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Some notes on the greedy basis for Banach spaces under ε-isometry

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Abstract

In this paper, we discuss some conditions of a greedy basis for Banach space X under a standard ε-isometry mapping. We show that if X and Y are Banach spaces, (x_n) is a greedy basis for X, and $f : X \to Y$ is a standard ε-isometry, then $(f(x_n))$ is a greedy basis for a subspace of Y. As a result, if f is a surjective standard ε-isometry, then $(f(x_n))$ is a greedy basis for Y. We also show that $span\{(f(x_n))\}^*$ is isomorphic with $\Psi \subset Y^*$ where Ψ is defined as

$$\Psi := \overline{span}\{\psi_n : \ \psi_n \in Y^* \ and \ |\langle x_n^*, x\rangle - \langle \psi_n, f(x)\rangle| < 3\varepsilon a\}$$

where $\|\psi_n\| = a = \|x_n^*\|$.

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1 Introduction

The study of ε-isometry emerged from Mazur-Ulam's paper showing that any surjective isometry mapping $g : X \to Y$ is affine, where X and Y are real Banach spaces [16]. Besides, if $g(0) = 0$, then g is a linear mapping. These results indicate that isometry mapping has an important role. Note that this result does not work for complex Banach spaces. Hence, X and Y always refer to real Banach spaces. The surjective conditions and $g(0) = 0$ in the Mazur-Ulam's theorem are weakened by Figiel who showed that for any isometric mapping, there is a bounded linear operator $F : \overline{span}(g(X)) \to X$ with $\|F\| = 1$ such that $Fg = Id_X$ which shows that the domain of Fg must be X, i.e., for any isometry mapping g, $Fg : X \to X$ is an isometry [11]. ε-isometry is a generalization of the concept of isometry, which was introduced in 1945 by Hyers and Ulam. Suppose that there is a mapping $f : X \to Y$ where X and Y are Banach spaces. If for any $\varepsilon \geq 0$, the mapping f satisfies

$$|\|f(x) - f(y)\| - \|x - y\|| \leq \varepsilon$$

for every $x, y \in X$, then the mapping f is called an *ε-isometry*. Obviously, 0-isometry is just an isometry. Also, f is called *standard* if $f(0) = 0$.

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Hyers-Ulam [14] showed that for any standard surjective ε-isometry mapping f between Hilbert spaces, there is always a surjective isometry mapping g such that

$$\|f(x) - g(x)\| \leq 10\varepsilon.$$

To this result, many mathematicians have been interested in looking for the ε-isometry conditions in more general spaces (see [3, 4]) or in reducing the value of 10 in the above inequality (see [12, 13]). Finally, Omladič and Šemrl [18] gave a general result on any real Banach spaces with

$$\|f(x) - g(x)\| \leq 2\varepsilon.$$

On the other hand, by providing a counterexample Qian showed that Figiel's theorem does not apply to any ε-isometry and any Banach spaces [19]. Therefore, Cheng *et al.* [6] first provided a solution to this problem, which is to find the stability of the nonsurjective standard ε-isometry under weak topology. The theorem is as follows.

Theorem 1.1. ([6], Lemma 2.4) Suppose that $f: X \to Y$ is a standard ε-isometry. Then for each $x^* \in X^*$, there is $\varphi \in Y^*$ that satisfies $\|\varphi\| = r = \|x^*\|$, such that

$$|\langle x^*, x\rangle - \langle \varphi, f(x)\rangle| \leq \kappa r\varepsilon, \text{ for all } x \in X,$$

where κ=4.

Recently, Cheng and Dong (2020) proved that the constant κ can be optimized to $\kappa = 3$ [5]. The usage of Theorem 1.1 can be found in ([7, 8, 9, 20, 21, 23, 24]). In this paper, the symbols w and w^* will refer to weak and weak* topology, respectively. B_X and S_X are unit ball and unit sphere of a Banach space X, respectively. The other used notions and symbols are commonly found in some textbooks (see [2, 10, 17]).

