easy to see that $\alpha(Sx)\beta(Sy) \geq s^3$, as such we have that

$$\begin{aligned}F(d(y, Ty)) = F(d(Tx, Ty)) &< \psi(d(Sx, Sy)) + F(d(Tx, Ty)) \leq \psi(d(Sx, Sy)) + F(\alpha(Sx)\beta(Sy)d(Tx, Ty))\\ &\leq F(d(Sx, Sy)) + L\min\{d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\},\\ &= F(d(y, Sy)),\end{aligned}$$

which is a contradiction, as such we have that

$$y = Ty = Sy.$$

Hence, y is the common fixed point for the pair (T, S). □

Theorem 2.11. Suppose that the hypothesis of Theorem 2.10 holds and in addition suppose $\alpha(x) \geq s^3$ and $\beta(y) \geq s^3$ for all $x, y \in C(T, S)$, where $C(T, S)$ is the set of common fixed point for the pair (T, S). Then (T, S) has a unique common fixed point.

Proof . Let $x, y \in F(T)$, that is $x = Tx = Sx$ and $y = Sy = Ty$ such that $x \neq y$. Since, $\alpha(x) \geq s^3$ and $\beta(y) \geq s^3$, we have $\alpha(x)\beta(y) \geq s^3$, we obtain that

$$\begin{aligned}F(d(x, y)) = F(d(Tx, Ty)) &< \psi(d(Sx, Sy)) + F(d(Tx, Ty)) \leq \psi(d(Sx, Sy)) + F(\alpha(Sx)\beta(Sy)d(Tx, Ty))\\ &\leq F(d(Sx, Sy)) + L\min\{d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\},\\ &= F(d(x, y)),\end{aligned}$$

which implies that

$$F(d(x, y)) < F(d(x, y)).$$

Clearly, we get a contradiction, thus, (T, S) have a unique common fixed point. □

Theorem 2.12. Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a generalized (ψ, F)-contraction type II mapping. Suppose the following conditions hold:

1. T is a cyclic $(\alpha_s, \beta_s) - S$-admissible mapping,
2. there exists $Sx_0 \in X$ such that $\alpha(Sx_0) \geq s^3$ and $\beta(Sx_0) \geq s^3$,
3. $T(X) \subseteq S(X)$,
4. $T(X)$ is complete in $S(X)$,
5. if for any sequence $\{Sx_n\}$ in X with $\beta(Sx_n) \geq s^3$, for all $n \geq 0$ and $Sx_n \to Sx$ as $n \to \infty$, then $\beta(Sx) \geq s^3$.

Then the pair (T, S) has a coincidence point in X. In addition, if the pair (T, S) is weakly compatible, then the pair (T, S) has a common fixed point.

Proof . The prove follow similar approach as in Theorem 2.10 as such we omit it. □

Theorem 2.13. Suppose that the hypothesis of Theorem 2.12 holds and in addition suppose $\alpha(x) \geq s^3$ and $\beta(y) \geq s^3$ for all $x, y \in C(T, S)$, where $C(T, S)$ is the set of common fixed point for the pair (T, S). Then (T, S) has a unique common fixed point.

Proof . The prove follow similar approach as in Theorem 2.13 as such we omit it. □

Theorem 2.14. Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a generalized (ψ, F)-contraction type III mapping. Suppose the following conditions hold:

1. T is a cyclic $(\alpha_s, \beta_s) - S$-admissible mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0) \geq s^3$ and $\beta(x_0) \geq s^3$,
3. if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq s^3$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq s^3$.

Then the pair T has a fixed point.

Proof . The prove follows similar approach as in Theorem 2.10 by taking $S = I$ (identity mapping). □

Theorem 2.15. Suppose that the hypothesis of Theorem 2.14 holds and in addition suppose $\alpha(x) \geq s^3$ and $\beta(y) \geq s^3$ for all $x, y \in F(T)$, where $F(T)$ is the set of fixed point of T. Then T has a unique fixed point.

Proof . The prove follows similar approach as in Theorem 2.13 by taking $S = I$ (identity mapping). □

Theorem 2.16. Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a generalized (ψ, F)-contraction type IV mapping. Then the T has a unique fixed point.

