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Effects of elasticity and cross-flow Reynolds on visco-elastic fluids across the ground and a porous elliptic plate

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Abstract

We are concerned with an analysis performed to simulate the steady-state Walter's B' viscoelastic fluid in a 3-D space across the ground and a porous elliptic plate. We study the effect of viscoelasticity and with the help of a suitable resemblance transformation for components of velocity, fundamental equations are then reduced to a set of ODEs which are then solved by the Homotopy analysis method (HAM). Impacts of elasticity and cross-flow Reynolds number are discussed.

Keywords: HAM, Walter's B' viscoelastic fluid, elliptic plate, velocity
2020 MSC: Primary 35Q70; Secondary 37N10

1 Introduction

The significance of fluids known as non-Newtonian became great importance in various fields of technology and industries recently thus the investigation of the dynamics of such fluids is needed. Some applications of this range of fluids are in the boundary layer control and reducing corrosion. Because of these applications, in the past century, further attention has been paid to discussing viscoelastic fluid flow [8]-[10]. It was not until 1953 that Berman [11] contributed extensively in the field. His research provided an important technique that was a keystone in solving the classical problem of simulating viscous flow equations. Other researchers such as Sellars (1955), Yuan (1956), Proudman (1960), Zaturska et al. (1988), Wang (1978), and a few more made a further contribution to this topic [23]-[7]. We study Walter B's viscous fluid flow type in three dimensional between the ground and an elliptic porous plate. The description of the problem is demonstrated in Fig. 1.

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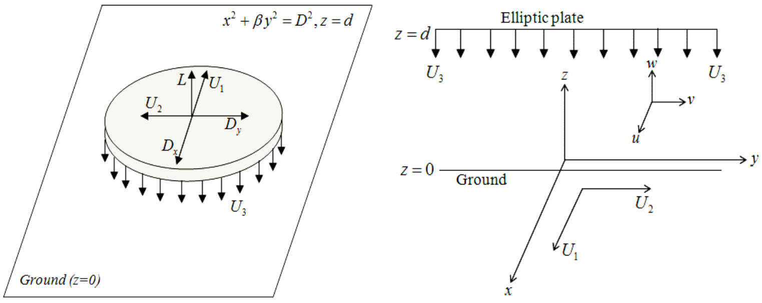


Figure 1: Schematic diagram of fluid flow between the ground and a Porous Elliptic Plate.

The problem used to simulate cushioned porous sliders of fluids (FCPS) is considered helpful in reducing friction between objects which are moving. This type of flow gained a lot of attention due to its advanced performance and applicability in the field of fluid softened moving pads.

These sliders have the advantage of reducing the friction resistance between these cushions. Due to this, there has been some previous research regarding the Newtonian fluid with a different types of sliders. For instance, Wang 1974 studied the porous circular pads, Skalak and Wang 1975 investigated the flat porous pads, elliptical sliders, and elliptic porous pads by Wang and Watson in 1978. In 1981, Ballat explored the dynamics of the flat porous sliders in a second-order viscoelastic fluid. In addition, in 1993, Ariel extended the work of Wang to Walter's B' viscoelastic fluid.

Mathematical modeling of these types of problems comes in handy when trying to simulate these experiments. Ordinary and partial differential equations are those type of equations that succeed in simulating this and some new techniques for finding their solution is needed. These techniques can provide analytical solutions unlike other methods because in most cases other techniques fail to provide accurate solutions. Currently, researchers are interested in such techniques to construct analytical solutions including Adomian decomposition [7] and perturbation techniques. Perturbation techniques depend mainly on what is so-called a small parameter [34]. Thus, the need to find strong and efficient techniques that are independent of any parameter is a must. One of these techniques that can achieve this property is the HAM which was first developed by Liao [24]-[33]. This technique has been used ever since for solving different classes of problems with different types [20]-[6]. Therefore, this paper aims into adapting this method for solving the problem of viscous Walter B's viscoelastic fluid with elliptic sliders and to evaluate the effects of elasticity and Reynolds number (cross-flow) of fluid on components of velocity utilizing a modern method called HAM. Figure (1) presents a schematic diagram for the problem under consideration.

