Int. J. Nonlinear Anal. Appl. 14 (2023) 7, 119–122
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.7160

Existence of solutions for a (p, q)-Laplace equation with Steklov boundary conditions

Abdolrahman Razani, Farzaneh Safari*

Department of Mathematics, Imam Khomeini International University, Qazvin, Iran

(Communicated by Asadollah Aghajani)

Abstract

Here, the existence of at least one nontrivial solution for the (p, q)-Laplacian problem

$$\begin{cases} div(|\nabla u|^{p-2}\nabla u) + div(|\nabla u|^{q-2}\nabla u) = f(x, u) & x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} + |\nabla u|^{q-2}\frac{\partial u}{\partial n} = g(x, u) & x \in \partial\Omega \end{cases}$$

is done, where Ω is a bounded domain in $\mathbb{R}^N, N \geq 3$ and $q, p \geq 2$, via variational methods.

Keywords: (p, q)-Laplace equation, Steklov boundary conditions, Mountain Pass Theorem
2020 MSC: Primary 49J24; Secondary 49J52

1 Introduction

Usually solutions to (p, q)-Laplacian problems are the steady state solutions of the reaction diffusion systems. Reaction–diffusion systems are mathematical models which correspond to several physical phenomena. This system has a wide range of applications in physics and related sciences like chemical reaction design, biophysics, plasma physics, geology, and ecology.
This equations also arise in the study of soliton-like solutions of the nonlinear Schrödinger equation as a model for elementary particles for example waves in a discrete electrical lattice. These problems have been intensively studied in the last decades.
In this note, we investigate the existence of at least one weak solutions for a (p, q)-Laplacian problem with Steklov boundary conditions and our starting point is introducing some notations and recalling a basic result which compose the tools that are needed for proving our claim.

2 Preliminaries

Thought this note Ω is a bounded domain in $\mathbb{R}^N, N \geq 3$, for $p > 1$, by $|.|_p$ we denote the norm on the Lebesgue space $L^p(\Omega)$, and $\|.\|_p$ denotes the norm of the Sobolev space $W_0^{1,p}(\Omega)$, i.e. $\|u\|_p = |\nabla u|_p$.

$f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions that hold in the following conditions

*Corresponding author
Email addresses: razani@sci.ikiu.ac.ir (Abdolrahman Razani), f.safari@edu.ikiu.ac.ir (Farzaneh Safari)

Received: March 2022 *Accepted:* September 2022

$(f1)$ The constants $a_1, a_2 \geq 0$ exist such that

$$|f(x,t)| \leq a_1 + a_2|t|^{\theta-1}, \qquad (x,t) \in \Omega \times \mathbb{R}$$

with $1 < \theta < p^*$, where

$$p^*(N) = \begin{cases} \frac{Np}{N-p} & p < N, \\ \infty & p \geq N; \end{cases}$$

$(f2)$ $f(x,t)t \geq 0$ for all $(x,t) \in \Omega \times \mathbb{R}$;

$(f3)$ $f(x,0) \neq 0$ for all $x \in \Omega$;

$(g1)$ The constant $b \geq 0$ exists such that

$$|g(x,t)| \leq b|t|^{\gamma-1}, \qquad (x,t) \in \partial\Omega \times \mathbb{R}$$

with $1 < \gamma < p_*^{\partial}(N)$, where

$$p_*^{\partial}(n) = \begin{cases} \frac{(N-1)p}{N-p} & p < N, \\ \infty & p \geq N; \end{cases}$$

$(g2)$ There exists constant $\mu > 0$ such that

$$\mu G(x,t) \leq t g(x,t), \quad (x,t) \in \partial\Omega \times \mathbb{R}$$

where $G(x,t) = \int_0^t g(x,s)ds$ and $p, q, \mu, \gamma, \theta$ satisfy the following relations

$$2 \leq p < \gamma < q < \mu \quad \& \quad \theta < \mu. \tag{2.1}$$

Definition 2.1. ((PS) compactness condition) Let X be a reflexive Banach space. We say that $I \in C(X, \mathbb{R})$ satisfies the Palais-Smale (PS) compactness condition if any sequence $\{u_k\} \subset X$ such that

- $\{I(u_k)\}$ is bounded, and
- $I'(u_k) \to 0$ in X,

has a convergent subsequence in X.

The Mountain Pass Theorem (MPT) is an existence theorem from the calculus of variations and is as follows.

Theorem 2.2. Let $(X, \|.\|_X)$ be a reflexive Banach space. Suppose that the functional $I : X \to (-\infty, +\infty]$ satisfies (PS) compactness condition and also the following assertions

(i) $I(0) = 0$;

(ii) There exists $e \in V$ such that $I(e) \leq 0$;

(iii) There exists positive constant ρ such that $I(u) > 0$, if $\|u\|_X = \rho$;

Then I has a critical value $c \geq \rho$ which is characterized by

$$c = \inf_{h \in \Gamma} \sup_{t \in [0,1]} I(h(t)),$$

where $\Gamma = \{h \in C([0,1], X) : h(0) = 0\ ,\ h(1) = e\}$.

3 Main result

We state now the main result of the paper.

Theorem 3.1. Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 3$, constants $p, q, \gamma, \mu, \theta$ hold the relations (2.1) and the functions f, g satisfy the assumptions $(f1), (f_2)$ and $(g1), (g2)$, respectively. Then the following Steklov problem

$$\begin{cases} div(|\nabla u|^{p-2}\nabla u) + div(|\nabla u|^{q-2}\nabla u) = f(x, u) & x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} + |\nabla u|^{q-2}\frac{\partial u}{\partial n} = g(x, u) & x \in \partial\Omega, \end{cases} \tag{3.1}$$

admits at least one nontrivial (weak) solution.