Proof . The prove follows similar approach as in Theorem 2.10 and Theorem 2.13 by taking $S = I$ (identity mapping)$L = 0$ and removing the condition $\alpha(x)\beta(y) \geq s^3$. □

3 Application

In this section, we establish the existence and uniqueness of the solution of a fractional differential equation involving the Caputo Atangan-Baleanu via fixed point technique.

$$\begin{aligned} D^\alpha f(t) &= g(t, f(t)), \quad t \in I = [0,1] \\ f(0) &= a, \end{aligned} \tag{3.1}$$

where $f : I \to \mathbb{R}, g \in C(I)$ are continuous functions such that $g(0, x(0)) = 0, \alpha \in (0,1)$ and a is a constant. Let $X = C(I, \mathbb{R})$ be the space of continuous function defined on I and $d(x,y) = \sup_{t \in I} |x(t) - y(t)|^2$, where $s = 2$. It is well-known that (X, d) is a complete b-metric space.

Definition 3.1. [6, 29, 23] Let $f \in H^1(a,b)$ with $a < b$ and $\alpha \in [0,1]$. The Caputo Atangana-Baleanu fractional derivative of f of oder α is defined by

$$D^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t f^{'}(t) E_\alpha[-\alpha \frac{(t-x)^\alpha}{1-\alpha}] dx, \tag{3.2}$$

where E_α is the Mittage-Leffler function defined by

$$E_\alpha(y) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(\alpha n + 1)} \tag{3.3}$$

and $B(\alpha)$ is a normalizing positive function satisfying $B(0) = B(1) = 1$. Then, the associated fractional integral is given by

$$I^\alpha f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)} I^\alpha f(t), \tag{3.4}$$

where ${}_aI^\alpha$ is the left Riemann-Liouville fractional integral given as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x) dx. \tag{3.5}$$

Proposition 3.2. [6] For $0<\alpha<1$, we have

$$I^{\alpha}D^{\alpha}f(x) = f(x) - f(a)E_{\alpha}(\lambda(x-a)^{\alpha}) - \frac{\alpha}{1-\alpha}f(a)x^{\alpha}E_{\alpha,\alpha+1}(\lambda(x-a)^{\alpha}) \\ = f(x) - f(a). \tag{3.6}$$

Similarly

$$I_b^{\alpha}D_b^{\alpha}f(x) = f(x) - f(b). \tag{3.7}$$

Theorem 3.3. Let $X = C(I,\mathbb{R})$ such that for all $t\in[0,1]$ and $f_1, f_2 \in C(I,\mathbb{R})$, we have that

$$|g(t,f_1(s)) - g(t,f_2(s))| \le \frac{B(\alpha)\Gamma(\alpha)}{6((1-\alpha)\Gamma(\alpha)+1)[1+\sup_{u\in[0,1]}|f_1(u)| + \sup_{u\in[0,1]}|f_2(u)|]} \times |f_1(s) - f_2(s)|. \tag{3.8}$$

Then, the initial value problem (3.1) has a unique solution $f(t)\in C([0,1],\mathbb{R})$.

Proof . Using Proposition 3.2, applying the Atangana–Baleanu integral to both sides of (3.1), we have that

$$f(t) = a + I^{\alpha}g(t,f(t)), \tag{3.9}$$

defining $T : X \to X$ by

$$(Tf)(t) = a + I^{\alpha}g(t,f(t)). \tag{3.10}$$

It is well-known that if $f \in C([0,1],\mathbb{R})$ is a fixed point of T then f is a solution of problem (3.1).