2 The HAM

Consider

$$\phi(\tau, 0) = u_0(\tau), \tag{2.1}$$

in which N is a nonlinear operator, τ represents an independent variable $u(\tau)$ and is a function (unknown) that needed to be determined. To simplify the computations, all boundary and initial conditions are ignored. Then, by generalizing the basic definition of the HAM method, Liao [26] constructed the equation:

$$(1-p)L\left[\phi(\tau,p) - u_0(\tau)\right] = p\,\hbar\,H(\tau)\mathrm{N}\left[\phi(\tau,p)\right] \tag{2.2}$$

in which $p \in [0,1]$ is a constant called embedding, $\hbar \neq 0$ a parameter, $H(\tau) \neq 0$ a function, L an operator, $u_0(\tau)$ an initial estimation of $u(\tau)$ and $\varphi(\tau;p)$ is a function needs to be determined later. If$p = 0$ and$p = 1$, then:

$$\phi(\tau, 0) = u_0(\tau), \qquad \textit{and} \qquad \phi(\tau, 1) = u(\tau). \tag{2.3}$$

Hence, the unknown solution $\varphi(\tau;p)$ apart from the initial guess, $u_0(\tau)$ to $u(\tau)$ as p changes the value from *0* to *1*. Then, by expanding the solution $\varphi(\tau;p)$ in terms of Taylor series expansion concerning p, we get

$$\phi(\tau;p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)p^m \tag{2.4}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \phi(\tau,p)}{\partial p^m}\bigg|_{p=0} \tag{2.5}$$

This series of Eq. (2.5) shall converge when $p = 1$ an appropriate value of operator, h is well chosen. Therefore,

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau). \tag{2.6}$$

Liao [26] proved that Eq. (2.6) must be one of the solutions to the original equations $H(\tau) = 1$. Thus, Eq. () is

$$(1-p)L[\phi(\tau;p) - u_0(\tau)] + p\hbar N[\phi(\tau;p)] = 0 \tag{2.7}$$

Define:

$$\underset{u\ n}{\rightarrow} = \{u_0(t), ..., u_n(t)\}. \tag{2.8}$$

By taking the derivative of Eq.(2.2) m times concerning p, letting $p = 0$, and dividing by $m!$, we get

$$L[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar H(\tau) R_m(\vec{u}_{m-1}), \tag{2.9}$$

where,

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(\tau;p)]}{\partial p^{m-1}}\bigg|_{p=0} \tag{2.10}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{2.11}$$

Notice that$u_m(\tau)$ for $m \geq 1$ is governed under Eq. (2.9) with boundary conditions solved by MAPLE 11.

3 Description of model

We consider the study of flow between an elliptic porous plate and ground of Walter's B' Fluid type. The flow described to be injected through the plate and the boundaries are defined by $x^2 + \beta y^2 = D^2(\beta < 1)z = d$ whichβ is the ratio of squares between minor and major axes.

Provided pressure is supposed big enough to produce U_3 is the almost constant velocity of injection through the plates. The plate is also assumed laterally moving with $U_1$and U_2 velocities in direction of both x, and y. The gab by d across ground and plates is supposed to be small compared with D, that is, $D >> d$ *and* hence, the effect of the edges can be ignored.

The Cauchy stress tensor T Walter's B' viscoelastic fluid can be connected to motion through the equation

$$T = -Ip + 2_0 e - 2k_0 \frac{de}{dt} \tag{3.1}$$

In which, p is defined as the pressure, I considered an identity tensor, and e is the strain tensor that can be calculated from the following

$$2e = \nabla v + (\nabla v)^T,\ \{(\nabla v)^{ij} = \frac{\delta v_j}{\delta x_i}\}. \tag{3.2}$$

In Eq.(3.2), v is the velocity vector, ∇ is the gradient operator, and $\delta/\delta t$denotes the convected differentiation by a tensor quantity concerning the material in motion. The convected differentiation of the rate of strain tensor is in the form

$$\frac{de}{dt} = \frac{e}{t} - e.\nabla v + v.\nabla e - (\nabla v)^T.e. \tag{3.3}$$

Finally,η_0 is the velocity in a small shear rate and k_0 is a short memory coefficient that can be defined as

$$\tau_0 = \int_0^8 N(t) \quad d\tau, \tag{3.4}$$

$$k_0 = \int_0^8 tN\left(t\right) \, d\tau, \tag{3.5}$$

here $N(\tau)$ is the distribution function and τ relaxation time. This is a suitable model for approximating Walter's B' viscoelastic fluid by taking short memory into account so the term involving

$$\int_0^8 t^n N(\tau) d\tau, n = 2, \tag{3.6}$$

can be ignored. Besides the above equation, the equation for the model can be summarized as follows