2 Greedy basis under ε-isometry

Let (x_n) be a Schauder basis for a Banach space X. Then clearly

$$\left\|\sum_{n=1}^{m_1} \alpha_n x_n\right\| \leq M \left\|\sum_{n=1}^{m_2} \alpha_n x_n\right\|$$

whenever $m_1, m_2 \in \mathbb{N}$, $m_1 \leq m_2$, and $\alpha_1, \alpha_2, \ldots, \alpha_{m_2} \in \mathbb{F}$. The scalar M is the basis constant for (x_n). If P_m is a natural projection for (x_n), i.e., $P_m\left(\sum_n \alpha_n x_n\right) = \sum_{n=1}^{m} \alpha_n x_n$, then

$$\|x - P_m(x)\| \leq (M+1) \inf_{\{\beta_n\}} \left\|x - \sum_{n=1}^{m} \beta_n x_n\right\| \tag{2.1}$$

which shows that $P_m(x)$ is a *near-best approximation* for $x \in X$.

Let x_n^* be the coordinate functional of x_n for each n and rearrange the order of (x_n) by choosing the biggest value of $|\langle x_n^*, x_n\rangle|$ from m elements as the first order. Next, choose the biggest value of $|\langle x_n^*, x_n\rangle|$ from $m-1$ elements as the second order. Continuing this process, the permutation ρ:$\mathbb{N} \to \mathbb{N}$ can be obtained such that $\left|\left\langle x^*_{\rho(j)}, x_{\rho(j)}\right\rangle\right| > \left|\left\langle x^*_{\rho(k)}, x_{\rho(k)}\right\rangle\right|$ whenever $j < k$. Let $A_m(x) \subset \mathbb{N}$ be the set of indices obtained from this process. In this case, the m-th greedy approximation of x is defined as

$$\mathcal{G}_m(x) = \sum_{n \in A_m(x)} x_n^*(x) x_n.$$

The sequence of maps $(\mathcal{G}_m(x))_{m=1}^{\infty}$ is called *greedy algorithm* associated with the basis (x_n). If the function $\sigma: \mathcal{G}_m(x) \to \mathbb{R}$ is defined as $\sigma(\mathcal{G}_m) = \sup_{\|x\|\leq 1} \|\mathcal{G}_m(x)\|$, then the homogeneity of $\mathcal{G}_m$ shows that the function σ is a norm-defined function on $(\mathcal{G}_m(x))_{m=1}^{\infty}$. For convenience, let $\sigma(\mathcal{G}_m) = \|\mathcal{G}_m\|$. Obviously $\|\mathcal{G}_m\| > 0$ and $\|\mathcal{G}_m\| = 0$ if and only if $\mathcal{G}_m$ is a zero mapping for all m. The triangle inequality follows from the fact that $\mathcal{G}_m(x)$ is just a series. Albiac and Kalton [2] and Temlyakov ([25, 26]) gave the further discussion of this greedy process.

There are two types of bases regarding for greedy approximation $\mathcal{G}_m(x)$. The first type is $\mathcal{G}_m(x) \to x$ as $m \to \infty$ without using unconditional condition of basis (x_n) for X and the second type is using the unconditionality of (x_n). In the first case, the basis (x_n) is called a quasi-greedy basis while (x_n) is called a greedy basis for the second case.

Definition 2.1. Let X be a Banach space and (x_n) be a basis for X. The basis (x_n) is a *quasi-greedy basis* if $(\mathcal{G}_m(x))_{m=1}^{\infty}$ converges to x in norm topology for all $x \in X$.

The previous discussion shows that the sequence $(\mathcal{G}_m(x))_{m=1}^{\infty}$ is related to the basis (x_n). Therefore, to get a quasi-greedy basis, firstly the Banach space X must contain a basis (x_n).

Definition 2.2. Let X be a Banach space and (x_n) be a basis for X. Assume that (p_n) and (q_n) are sequences of positive integers such that $p_n < q_n$ for each $n \in \mathbb{N}$. Then *Block basic sequence* is a sequence (y_n) such that $y_n = \sum_{i=p_n}^{q_n} x_i^*(x)x_i$ for each n.