$$\begin{aligned}
|(Tf_1)(t) - (Tf_2)(t)|^2 &= | \ \ I^{\alpha}[g(s,f_1(s)) - g(s,f_2(s))]|^2 \\
&= \left|\frac{1-\alpha}{B(\alpha)}[g(t,f_1(t)) - g(t,f_2(t))] + \frac{\alpha}{B(\alpha)} \ \ I^{\alpha}[g(s,f_1(s)) - g(s,f_2(s))]\right|^2 \\
&\le \left\{\frac{1-\alpha}{B(\alpha)}|g(t,f_1(t)) - g(t,f_2(t))| + \frac{\alpha}{B(\alpha)} \ \ {}_0I^{\alpha}|g(s,f_1(s)) - g(s,f_2(s))|\right\}^2 \\
&\le \left\{\frac{B(\alpha)\Gamma(\alpha)}{6((1-\alpha)\Gamma(\alpha)+1)[1+\sup_{u\in[0,1]}|f_1(u)| + \sup_{u\in[0,1]}|f_2(u)|]}\right\}^2 \\
&\times \left\{\frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{B(\alpha)} \ \ {}_0I^{\alpha}(1)\right\}|f_1(s) - f_2(s)|\}^2 \\
&= \left\{\frac{B(\alpha)\Gamma(\alpha)}{6((1-\alpha)\Gamma(\alpha)+1)[1+\sup_{u\in[0,1]}|f_1(u)| + \sup_{u\in[0,1]}|f_2(u)|]}\right\}^2 \\
&\times \left\{\frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{B(\alpha)}\frac{1}{\alpha\Gamma(\alpha)}\right\}|f_1(s) - f_2(s)|\}^2 \\
&\le \left\{\frac{B(\alpha)\Gamma(\alpha)}{6((1-\alpha)\Gamma(\alpha)+1)[1+\sup_{u\in[0,1]}|f_1(u)| + \sup_{u\in[0,1]}|f_2(u)|]}\right\}^2 \\
&\times \left\{\frac{1-\alpha}{B(\alpha)} + \frac{1}{B(\alpha)\Gamma(\alpha)}\right\}\{\sup_{s\in[0,1]}|f_1(s) - f_2(s)|\}^2 \\
&\le \frac{1}{36[1+\sup_{u\in[0,1]}|f_1(u)| + \sup_{u\in[0,1]}|f_2(u)|]^2}\{\sup_{s\in[0,1]}|f_1(s) - f_2(s)|^2\} \\
&\le \frac{1}{36[1+\sup_{u\in[0,1]}|f_1(u) - f_2(u)|^2]}\{\sup_{s\in[0,1]}|f_1(s) - f_2(s)|^2\} \\
&= \frac{d(f_1,f_2)}{36[1+d(f_1,f_2)]}.
\end{aligned} \tag{3.11}$$

Taking $f_1 = x$ and $f_2 = y$, we have that $d(Tx,Ty) \le \frac{d(x,y)}{25(1+d(x,y))}$, which implies that $36d(Tx,Ty) \le \frac{d(x,y)}{1+d(x,y)}$, it follows that

$$32d(Tx,Ty) \le 36d(Tx,Ty) \le \frac{d(x,y)}{1+d(x,y)} \Rightarrow 32d(Tx,Ty) \le \frac{d(x,y)}{1+d(x,y)}, \tag{3.12}$$

taking natural logarithm of both sides, we have that

$$\ln(1 + d(x, y)) + \ln(s^5 d(x, y)) \leq \ln(d(x, y)), \tag{3.13}$$

taking $F(t) = \ln(t)$ and $\psi(t) = \ln(1 + t)$, we have

$$\psi(d(x, y)) + F(s^5 d(x, y)) \leq F(d(x, y)), \tag{3.14}$$

that is T is a generalized (ψ, F)-contraction type IV mapping and all conditions in Theorem 2.16 are satisfied, so T has a unique fixed point and so problem (3.1) has a unique solution. □

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References

[1] C.T. Aage and J.N .Salunke, *Fixed points for weak contractions in G-metric spaces*, Appl. Math. E-Notes. **12** (2012), 23–28.

[2] T. Abdeljawad and D. Baleanu, *Integration by parts and its applications of a new nonlocal fractional derivative with Mittag–Leffler nonsingular kernel*, *J. Nonlinear Sci Appl.* **10** (2017), no. 3, 1098–107.

[3] Ö. Acar, *Some fixed-point results via mix-type contractive condition*, J. Funct. Spaces **2021** (2021), 1–20.

[4] Ö. Acar, *Fixed point theorems for rational type F-contraction*, Carpath. Math. Pub., **13** (2021), no. 1, 39–47.

[5] Ö. Acar, *Rational type multivalued F_G-contractive mappings with a graph*, Results Math. **73** (2018), no. 2, 1–9.

[6] A. Atangana, D. Baleanu, *New fractional derivative with non-local and non-singular kernal*, Thermal Sci. **20** (2016), no. 2, 757–763.

[7] A.S. Bubu, T. Dosenovic, M.A. Md, S. Radenovic and K.P.R. Rao, *Some presic type results in b− dislocated metric spaces*, Construct. Math. Anal., **2** (2019), no. 1, 40–48.

[8] G.V.R. Babu and T.M. Dula, *Fixed points of generalized TAC-contractive mappings in b-metric spaces*, Mat. Vesnik **69** (2017), no. 2, 75–88.