1. Continuity equation:

$$\nabla.\nu = 0, \tag{3.7}$$

1. Equation of motion:

$$\rho(\nu.\nabla\nu) = \nabla.T, \tag{3.8}$$

where $\rho$is considered as the density of the fluid. The flow for the basic equations in Eqs. (3.7) and (**??**) can be assumed steady and laminar, incompressible and the body forces may be negligible. Then, by Cauchy tensor from Eq. (3.1), (3.2), (3.3) we get

$$\rho(\nu.\nabla\nu) = -\nabla p + \eta_0 \nabla^2 \nu - 2k_0 \nu.\nabla\nabla^2 \nu + k_0 \nabla^2(\nu.\nabla\nu), \tag{3.9}$$

We search for an equation

$$u = U_1 f(\eta) + \frac{U_3}{d} h'(\eta), v = U_2 g(\eta) + \frac{U_3 y}{d} k'(\eta), w = -U_3\{h(\eta) + k(\eta)\}. \tag{3.10}$$

in here $\eta = z/d$. The velocity field boundary conditions are

$$u(0) = U_1, v(0) = U_2, w(0) = 0, \tag{3.11}$$

Then, with the aid of Eq. (2.10) transformation can be done and the general form of the pressure distribution is

$$\begin{aligned} p(x, y, \eta) = {} & C_1 y + C_3 x + C_2 \tfrac{y^2}{2} + C_4 \tfrac{x^2}{2} + \tfrac{1}{2}\rho w^2 + \eta_0 \tfrac{dw}{dz} + 2k_0 (\tfrac{dw}{dz})^2 \\ & - k_0 w \tfrac{d^2 w}{dz^2} + p_0, \end{aligned} \tag{3.12}$$

$$\begin{aligned} C_1 = {} & \tfrac{\eta_0 U_2}{d^2}(g'' + R\{(h+k)g' - k'g\} + RN\{(h+\ k)g''' - (k' + 2h')g'' \\ & + (k'' - h'')g' - k'''g\}) \end{aligned} \tag{3.13}$$

$$\begin{aligned} C_2 = {} & \tfrac{\eta_0 U^3}{d^3}(k''' + R\{(h+\ k)k'' - k'^2\} + RN\{(h+\ k)k^{IV} - 2(h' + k')k''' \\ & - h''k'' +\ k''^2\}) \end{aligned} \tag{3.14}$$

$$\begin{aligned} C_4 = {} & \tfrac{\eta_0 U_3}{d^3}(h''' + R\{(h+\ k)h'' - h'^2\} + RN\{(h+k)h^{IV} - 2(h' + k')h''' \\ & - k''h'' + h''^2\}), \end{aligned} \tag{3.15}$$

The constant of integration in the above equation is$p_0$and R is a cross-flow Reynolds number

$$R = \frac{\rho U_3 d}{\eta_0}, \tag{3.16}$$

$$N = \frac{k_0}{\rho d^2}. \tag{3.17}$$

C_1 to C_4 have to satisfy:

$$C_1 = 0, C_3 = 0, C_2 = \beta C_4. \tag{3.18}$$

Writing Eq. (3.18) in Eqs. (**??**)-(**??**) , get

$$\begin{aligned} p(x, y, \eta) = {} & \tfrac{\rho {U_3}^2 A}{2d^2 R}(x^2 + \beta y^2) - \tfrac{1}{2}\rho w^2 + \eta_0 \tfrac{dw}{dz} + 2k_0(\tfrac{dw}{dz})^2 \\ & - k_0 w \tfrac{d^2 w}{dz^2} + p_0, \end{aligned} \tag{3.19}$$

$$\begin{aligned} & h''' + R\{(h + k)h'' - h'^2\} + RN\{(h + k)h^{IV} - 2(h' + k')h''' \\ & -k''h'' + h''^2\} - A = 0, \end{aligned} \tag{3.20}$$

$$\begin{aligned} & k''' + R\{(h + k)k'' - k'^2\} + RN\{(h + k)k^{IV} - 2(h' + k')k''' \\ & -k''h'' + k''^2\} - \beta A = 0, \end{aligned} \tag{3.21}$$

$$\begin{aligned} & f'' + R\{(h + k)f' - h'f\} + RN\{(h + k)f''' - (h' + 2k')f'' \\ & +(h'' - k'')f' + h'''f\} = 0, \end{aligned} \tag{3.22}$$

$$\begin{aligned} & g'' + R\{(h + k)g' - k'g\} + RN\{(h + k)g''' - (k' + 2h')g'' \\ & +(k'' - h'')g' + k'''g\} = 0, \end{aligned} \tag{3.23}$$

