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Well-posedness and stability results for a class of nonlinear fourth-order wave equation with variable-exponents

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Abstract

In this paper, we consider a class of nonlinear fourth-order wave equation with damping and source terms of variable-exponent types. First, by the Faedo-Galerkin approximation method with positive initial energy and suitable conditions on the variable exponents $m(.)$ and $r(.)$, we established the local existence. We also prove that the local solution is global. Finally, the stability estimate of the solution was obtained by using the Komornik inequality

Keywords: Nonlinear fourth-order wave equation, Faedo-Galerkin method, Global existence, Stability result
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1 Introduction

We consider the following boundary value problem:

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta u + |u_t|^{m(x)-2} u_t = |u|^{r(x)-2} u, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with smooth boundary $\partial\Omega$. $m(.)$ and $r(.)$ are given measurable functions on Ω, satisfying

$$\begin{aligned} &2 < r_1 \leq r(x) \leq r_2 < 2\frac{n-1}{n-2}, && \text{if } n \geq 3, \\ &2 < r(x) < \infty, && \text{if } n = 1,\ 2, \end{aligned}$$

$$\begin{aligned} &2 < m_1 \leq m(x) \leq m_2 < 2\frac{n}{n-2}, && \text{if } n \geq 3, \\ &2 < m(x) < \infty, && \text{if } n = 1,\ 2, \end{aligned}$$

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and

$$r_1 := ess\inf_{x\in\Omega} r\left(x\right), \quad r_2 := ess\sup_{x\in\Omega} r\left(x\right),$$
$$m_1 := ess\inf_{x\in\Omega} m\left(x\right), \quad m_2 := ess\sup_{x\in\Omega} m\left(x\right),$$

We also assume that $m(.)$ and $r(.)$ satisfy the log-Hölder continuity condition:

$$\left|q\left(x\right)-q\left(y\right)\right| \leq -\frac{A}{\log\left|x-y\right|}, \text{ for a.e. } x,\ y\in\Omega, \text{ with } \left|x-y\right|<\delta, \tag{1.2}$$

$A>0$, $0<\delta<1$.

Equation (1.1) can be viewed as a generalization of the following equation

$$u_{tt}+\Delta^2 u-\Delta u+g\left(u_t\right)=f\left(u\right), \tag{1.3}$$

have been discussed by many authors. For $g\left(u_t\right)=\left|u_t\right|^{p-2}u_t$, the global existence and blow up results can by found in **[6]**. Recently, Pişkin and Polat **[13]** proved decay of the solution of problem (1.3).

Messaoudi **[8]** studied the following equation

$$u_{tt}+\Delta^2 u+\left|u_t\right|^{p-2}u_t=\left|u\right|^{r-2}u, \tag{1.4}$$

he proved the local existence and blow up of the solution. Also, Wu and Tsai **[15]** obtained global existence and blow up of the solution of the problem (1.4). Later, Chen and Zhou **[1]** studied blow up of the solution of the problem (1.4) for positive initial energy.

Many authors studied the existence and nonexistence of solutions for problem with variable exponents, can refer **[4, 5, 10, 12, 14, 16, 17]**. Messaoudi et al. **[11]** considered the following equation:

$$u_{tt}-\Delta u+a\left|u_t\right|^{m(x)-2}u_t=b\left|u\right|^{p(x)-2}u,$$

and used the Faedo-Galerkin method to establish the existence of a unique weak local solution. They also proved that the solutions with negative initial energy blow up in finite time. Messaoudi and Talahmeh **[9]**, considered the following equation:

$$u_{tt}-div\left(\left|\nabla u\right|^{r(x)-2}\nabla u\right)+a\left|u_t\right|^{m(x)-2}u_t=b\left|u\right|^{p(x)-2}u,$$

where a, b are the nonnegative constants. They proved a finite-time blow-up result for the solution with negative initial energy as well as for certain solutions with positive initial energy; in the case where $m\left(x\right)=2$ and under suitable conditions on the exponents, they established a blow-up result for solutions with arbitrary positive initial energy.

Our objective in this paper is to study: In section 2, some notations, assumptions and preliminaries are introduced, in section 3, we proved the local solution, the global existence of solution is proved and the main results of this article are shown in section 4.