Since $\|y_n\| = \left\|\sum_{i=p_n}^{q_n} x_i^*(x)x_i\right\| \leq M \left\|\sum_{i=1}^{n} x_i^*(x)x_i\right\|$ for all $q_n \leq n$, (y_n) is a basic sequence taken with respect to (x_n) ([2], Lemma 1.3.5). The following theorem gives the rule to decide when a basis is quasi-greedy.

Theorem 2.3. ([27], Theorem 1) A basis (x_n) for a Banach space X is quasi-greedy if and only if there is $C_{qg} \geq 1$ such that $\|\mathcal{G}_m(x)\| \leq C_{qg}\|x\|$ for all $x \in X$ and $m \in \mathbb{N}$.

The following proposition is similar to Schauder basis.

Theorem 2.4. ([27], Proposition 3) Let (x_n) be a quasi-greedy basis for a Banach space X and (β_n) be a bounded sequence of nonzero scalars. Then $(\beta_n x_n)$ is also a quasi-greedy basis for X.

Since $0 < \|x_n\| < \infty$ for all elements of a basis, the following definition is reasonable (see [2, 15, 28]).

Definition 2.5. Let X be a Banach space and (x_n) be a basis for X. (x_n) is called a *democratic basis* if blocks of the same size are uniformly comparable under the norm, that is, there is a *democracy constant* $C_d \geq 1$ such that $\left\|\sum_{n\in A} x_n\right\| \leq C_d \left\|\sum_{n\in B} x_n\right\|$ for every $A, B \subset \mathbb{N}$ with $|A| = |B|$.

The constant C_d shows how far a basis being a democracy is. Let an *upper democracy function* be defined as

$$\chi_u(m) = \sup_{|A|\leq m} \left\|\sum_{n\in A} x_n\right\|$$

and a *lower democracy function* be defined as

$$\chi_l(m) = \inf_{|A|\geq m} \left\|\sum_{n\in A} x_n\right\|.$$

By this new definition, a basis (x_n) is democratic if and only if $\chi_u(m) \approx \chi_l(m)$, i.e., $\sup\frac{\chi_u(m)}{\chi_l(m)} < \infty$ and $\sup\frac{\chi_l(m)}{\chi_u(m)} < \infty$ (see [2]).

As in inequality (2.1), we have the same result for greedy basis, that is,

$$\|x - \mathcal{G}_m(x)\| \leq (K+1) \inf_{\substack{\{\beta_n\} \\ A_m(x)}} \left\|x - \sum_{n\in A_m(x)} \beta_n x_n\right\|.$$

Note that the infimum is taken over scalar β_n and the set $A_m(x)$. Therefore, the following definition emerges.

Definition 2.6. Let X be a Banach space and (x_n) be a basis for X. (x_n) is called a *greedy basis* if there is a *greedy constant* $C_g \geq 1$ such that

$$\|x - \mathcal{G}_m(x)\| \leq C_g \inf_{A_m(x),\quad \beta_n} \left\| x - \sum_{n \in A_m(x)} \beta_n x_n \right\|$$

where $\beta_n \in \mathbb{R}$ and A is an index set with $|A_m(x)| = m$.

For simplicity, let $\sum_m = \sum_{n \in A_m(x)} \beta_n x_n$. Now we are ready to discuss the stability of greedy basis under ε-isometry mapping.

Theorem 2.7. Let (x_n) be a greedy basis for X and $f : X \to Y$ be a standard ε-isometry. Then $(f(x_n))$ is a greedy basis for *span* $\{(f(x_n))\} \subset Y$ equivalence to (x_n).

Proof . Since (x_n) is a greedy basis for X, there exists a set $A_m(x) \subset \mathbb{N}$ with $|A_m(x)| = m$ such that

$$\|x - \mathcal{G}_m(x)\| \leq C_g \inf_{\mathrm{z}_m \in \Sigma_m} \|x - \mathrm{z}_m\|$$