BCs in Eq.(3.11) implies

$$h'(0) = 0, h^{'}(1) = 0,\ h(0) = 0, \tag{3.24}$$

$$k'(0) = 0, k^{'}(1) = 0,\ k(0) = 0, \tag{3.25}$$

$$f\,(1) = 0,\ f(0) = 1, \tag{3.26}$$

$$g\,(1) = 0, g(0) = 1. \tag{3.27}$$

4 Solutions

Let the initial guesses and linear operators be defined as:

$$\begin{aligned} & H_0(\eta) = \tfrac{1}{1+\beta}(3\eta^2 - 2\eta^3), \\ & K_0(\eta) = \tfrac{\beta}{1+\beta}(3\eta^2 - 2\eta^3), \\ & f_0(\eta) = 1 - \eta, \\ & g_0(\eta) = 1 - \eta. \end{aligned} \tag{4.1}$$

and

$$(1 - P)L_1\left[H(\eta; p) - H_0(\eta)\right] = p\hbar_1 N_1\left[H(\eta; p)\right] \tag{4.2}$$

$$(1 - P)L_2\left[K(\eta; p) - K_0(\eta)\right] = p\hbar_2 N_2\left[K(\eta; p)\right] \tag{4.3}$$

$$(1 - P)L_3\left[f(\eta; p) - f_0(\eta)\right] = p\hbar_2 N_2\left[f(\eta; p)\right] \tag{4.4}$$

$$(1 - P)L_4\left[g(\eta; p) - g_0(\eta)\right] = p\hbar_2 N_2\left[g(\eta; p)\right] \tag{4.5}$$

$$\begin{aligned} & H(0; p) = 0\,; \quad H'(0; p) = 0; \quad H'(1; p) = 0; \\ & K(0; p) = 0\,; \quad K'(0; p) = 0; \quad K'(1; p) = 0; \\ & f(0; p) = 1\,; \quad f(1; p) = 0; \\ & g(0; p) = 1\,; \quad g(1; p) = 0; \end{aligned} \tag{4.6}$$

$$\begin{aligned} N_1\left[H(\eta; p)\right] = {} & H''' + R\{(H + K)H'' - H'^2\} \\ & + RN\{(H + K)H^{IV} - 2(H' + K')H''' - K''H'' + H''^2\} - A = 0, \end{aligned} \tag{4.7}$$

$$N_2\left[K(\eta;p)\right] = K''' + R\{(H+K)K'' - K'^2\} \\ + RN\{(H+ K)K^{IV} - 2(H'+K')K''' - K''H'' + K''^2\} - \beta A = 0, \tag{4.8}$$

$$N_3\left[f(\eta;p)\right] = f'' + R\{(H+ K)f' - H'f\} \\ + RN\{(H+K)f''' - (H'+2K')f'' + (H''-K'')f' + H'''f\} = 0, \tag{4.9}$$

$$N_4\left[g(\eta;p)\right] = g'' + R\{(H+ K)g' - K'g\} \\ + RN\{(H+K)g''' - (K'+2H')g'' + (K''-H'')g' + K'''g\} = 0, \tag{4.10}$$

For$p = 0$ and $p = 1$ we have

$$H(\eta;0) = H_0(\eta); \quad H(\eta;1) = H(\eta); \quad G(\eta;0) = G_0(\eta); \quad G(\eta;1) = G(\eta)f(\eta;0) = f_0(\eta); \quad f(\eta;1) = f(\eta); \quad g(\eta;0) = g_0(\tag{4.11}$$

When p increases from 0 to 1 then $H(\eta;p)$ vary from $H_0(\eta)$ to $H(\eta)$, $K(\eta)$ vary from $K_0(\eta)$ to $K(\eta)$, $f(\eta)$ vary from $f_0(\eta)$ to $f(\eta)$, and $g(\eta)$ vary from $g_0(\eta)$ to $g(\eta)$. By Taylor's theorem and using Eqs. (4.7) to (4.10),$H(\eta;p)$, $K(\eta;p)$, $f(\eta;p)$ and $g(\eta;p)$ can be expanded in a power series of p as follows:

$$H(\eta;p) = H_0(\eta) + \sum_{m-1}^{\infty} H_m(\eta)p^m, \quad H_m(\eta) = \frac{1}{m!}\frac{\partial^m(H(\eta;p))}{\partial p^m} \tag{4.12}$$