2 Preliminaries

We begin this section with some notations and definitions about Lebesgue and Sobolev spaces with constant exponents and variable exponents (see **[3]**). Denote by $\|.\|_p$, the $L^p\left(\Omega\right)$ norm of a Lebesgue function $u\in L^p\left(\Omega\right)$ endowed with the norm

$$\|u\|_p^p=\int_\Omega \left|u\left(x\right)\right|^p dx.$$

We also consider the Sobolev space equiped with the scalar product

$$\left(u,\ v\right)_{H^2(\Omega)}=\left(u,\ v\right)+\left(\Delta u,\ \Delta v\right).$$

We define the subspace of $H^2\left(\Omega\right)$, denoted by $H_0^2\left(\Omega\right)$, as the closure of $C_0^\infty\left(\Omega\right)$ in the strong topology of $H^2\left(\Omega\right)$. This space endowed with the norm induced by the scalar product

$$\left(u,\ v\right)_{H_0^2(\Omega)}=\left(\Delta u,\ \Delta v\right).$$

is a Hilbert space. We use $W_0^{1,p}(\Omega)$ to the well-known Sobolev space such that u and $|\nabla u|$ are in $L^p(\Omega)$ equipped with the norm $\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_p$.

Let $q : \Omega \to [1, \ +\infty]$ be a measurable function, where Ω is a domain of $\mathbb{R}^n$.

We define the Lebesque space with a variale exponent $q(.)$ by:

$$L^{q(.)}(\Omega) := \left\{ v : \Omega \to \mathbb{R} : \text{ measurable in } \Omega, \ \varrho_{q(.)}(\lambda v) < +\infty, \text{ for some } \lambda > 0 \right\},$$

where $\varrho_{q(.)}(v) = \int\limits_{\Omega} |v(x)|^{q(x)} dx$.

The set $L^{q(.)}(\Omega)$ equipped with the norm (Luxemburg's norm)

$$\|v\|_{q(.)} := \inf \left\{ \lambda > 0 : \int\limits_{\Omega} \left| \frac{v(x)}{\lambda} \right|^{q(x)} dx \leq 1 \right\},$$

$L^{q(.)}(\Omega)$ is a Banach space **[3]**.

We next, define the variable-exponent Sobolev space $W^{1,q(.)}(\Omega)$ as follows:

$$W^{1,q(.)}(\Omega) := \left\{ v \in L^{q(.)}(\Omega) \text{ such that } \nabla v \text{ exists and } |\nabla v| \in L^{q(.)}(\Omega) \right\}.$$

This is a Banach space with respect to the norm $\|v\|_{W^{1,q(.)}(\Omega)} = \|v\|_{q(.)} + \|\nabla v\|_{q(.)}$.

Furthemore, we set $W_0^{1,q(.)}(\Omega)$ to be the closure of $C_0^{\infty}(\Omega)$ in the space $W^{1,q(.)}(\Omega)$. Let us note that the space $W_0^{1,q(.)}(\Omega)$ has a differenet definition in the case of variable exponents.

However, under the log-Hölder continuity condition, both definitions are equivalent **[3]**. The space $W^{-1,q'(.)}(\Omega)$, dual of $W_0^{1,q(.)}(\Omega)$, is defined in the same way as the classical Sobolev spaces, where $\frac{1}{q(.)} + \frac{1}{q'(.)} = 1$.

Lemma 2.1. **[3]** If

$$1 \leq q_1 := ess \inf_{x \in \Omega} q(x) \leq q(x) \leq q_2 := ess\sup_{x \in \Omega} q(x) < \infty,$$

then we have

$$\min \left\{ \|u\|_{q(.)}^{q_1}, \ \|u\|_{q(.)}^{q_2} \right\} \leq \varrho_{q(.)}(u) \leq \max \left\{ \|u\|_{q(.)}^{q_1}, \ \|u\|_{q(.)}^{q_2} \right\},$$

for any $u \in L^{q(.)}(\Omega)$.

Lemma 2.2. **[3]**(Hölder's Inequality) Suppose that $p, q, s \geq 1$ are measurable functions defined on Ω such that

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \quad \text{for } a.e. \quad y \in \Omega.$$

If $u \in L^{p(.)}(\Omega)$ and $v \in L^{q(.)}(\Omega)$, then $uv \in L^{s(.)}(\Omega)$, with

$$\|uv\|_{s(.)} \leq 2 \|u\|_{p(.)} \|v\|_{q(.)}.$$

Lemma 2.3. (Lars et al) **[3]** If p is a measurable function on Ω satisfying (1.2), then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact.