We point out in [2, 3, 4, 5, 6, 7, 8] authors have probed some elliptic equations with different boundary conditions usually on the Heisenberg groups.

We continue by the definition of weak solution for the problem (3.1).

Definition 3.2. (Weak solution) We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (3.1) if the following integral equality is true

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla v dx + \int_\Omega f(x, u) v dx = \int_{\partial\Omega} g(x, u) v d\sigma,$$

for any $v \in W_0^{1,p}(\Omega)$.

We consider the Euler-Lagrange energy functional corresponding to the problem (3.1); i.e.,

$$I(u) := \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{q}\int_\Omega |\nabla u|^q dx + \int_\Omega F(x, u) dx - \int_{\partial\Omega} G(x, u) d\sigma,$$

that in which

$$F(x, t) = \int_0^t f(x, s) ds \quad \& \quad G(x, t) = \int_0^t g(x, s) ds.$$

Clearly, every critical point of I is a weak solution of the problem (3.1). To prove that I has a critical point we apply MPT (Theorem 2.2):

Firstly, we verify that I satisfies MPT conditions:

Proof . It is clear that $I(0) = 0$. Since $p < q$, so $W^{1,q}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ and from $(g2)$, one has

$$C|t|^\mu \leq G(x, t), \quad (x, t) \in \partial\Omega \times \mathbb{R}.$$

for some suitable $C > 0$. Take $e \in W^{1,q}(\Omega) \subset W^{1,p}(\Omega)$, then for appropriate $a_1', a_2' > 0$, we have

$$\begin{aligned} I(te) &= \tfrac{t^p}{p}\|e\|_p^p + \tfrac{t^q}{q}\|e\|_q^q + \textstyle\int_\Omega F(x, te) dx - \int_{\partial\Omega} G(x, te) d\sigma \\ &\leq \tfrac{t^p}{p}\|e\|_q^p + \tfrac{t^q}{q}\|e\|_q^q + a_1'\|u\|_q + a_2'|t|^\theta \|u\|_q^\theta - C|t|^\mu \textstyle\int_{\partial\Omega} |e|^\mu d\sigma. \end{aligned}$$

Now, since $\mu > q > \gamma > p > 1$ and $\mu > \theta$ for t large enough $I(te)$ is negative. We now prove condition (iii) of MPT. From $(f2)$ one gains that $F(x, t) \geq 0$ for $t \in \mathbb{R}$. Take u with $\|u\|_p = \rho > 0$. Using standard embedding it follows that

$$I(u) \geq \frac{1}{p}\rho^p + \frac{1}{q}\rho^q - b'\rho^\gamma > 0,$$

provided $\rho > 0$ is small enough. Therefore, MPT conditions are held for the functional I. $\square$

Now, we verify Palais-Smale compactness condition.

Proof . In fact we show that any (PS)-sequence is bounded. To this end, suppose that $\{u_k\}$ is a sequence in $W_0^{1,p}(\Omega)$ such that

$$\{I(u_k)\} \text{ is bounded} \quad \& \quad I'(u_k) \to 0 \quad \text{in } W_0^{1,p}(\Omega).$$

Using the standard embedding, there exists $b' > 0$ such that

$$\begin{aligned} I'(u_k)u_k &= \|u_k\|_p^p + \|u_k\|_q^q + \int_\Omega f(x,u_k)u_k dx - \int_{\partial\Omega} g(x,u_k)u_k d\sigma, \\ &\geq \|u_k\|_p^p + \|u_k\|_p^q - b'\|u_k\|_p^\gamma \end{aligned}$$

Thus, for large enough k we have

$$\|u_k\|_p^q \leq \|u_k\|_p^p + \|u_k\|_p^q \leq b'\|u_k\|_p^\gamma.$$

Since $\gamma < q$, $\{u_k\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$ as desired. □

References

[1] M. Badiale and E. Serra, *Semilinear Elliptic Equations for Beginners, Existence results via the variational approach*, Springer, 2011.

[2] M.A. Ragusa, A. Razani and F. Safari, *Existence of radial solutions for a $p(x)$-Laplacian Dirichlet problem*, Adv. Differential Equations **2021** (2021), no. 1, 1–14.

[3] F. Safari and A. Razani, *Existence of positive radial solutions for Neumann problem on the Heisenberg group*, Bound. Value Probl. **2020** (2020), no. 1, 1–14.

[4] F. Safari and A. Razani, *Nonlinear nonhomogeneous Neumann problem on the Heisenberg group*, Appl. Anal. **101** (2020), no. 7, 2387–2400.

[5] F. Safari and A. Razani, *Existence of radial solutions of the Kohn-Laplacian problem*, Complex Var. Elliptic Equ. **67** (2022), no. 2, 259–273.

[6] F. Safari and A. Razani, *Positive weak solutions of a generalized supercritical Neumann problem*, Iran J. Sci. Technol. Trans. Sci. **44** (2020), no. 6, 1891–1898,

[7] F. Safari and A. Razani, *Radial solutions for a general form of a p-Laplace equation involving nonlinearity terms*, Complex Var. Elliptic Equ. (2021), 1–11.`https://doi.org/10.1080/17476933.2021.1991331`.

[8] F. Safari and A. Razani, *Existence of radial solutions for a weighted p-biharmonic problem with Navier boundary condition on the Heisenberg group*, Math. Slovaca **72** (2022), no. 3, 677–692.