$$K(\eta;p) = K_0(\eta) + \sum_{m-1}^{\infty} K_m(\eta)p^m, \quad K_m(\eta) = \frac{1}{m!}\frac{\partial^m(K(\eta;p))}{\partial p^m} \tag{4.13}$$

$$f(\eta;p) = f_0(\eta) + \sum_{m-1}^{\infty} f_m(\eta)p^m, \quad f_m(\eta) = \frac{1}{m!}\frac{\partial^m(f(\eta;p))}{\partial p^m} \tag{4.14}$$

$$g(\eta;p) = g_0(\eta) + \sum_{m-1}^{\infty} g_m(\eta)p^m, \quad g_m(\eta) = \frac{1}{m!}\frac{\partial^m(g(\eta;p))}{\partial p^m} \tag{4.15}$$

Let $\hbar_1 = \hbar_2 = \hbar_3 = \hbar_4 = \hbar$, for $\hbar$ is the series are convergent at$p = 1$. Therefore, by Eqs. (4.12) to (4.15):

$$H(\eta) = H_0(\eta) + \sum_{m-1}^{\infty} H_m(\eta), \tag{4.16}$$

$$G(\eta) = G_0(\eta) + \sum_{m-1}^{\infty} G_m(\eta), \tag{4.17}$$

$$f(\eta) = f_0(\eta) + \sum_{m-1}^{\infty} f_m(\eta), \tag{4.18}$$

$$g(\eta) = g_0(\eta) + \sum_{m-1}^{\infty} g_m(\eta), \tag{4.19}$$

mth-order equations

$$L\left[H_m(\eta) - \chi_m H_{m-1}(\eta)\right] = \hbar R_m^H(\eta) \tag{4.20}$$

$$H_m(0) = 0\,; \quad H'_m(0) = 0; \quad H'_m = 0\,; \tag{4.21}$$

$$L\left[K_m(\eta) - \chi_m K_{m-1}(\eta)\right] = \hbar R_m^K(\eta) \tag{4.22}$$

$$G_m(0) = 0\,; \quad G'_m(0) = 0; \quad G'_m(1) = 0\,; \tag{4.23}$$

$$L\left[f_m(\eta)-\chi_m f_{m-1}(\eta)\right]=\hbar R_m^f(\eta) \tag{4.24}$$

$$f_m(0)=0\,;\qquad f_m(1)=0; \tag{4.25}$$

$$L\left[g_m(\eta)-\chi_m g_{m-1}(\eta)\right]=\hbar R_m^g(\eta) \tag{4.26}$$

$$g_m(0)=0\,;\qquad g_m(1)=0 \tag{4.27}$$

By Maple, we get

$$\begin{aligned}H_1(\eta)=&-\tfrac{6\hbar R}{(1+\beta)^2}\big(\tfrac{1}{20}\beta\eta^5-\tfrac{1}{15}\beta\eta^6+\tfrac{2}{105}\beta\eta^7-\tfrac{1}{20}\eta^5+\tfrac{1}{30}\eta^6-\tfrac{1}{105}\eta^7\\&+N\eta^3+2N\beta\eta^4-\tfrac{4}{5}N\beta\eta^5-N\beta\eta^3\big)+\tfrac{1}{6}\big(-\tfrac{12}{1+\beta}-\tfrac{12\hbar}{1+\beta}-\hbar A\big)\eta^3\\&-\tfrac{1}{20(1+2\beta+\beta^2)}\big(-\hbar R\beta-7\hbar R+180\hbar RN+60\hbar RN\beta-60-60\beta\\&-60\hbar-60\hbar\beta-5\hbar A-10\hbar\beta A-5\hbar\beta^2A\big)\eta^2,\end{aligned} \tag{4.28}$$

$$\begin{aligned}K_1(\eta)=&-\tfrac{6\hbar R\beta}{(1+\beta)^2}\big(\tfrac{1}{20}\beta\eta^5-\tfrac{1}{30}\beta\eta^6+\tfrac{2}{105}\beta\eta^7-\tfrac{1}{20}\eta^5+\tfrac{1}{15}\eta^6-\tfrac{2}{105}\eta^7\\&+N\eta^3-2N\eta^4+\tfrac{4}{5}N\eta^5-N\beta\eta^3\big)+\tfrac{1}{6}\big(-\tfrac{12\beta}{1+\beta}-\tfrac{12\hbar\beta}{1+\beta}-\hbar\beta A\big)\eta^3\\&-\tfrac{\beta}{20(1+2\beta+\beta^2)}\big(-60\hbar RN+7\hbar R\beta+\hbar R-180\hbar RN\beta+60\hbar\\&+60\hbar\beta+5\hbar A+10\hbar\beta A+5\hbar A\beta^2+60+60\beta\big)\eta^2,\end{aligned} \tag{4.29}$$