From the Lemma 2.3, there exists the positive constant c_* satisfying

$$\|u\|_{p(.)} \leq c_* \|\nabla u\|_2, \quad \text{for } u \in H_0^1(\Omega).$$

Lemma 2.4. **[2]**Let $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-increasing function and assume that there are two constants $\alpha > 0$ and $C > 0$ such that

$$\int\limits_t^{\infty} G^{\alpha+1}(s)\, ds \leq CG^{\alpha}(0)\, G(s), \quad \forall t \in \mathbb{R}_+.$$

Then we have

$$G(t) \leq G(0)\left(\frac{C+\alpha t}{C+\alpha C}\right)^{\frac{-1}{\alpha}}, \quad \forall\, t \geq C.$$

Theorem 2.5. Suppose that $r,\, m \in C\left(\overline{\Omega}\right)$ and satisfies (1.2). Then, for any $(u_0,\ u_1) \in H^2(\Omega) \cap H^4(\Omega) \times L^2(\Omega)$, problem (1.1) has a unique weak local solution

$$\begin{aligned} u &\in L^\infty\left((0,T),\ H^2(\Omega)\right), \\ u_t &\in L^\infty\left((0,T),\ L^2(\Omega)\right) \cap L^{m(.)}(\Omega \times (0,\ T)), \\ u_{tt} &\in L^\infty\left((0,T),\ L^2(\Omega)\right). \end{aligned}$$

In the order to state and prove our result, we define the potential energy functional and Nehari's functional, by the following

$$E(t) = E(u(t)) = \frac{1}{2}\left(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2\right) - \int_\Omega \frac{1}{r(x)}|u(t)|^{r(x)}\,dx. \tag{2.1}$$

$$I(t) = I(u(t)) = \|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2 - \int_\Omega |u(t)|^{r(x)}\,dx. \tag{2.2}$$

$$J(t) = J(u(t)) = \frac{1}{2}\left(\|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2\right) - \int_\Omega \frac{1}{r(x)}|u(t)|^{r(x)}\,dx. \tag{2.3}$$

Lemma 2.6. Under the assumptions of theorem 2.5, we have

$$E'(t) = -\int_\Omega |u_t(t)|^{m(x)}\,dx \leq 0, \quad t \in [0,\ T].$$

and

$$E(t) \leq E(0).$$

Proof . We multiply the first equation of (1.1) by u_t and integrating over the domain Ω, we get

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\left(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2\right) - \int_\Omega \frac{1}{r(x)}|u(t)|^{r(x)}\,dx\right) \\ &= -\int_\Omega |u_t(t)|^{m(x)}\,dx, \end{aligned}$$

then

$$E'(t) = -\int_\Omega |u_t(t)|^{m(x)}\,dx \leq 0. \tag{2.4}$$

Integratying (2.4) over $(0,\ t)$, we obtain

$$E(t) \leq E(0).$$

□

Lemma 2.7. Assume that the assumptions of theorem 2.5 and $E(0) > 0$ hold,

$$I(0) > 0,$$

and

$$\theta_1 + \theta_2 < 1, \tag{2.5}$$

where

$$\theta_1 := \alpha \max\left\{ c_{1,*}^{r_1}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_1-2}{2}},\ c_{1,*}^{r_2}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_2-2}{2}}\right\},$$

$$\theta_2 := (1-\alpha)\max\left\{ c_{2,*}^{r_1}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_1-2}{2}},\ c_{2,*}^{r_2}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_2-2}{2}}\right\},$$

with $0<\alpha<1$, $c_{1,*}$ and $c_{2,*}$ are the bests embedding constants of $H_0^1(\Omega) \hookrightarrow L^{r(.)}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{r(.)}(\Omega)$ respectively, then $I(t)>0$, for all $t\in[0,\ T]$.