whenever $C_g \geq 1$. Recall that $\mathcal{G}_m(x)$ is a greedy approximation for each $x \in X$. Hence there is a unique decreasing sequence $(|x_n^*(x)|)$ of scalars such that $\mathcal{G}_m(x) = \sum_{n \in A_m(x)} x_n^*(x) x_n$. By the definition of $A_m(x)$, each $|x_n^*(x)| > 0$ for all $n \in A_m(x)$, otherwise $|x_n^*(x)| = 0$. Thus $\lim_n x_n^*(x) = 0$. Since f is a standard ε-isometry and there is $\psi_n \in Y^*$ for any $x_n^* \in X^*$ with $\|\psi_n\| = \|x_n^*\|$ (Theorem 1.1), $\sum_{n \in A_m(x)} x_n^*(x) f(x_n)$ must be convergent in Y. This shows that $(f(x_n))$ is a quasi-greedy basis for *span* $\{(f(x_n))\}$. Let $\delta > 0$. Since $\sum_{n \in A_m(x)} x_n^*(x) f(x_n)$ is convergent to some member of Y, it has some convergent subseries $\sum_{i=1}^{\infty} x_{n_i}^*(x) f(x_{n_i})$. Clearly, every greedy basis is unconditional. Thus, there is an $N(\delta) = N \in \mathbb{N}$ such that for every $m_2 > m_1 \geq N$,

$$\left\| \sum_{n=m_1+1}^{m_2} x_n^*(x) x_n \right\| < \frac{\delta}{M_0}$$

for some $M_0 < \infty$. Since f is a standard ε-isometry,

$$\left\| \sum_{n=m_1+1}^{m_2} x_n^*(x) f(x_n) \right\| < \frac{\delta}{M}$$

for some $M < \infty$. Hence if $N \leq n_k < \cdots < n_{k+l}$, then

$$\left\| \sum_{i=k+1}^{k+l} x_{n_i}^*(x) f(x_{n_i}) \right\| \leq M \left\| \sum_{i=n_k+1}^{n_{k+l}} x_i^*(x) f(x_i) \right\| < \delta$$

which shows that $\sum_i x_{n_i}^*(x) f(x_{n_i})$ is Cauchy. If $n_i \notin A_m(x)$ is taken, then

$$min\{|x_n^*(x)| : n \in A_m(x)\} > max\left\{\left|x_{n_i}^*(x)\right| : n_i \notin A_m(x)\right\}.$$

Since the construction of $A_m(x)$ uses greedy approximation, $(f(x_n))$ is an unconditional basis for *span* $\{(f(x_n))\}$. Besides, the Cauchy condition of $\sum_i x_{n_i}^*(x) f(x_{n_i})$ implies that for some $r \in \mathbb{N}$

$$\sup_{i \geq r} \frac{\sup_{|A_m(x)| \leq m} \left\| \sum_{n_i \in A_m(x)} x_{n_i}^*(x) f(x_{n_i}) \right\|}{\inf_{|A_m(x)| \geq m} \left\| \sum_{n_i \in A_m(x)} x_{n_i}^*(x) f(x_{n_i}) \right\|} < \infty.$$

Hence, $(f(x_n))$ is a democratic basis for *span* $\{(f(x_n))\}$. These two facts show that $(f(x_n))$ is a greedy basis for *span* $\{(f(x_n))\}$.

What is left to prove is that (x_n) and $(f(x_n))$ are equivalent greedy bases for X and $span\{(f(x_n))\}$, respectively. Note that if $T : X \to span\{(f(x_n))\}$ is an isomorphism, then

$$T(\mathcal{G}_m(x)) = \mathcal{G}_m(T(x))$$

and so we just need to prove the existence of such isomorphism. For any coordinate functionals $\psi_n \in Y^*$ and $y \in span\{(f(x_n))\}$, let $T : X \to span\{(f(x_n))\}$ be defined as

$$T\left(\sum_n x_n^*(x)\, x_n\right) = \sum_n \psi_n(y)\, f(x_n).$$

By uniqueness of a greedy approximation $\mathcal{G}_m$, it is easy to show that T is well-defined, linear, and injective. Let (x_n^*, x_n) and $(\psi_n, f(x_n))$ be the orthogonal systems for greedy bases (x_n) and $(f(x_n))$, respectively. Suppose that $u_i \to u \in X$ and $Tu_i \to v \in span\{(f(x_n))\}$. If $Tu = v$, then T is bounded by the Closed Graph Theorem. Since $\mathcal{G}_m(x)$ and $\mathcal{G}_m(f(x))$ are greedy approximations, we have