$$\begin{aligned}f_1(\eta)=&\tfrac{\hbar R}{1+\beta}\big(\tfrac{3}{4}\beta\eta^4-\tfrac{1}{5}\eta^5-\tfrac{1}{4}\beta\eta^4+\tfrac{1}{10}\beta\eta^5-\eta^3+3N\eta^2+3N\beta\eta^2-2N\beta\eta^3\big)\\&-\tfrac{1}{20}\tfrac{\hbar R}{1+\beta}(-9-3\beta+60N+20N\beta)\eta,\end{aligned} \tag{4.30}$$

$$\begin{aligned}g_1(\eta)=&-\tfrac{\hbar R}{1+\beta}\big(-3N\beta\eta^2-3N\eta^2+2N\eta^3-\tfrac{3}{4}\beta\eta^4+\tfrac{1}{5}\beta\eta^5+\beta\eta^3+\tfrac{1}{4}\eta^4-\tfrac{1}{10}\eta^5\big)\\&-\tfrac{1}{20}\tfrac{\hbar R}{1+\beta}(-3-9\beta+20N+60N\beta)\eta.\end{aligned} \tag{4.31}$$

The solutions$H_2(\eta)$,$K_2(\eta)$,$f_2(\eta)$ and $g_2(\eta)$ are depicted graphically since they are too long.

5 Results

In Figs. (5)-(5), we prove that HAM is an acceptable method to solve this similarity solution. A comparison is made between HAM and other methods to verify the results. Both two figures show the high accuracy of HAM. These figures are shown below. The numerical method used in these figures is Runge-Kutta which is solved by Maple 11.

In Figs. (5) to (5) have been shown the results of HAM for lateral profiles of velocity in x - y directions and vertical velocity profiles by considering various values of parameters such as viscoelasticity (N), cross-flow Reynolds (R), and (β).

In Fig. (5), the difference of velocity profile in x-direction versus η has been shown. As shown in this figure, the velocity profile in the x-direction has the same profile for different values of (R). But increasing or decreasing the value of lateral velocity is depending on the elasticity number (N). When $N = 0$ (this means that the fluid is Newtonian fluid), increasing the cross-flow Reynolds number (R), causes a decrease in the value of lateral velocity. But if we consider $N \neq 0$ (this means that the fluid is viscoelastic fluid), the obtained results are inverse. In this manner, increasing in (R), increase values of lateral velocity in the x-direction.

From Fig. (5), it is clear that the same results can be obtained for the lateral velocity profile in the y-direction. So, the previous results are correct for this section, exactly. In these below figures, we evaluate the profiles of velocity in z-direction or vertical velocity profile. From Fig. 6, we can obtain that difference in cross-flow Reynolds number (R) has no significant effect on vertical velocity profiles.

Eventually, we consider Fig.(5). From this figure, we can obtain the result that we mentioned in the previous figure. However, in this section, we change the elasticity number (N) with a constant cross-flow Reynolds number (R).