[9] G. V.R. Babu and P.D. Sailaja, *A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces*, Thai J. Math. **9** (2011), no. 1, 1–10.

[10] S. Banach, *Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales*, Fund. Math., **3** (1922), 133–181.

[11] V. Berinde, *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Anal. Forum, **9** (2004), 43–53.

[12] V. Berinde, *General constructive fixed point theorem for Ciric-type almost contractions in metric spaces*, Carpath. J. Math., **24** (2008), 10–19.

[13] M. Boriceanu, M. Bota and A. Petrusel, *Mutivalued fractals in b-metric spaces*, Cent. Eur. J. Math., **8** (2010), 367–377.

[14] S. Chandok, K. Tas, A. H. Ansari, Some fixed point results for TAC-type contractive mappings, J. Funct. Spaces **2016** (2016), Article ID 1907676, 1–6.

[15] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inf. Univ. Ostraviensis, **1** (1993), 5–11.

[16] G. Jungck, *Contraction mappings in b-metric spaces*, Acta Math. Inf. Univ. Ostraviensis, **1** (1993), 5–11.

[17] G. Jungck, *Commuting mappings and fixed points*, *Amer. Math. Month.* **83** (1976), no. 4, 261–263.

[18] G. Jungck and N. Hussain, *Compatible maps and invariant approximations*, J. Math. Anal. Appl., **325** (2007), no. 2, 1003-1012.

[19] N. Hussain, M. A. Kutbi and P. Salimi, Fixed point theory in α-complete metric spaces with applications, Abstr. Appl. Anal. **2014** (2014), Article ID 280817.

[20] D.S. Jaggi, *Some unique fixed point theorems*, Indian J. Pure Appl. Math., **8** (1977), no. 1, 223–230.

[21] Z. Kadelburg and S. Radenović, *Notes on some recent papers concerning F-contractions in b-metric spaces*, Construct. Math. Anal. **1** (2018), no. 2, 108–112.

[22] E. Karapinar, *A short survey on the recent fixed point results on b-metric spaces*, Construct. Math. Anal. **1** (2018), no. 1, 15–44.

[23] A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and application of fractional differential equations*, North Holland Math Stud, 2006.

[24] A. Pansuwon, W. Sintunavarat, V. Parvaneh and Y.J. Cho, *Some fixed point theorems for (α, θ, k)-contractive multi-valued mappings with some applications*, Fixed Point Theory Appl. **132** (2015), 1–20.

[25] H. Piri and P. Kumam, *Some fixed point theorems concerning F-contraction in complete metric spaces*, Fixed Point Theory Appl. **210** (2014), 1–17.

[26] Z. Qingnian and S. Yisheng, *Fixed point theory for generalized ϕ-weak contractions*, Appl. Math. Lett. **22** (2009), no. 1, 75–78.

[27] J. R. Roshan, V. Parvaneh and Z. Kadelburg, *Common fixed point theorems for weakly isotone increasing mappings in ordered b-metric spaces*, J. Nonlinear Sci. and Appl. **7** (2014), 229–245.

[28] P.Salimi and V. Pasquale, *A result of Suzuki type in partial G-metric spaces*, Acta Math. Sci. Ser. B., (Engl. Ed.). **34** (2014), no. 2, 274–284.

[29] S. G. Samko, A. A. Kilbas, and O. Marichev, *Fractional integrals and derivatives: theory and applications*, Yverdon: Gordon and Breach, 1993.

[30] N. A. Secelean, *Iterated function systems consisting of F -contractions*, Fixed Point Theory Appl. **277** (2013), 1–20.

[31] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl. **340** (2008), no. 2, 1088–1095.

[32] T. Suzuki, *A new type of fixed point theorem in metric spaces*, Nonlinear Anal. **71** (2009), no. 11, 5313–5317.

[33] M. Turinici, *A Wardowski implicit contractions in metric spaces*, (2013) arXiv:1212.3164v2 [Math.GN].

[34] D. Wardowski, *Solving existence problems via F-contractions*, Proc. Amer. Math. Soc. **146** (2018), no. 4,1585–1598.

[35] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. **94** (2012), 1–20.

[36] O. Yamaoda and W. Sintunavarat, *Fixed point theorems for $(\alpha, \beta) - (\psi, \varphi)$-contractive mappings in b-metric spaces with some numerical results and applications*, J. Nonlinear Sci. Appl. **9** (2016), 22–33.