$$\lim_m \|u - \mathcal{G}_m(u)\| = 0$$

and

$$\lim_m \|v - \mathcal{G}_m(f(x_n))\| = 0$$

for any $u \in X$ and $v \in span\{(f(x_n))\}$. Therefore

$$\sum_{n\in A_m(u_i)} x_n^*(u_i)\, x_n = u_i \to u = \sum_{n\in A_m(u)} x_n^*(u)\, x_n.$$

Since $\psi_n \in Y^*$ is a coordinate functional for every n,

$$\sum_{n\in A_m(Tu_i)} \psi_n(Tu_i)\, f(x_n) = Tu_i \to v = \sum_{n\in A_m(v)} \psi_n(v)\, f(x_n).$$

The continuity of x_n^* and ψ_n implies $Tu = v$.

By Theorem 1.1, for any coordinate functional $x_n^* \in X^*$ there is $\psi_n \in Y^*$ with $\|x_n^*\| = a = \|\psi_n\|$ such that

$$|\langle x_n^*, x\rangle - \langle \psi_n, f(x)\rangle| < 3\varepsilon a$$

for every $x \in X$. Thus, T^{-1} is bounded and so $(f(x_n))$ is a greedy basis for $span\{(f(x_n))\}$ that is equivalent to greedy basis (x_n). □

If f is a standard surjective ε-isometry, then $(f(x_n))$ is a greedy basis for Y. Since T is an isomorphism, there is an isomorphism $T^* : span\{(f(x_n))\}^* \to X^*$ with $\|T\| = \|T^*\|$ (see [17], Theorem 1.10.12). Hence, the following is just a consequence of Theorem 2.7.

Corollary 2.8. Let (x_n) be a greedy basis for X and $f : X \to Y$ be an ε-isometry with $f(0) = 0$. If (x_n^*) and (ψ_n) are sequences of coordinate functionals for (x_n) and $(f(x_n))$, respectively, then (x_n^*) and (ψ_n) are greedy basis for X^* and $span\{(f(x_n))\}^*$, respectively.

Theorem 2.9. ([22], Rosenthal) Every bounded sequence in a real or complex Banach space has a weakly Cauchy subsequence.

Theorem 2.10. Let (x_n) be a greedy basis for X and $f : X \to Y$ be an ε-isometry with $f(0) = 0$. Then there is an isometry mapping $U : X \to Y^{**}$.

Proof . Since (x_n) is a greedy basis, (x_n) is a bounded sequence. Hence, by Theorem 2.9 and Theorem 2.4, the sequence (αx_n) has a subsequence $(\alpha_k x_n)$ which is weakly Cauchy. Since (x_n) and $(f(x_n))$ are equivalent greedy bases, the sequence $\left(f\frac{(\alpha x_n)}{\alpha}\right)$ also has a weakly Cauchy subsequence $\left(f\frac{(\alpha_k x_n)}{\alpha_k}\right)$. Therefore, the subsequence $\left(f\frac{\left(\alpha_k^{(n)} x_n\right)}{\alpha_k^{(n)}}\right)$

is a weakly Cauchy for any $n \in A_m(f(x))$ where $A_m(f(x))$ is related to $A_m(x)$. Note that $\alpha_k^{(n)} = \sum_{n\in A_m(x)} x_n^*(x)$ whenever x_n^* is a coordinate functional for each n. We can choose $n = k$ such that $\alpha_k^{(k)} = \sum x_k^*(x)$. This shows that $\left(f\frac{\left(\alpha_k^{(k)}x_n\right)}{\alpha_k^{(k)}}\right)$ is a weakly Cauchy subsequence independent from x_n. Thus, Theorem 2.7 implies that for $m = 1, 2, 3, \ldots,$ $\mathcal{G}_m(y) = \sum_{n\in A_m(y)} \psi_n(y)\left(f\frac{\left(\alpha_k^{(k)}x_n\right)}{\alpha_k^{(k)}}\right)$ is a weakly Cauchy sequence of greedy approximation for any $y \in span\{(f(x_n))\}$.