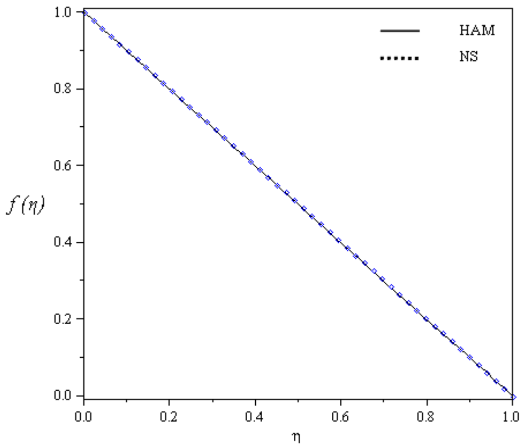


Figure 2: Comparison of the method with HAM for β*=0.5, N=0*, *R=0.2*

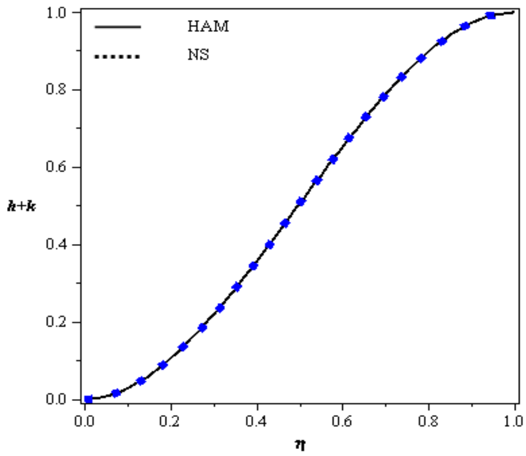
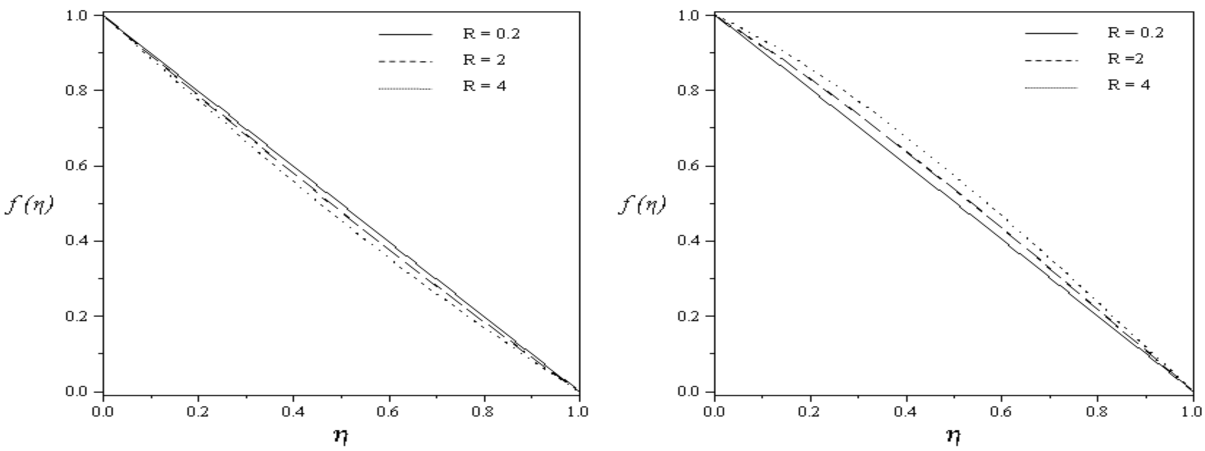
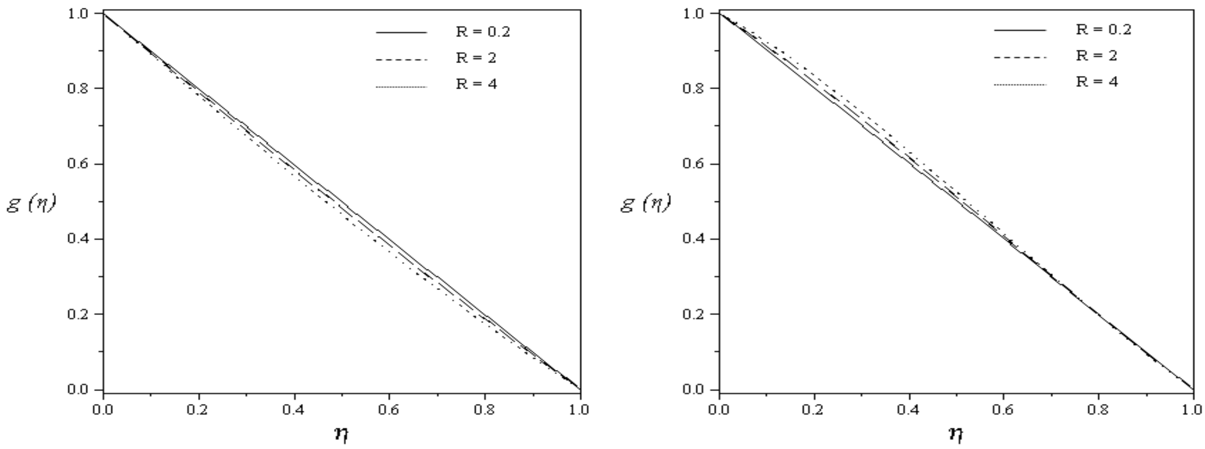


Figure 3: Comparison of the method with HAM for β*=0.5, N=0*



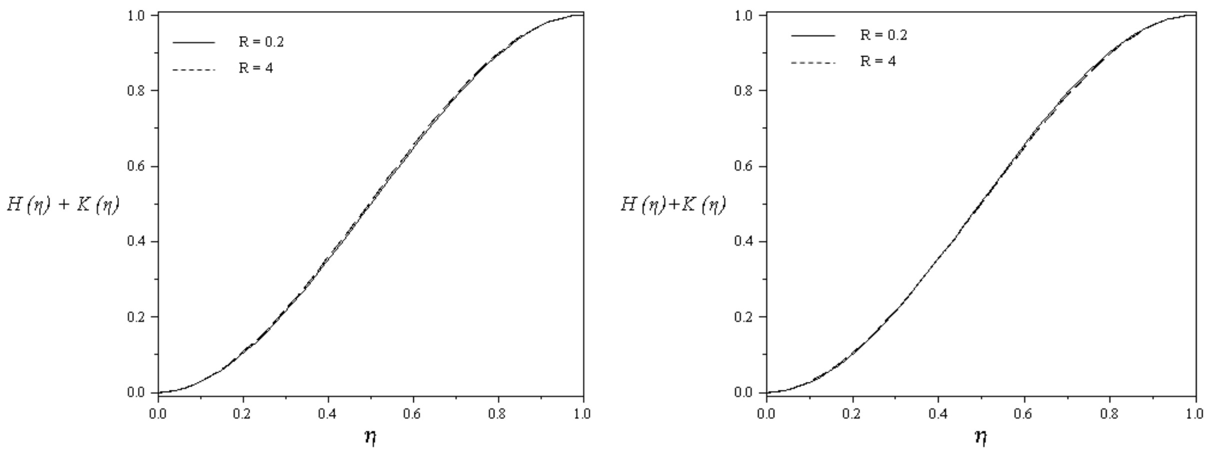
(a) (b)

Figure 4: Lateral velocity profile in x direction by $\beta = 0.5$(a) $N = 0$ (b)$N = 0.6$



(a) (b)

Figure 5: Lateral velocity profile in y direction by $\beta = 0.5$(a) $N = 0$ (b)$N = 0.6$



(a) (b)

Figure 6: Vertical velocity profile for $\beta = 0.5$ (a) $N = 0$ (b) $N = 0.6$

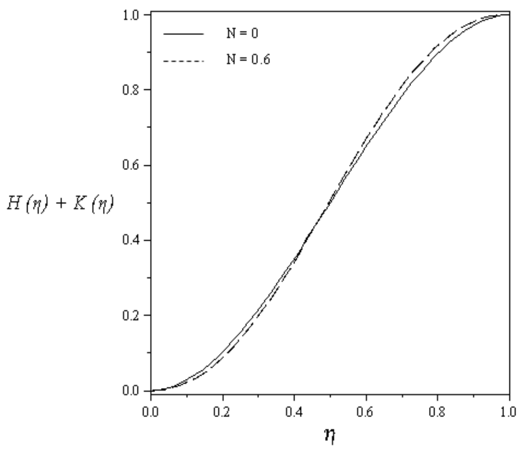


Figure 7: Vertical velocity profile for $\beta = 0.5$, $R = 0.8$

6 conclusion

We investigated a viscoelastic fluid of Walter B's fluid type in three-dimensional across a porous elliptic plate and ground. Using similarity transformation, these governing equations are converted into ODEs. In viscoelastic fluids, an increase in values of (R) can give us increasing the lateral velocity in both x and y-direction. However, for Newtonian fluids, we obtain an inverse result. The other result is that the lateral velocity profile in both x and y directions are similar. Also, by evaluating the last profiles for vertical velocities, we obtain that the change of the elasticity number (N) and (R) is insignificant for the vertical velocity profile of the Walter's B' fluid. All these analyses are calculated by HAM.

Nomenclature

NS	Numerical Solution	$\boldsymbol{T}$	Cauchy Stress Tensor
HPM	Homotopy Perturbation Method	t	Time
ρ	Density	U_1, U_2	Constant lateral Velocity Components
$\boldsymbol{e}$	Rate of Strain Tensor	U_3	The velocity of Uniform Injection
I	Identity Tensor,	v, u, w	Velocity Vector Components
k_0	Short memory Coefficient	ν	Velocity Vector,
N	Elastic Number	β	Minor axis square ration to Major
P	Pressure	η	Coordinate Axial (Normalized)
R	Cross-flow Reynolds Number	τ	Relaxation Times

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