Take a sequence $(y_n^{**}) \subset Y^{**}$ which is weak* convergent to $y^{**} \in Y^{**}$, that is, $y^{**}y^* = w^* - \lim y_n^{**}y^*$. Since each y_n^{**} is a bounded linear functional, the Uniform Bounded Principle implies that Y^{**} is w^*-complete. Put $y_k^{**} = Q\left(f\frac{\left(\alpha_k^{(k)}x_n\right)}{\alpha_k^{(k)}}\right)$ where $Q : Y \to Y^{**}$ is canonical embedding. So w^*-completeness of Y^{**} implies that the weak* limit of $\left(f\frac{\left(\alpha_k^{(k)}x_n\right)}{\alpha_k^{(k)}}\right)$ exists. Denote this weak* limit by U, that is,

$$U(x_n) = w^* - \lim_k \left(f\frac{\left(\alpha_k^{(k)}x_n\right)}{\alpha_k^{(k)}}\right).$$

Since Q is an isometric isomorphism from Y into Y^{**}, the mapping $U : X \to Y^{**}$ is well defined by Theorem 2.7. For each $n \in \mathbb{N}$, Theorem 1.1 shows that for every $x^* \in S_{X^*}$ there is $\psi \in S_{Y^*}$ such that

$$\left|\left\langle \psi, f\frac{\left(\alpha_k^{(k)}x_n\right)}{\alpha_k^{(k)}}\right\rangle - \langle x^*, x_n\rangle\right| \le \frac{3\varepsilon}{\alpha_k^{(k)}}.$$

If we take the limit as $k \to \infty$, then it is easy to see that

$$\langle \psi, U(x_n)\rangle = \langle x^*, x_n\rangle .$$

As a consequence of the Hanh-Banach Theorem, there is an $x^* \in S_{X^*}$ such that $x^*(x) = \|x\|$ for any nonzero $x \in X$ (see[17], Corollary 1.9.8). If we choose $x^* \in S_{X^*}$ such that

$$\left\langle x^*, x_{n_q} - x_{n_p}\right\rangle = \left\|x_{n_q} - x_{n_p}\right\|$$

then

$$\begin{aligned}
\left\|x_{n_q} - x_{n_p}\right\| &= \left\langle x^*, x_{n_q} - x_{n_p}\right\rangle \\
&= \left\langle \psi, U(x_{n_q}) - U(x_{n_q})\right\rangle \\
&= \left\langle \psi, U(x_{n_q})\right\rangle - \left\langle \psi, U(x_{n_q})\right\rangle \\
&\le \left\|U\left(x_{n_q}\right) - U\left(x_{n_p}\right)\right\|.
\end{aligned}$$

On the contrary,

$$\begin{aligned}\left\|U\left(x_{n_q}\right)-U\left(x_{n_p}\right)\right\| &= \left\|w^*-lim\left(\frac{f\left(m_k^{(k)}x_{n_q}\right)-f\left(m_k^{(k)}x_{n_p}\right)}{m_k^{(k)}}\right)\right\| \\ &\leq lim\ inf_k\left\|\left(\frac{f\left(m_k^{(k)}x_{n_q}\right)-f\left(m_k^{(k)}x_{n_p}\right)}{m_k^{(k)}}\right)\right\| \\ &\leq lim\ inf_k\frac{\left\|m_k^{(k)}x_{n_q}-m_k^{(k)}x_{n_p}\right\|+\varepsilon}{m_k^{(k)}} \\ &= \left\|x_{n_q}-x_{n_p}\right\|.\end{aligned}$$

Combining the last two inequalities, one gets that $U(x_n)$ is an isometry from X into Y^{**}. □

Corollary 2.11. Let (x_n) be a greedy basis for X and $f: X \to Y$ be an ε-isometry with $f(0)=0$. Then $(f(x_n))$ is a greedy basis for $span\{(f(x_n))\}^{**} \subset Y^{**}$.

Note that if $\varepsilon = 0$ in Theorem 1.1, then obviously $\langle \psi_n, f(x_n)\rangle = \langle x_n^*, x_n\rangle$. Hence $\psi_n : X^* \to \overline{span}\{(f(x_n))\}^*$ is a linear isometry for each n. Let $\Psi \subset Y^*$ be defined as

$$\Psi{:=}\overline{span}\left\{\psi_n :\ \psi_n \in S_{Y^*}\ satisfies\ Theorem\ 1.1\right\}$$

and every $y \in span\{(f(x_n))\} \subset Y$ be renormed by

$$|||y||| = \sup_{\psi \in S_\Psi} \psi(y).$$

Since ψ is a linear isometry for $\varepsilon = 0$, we have $|||y||| = \sup_{x^* \in S_{X^*}} \langle x^*, x\rangle$. Combining this fact and Theorem 2.7, we can deduce that $(f(x_n))$ is isometrically equivalent to the greedy basis (x_n).

Let $F = \overline{span}\{(f(x_n))\}$ for a greedy basis $(x_n) \subset X$. Since each $\psi \in \Psi$ depends on $x^* \in X^*$, if x^* separates points of X then ψ separates points of F. This fact gives the following theorem.

Theorem 2.12. Let X and Y be Banach spaces with Y reflexive, (x_n) be a greedy basis for X and $f: X \to Y$ be an ε-isometry with $f(0) = 0$. Let $F = \overline{span}\{(f(x_n))\}$ and

$$\Psi{:=}\overline{span}\left\{\psi_n :\ \psi_n \in Y^*\ and\ \left|\langle x_n^*, x\rangle - \langle \psi_n, f(x)\rangle\right| < 3\varepsilon a.\right\}$$

If Ψ separates the points of F, then Ψ is linearly isomorphic to F^*.

Proof . Since Ψ separates the points of F, Ψ is a Hausdorff subspace. Hence for any $y^* \in \Psi$, there is a unique $x^* \in X^*$ such that $y^* = \psi_{x^*}$. Now, take any $y \in F$ and define $T : \Psi \to F^*$ as

$$T(y^*)(y) = T(\psi_{x^*})(y) = \psi_{x^*}(y).$$

Clearly, T is a linear operator. Since $(f(x_n))$ is a greedy basis for F (Theorem 2.7), T is bounded by the definition of greedy basis. If $\psi_{x^*}(f(x_n)) = 0$ for all n, then $\psi_{x^*} = 0$ and so Theorem 1.1 says that $x^* = 0$. Therefore, T is a one-to-one, bounded and linear operator.

The definition of T implies that $T(\Psi)\subset Y^*$ is a w^*-closed subspace. Besides, the Hanh-Banach Theorem shows that $\overline{T(\Psi)}^{w^*} = F^*$. Since Y is a reflexive space, $\overline{T(\Psi)}^{w^*} = \overline{T(\Psi)}^{w} = \overline{T(\Psi)}^{\|\cdot\|}$. Note that Ψ is a closed subspace of Y^*. If $F^*\subset Y^*$ is also a closed subspace, then the proof will be completed by deploying the Inverse Mapping Theorem.

Let $(y_n^*) \subset T(\Psi)$ be a sequence that converges to $y^* \in F^*$. Then for any $y \in F$

$$\lim_n T\ \left(y_n^*\right)(y) = \lim_n \psi_{x_n^*}(y)$$

exists for all n. Since $(x_n^*) \subset X^*$, its limit exists and say that $x_n^* \to x^*$. Thus, there is a subsequence $\left(x_{n_i}^*\right) \subset (x_n^*)$ such that

$$\lim_i \psi_{x_{n_i}^*}\left(f\left(x_m\right)\right) = \lim_n \psi_{x_n^*}\left(f\left(x_m\right)\right) = \psi_{x^*}\left(f\left(x_m\right)\right)$$

for all m. Therefore the following limit

$$\lim_i \psi_{x_{n_i}^*}(y) = \psi_{x^*}(y)$$

exists for all $y \in F$ and so $\overline{T(\Psi)}^{w^*} = T(\Psi) = F^*$. As a result, this shows that F^* is a norm-closed subspace. □

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