Proof . By continuity, there exists T_*, such that

$$I(t)\geq 0, \quad \text{for all } t\in[0,\ T_*]. \tag{2.6}$$

Now, we have for all $t\in[0,\ T_*]$:

$$\begin{aligned} J(t) = J(u(t)) &= \frac{1}{2}\left(\|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2\right) - \int_\Omega \frac{1}{r(x)}|u(t)|^{r(x)}dx \\ &\geq \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}\|\Delta u(t)\|_2^2 - \frac{1}{r_1}\left(\|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2 - I(t)\right) \\ &\geq \frac{r_1-2}{2r_1}\left(\|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2\right) + \frac{1}{r_1}I(t), \end{aligned}$$

using (2.6), we obtain

$$\|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2 \leq \frac{2r_1}{r_1-2}J(t), \qquad \text{for all } t\in[0,\ T_*]. \tag{2.7}$$

By Lemma 2.6, we get

$$\|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2 \leq \frac{2r_1}{r_1-2}E(t) \leq \frac{2r_1}{r_1-2}E(0) \tag{2.8}$$

On the other hand, by Lemma 2.1, we have

$$\begin{aligned} \int_\Omega |u(t)|^{r(x)}dx \leq\ & Max\left\{\|u(t)\|_{r(.)}^{r_1},\ \|u(t)\|_{r(.)}^{r_2}\right\} \\ &= \alpha\ Max\left\{\|u(t)\|_{r(.)}^{r_1},\ \|u(t)\|_{r(.)}^{r_2}\right\} \\ &+ (1-\alpha)\,Max\left\{\|u(t)\|_{r(.)}^{r_1},\ \|u(t)\|_{r(.)}^{r_2}\right\}. \end{aligned}$$

By the embedding of $H_0^1(\Omega) \hookrightarrow L^{r(.)}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{r(.)}(\Omega)$, we obtain

$$\begin{aligned} \int_\Omega |u(t)|^{r(x)}dx &\leq \alpha\ Max\left\{c_{1,*}^{r_1}\|\nabla u(t)\|_2^{r_1},\ c_{1,*}^{r_2}\|\nabla u(t)\|_2^{r_2}\right\} \\ &+ (1-\alpha)\,Max\left\{c_{2,*}^{r_1}\|\Delta u(t)\|_2^{r_1},\ c_{2,*}^{r_2}\|\Delta u(t)\|_2^{r_2}\right\} \\ &\leq \alpha\ Max\left\{c_{1,*}^{r_1}\|\nabla u(t)\|_2^{r_1-2},\ c_{1,*}^{r_2}\|\nabla u(t)\|_2^{r_2-2}\right\}\times\|\nabla u(t)\|_2^2 \\ &+ (1-\alpha)\,Max\left\{c_{2,*}^{r_1}\|\Delta u(t)\|_2^{r_1-2},\ c_{2,*}^{r_2}\|\Delta u(t)\|_2^{r_2-2}\right\}\times\|\Delta u(t)\|_2^2 \\ &\leq \alpha\ Max\left\{c_{1,*}^{r_1}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_1-2}{2}},\ c_{1,*}^{r_2}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_2-2}{2}}\right\} \\ &\times\|\nabla u(t)\|_2^2 \\ &+ (1-\alpha)\,Max\left\{c_{2,*}^{r_1}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_1-2}{2}},\ c_{2,*}^{r_2}\left(\frac{2r_1}{r_1-2}E(0)\right)^{\frac{r_2-2}{2}}\right\} \\ &\times\|\Delta u(t)\|_2^2 \end{aligned}$$

Then, we get

$$\int_{\Omega} |u(t)|^{r(x)}\, dx \le \theta_1 \|\nabla u(t)\|_2^2 + \theta_2 \|\Delta u(t)\|_2^2, \qquad \text{for all } t \in [0,\ T_*]. \tag{2.9}$$

Since $\theta_1 + \theta_2 < 1$, then

$$\int_{\Omega} |u(t)|^{r(x)}\, dx < \|\nabla u(t)\|_2^2 + \|\Delta u(t)\|_2^2, \qquad \text{for all } t \in [0,\ T_*]. \tag{2.10}$$

This implies that

$$I(t) > 0, \qquad \text{for all } t \in [0,\ T_*].$$

By repeating the above procedure, we can extend T_* to T. □

3 Local existence

In this section we are going to obtain the existence of local solution to the problem (1.1). We will use the Faedo-Galerkin's method approximation. Let $\{v_l\}_{l=1}^{\infty}$ be a basis of $H_0^2(\Omega)$ wich constructs a complete orthonormal system in $L^2(\Omega)$. Denote by $V_k = span\{v_1,\ v_2, ..., v_k\}$ the subspace generated by the first k vectors of the basis $\{v_l\}_{l=1}^{\infty}$. By the normalization, we have $\|v_l\| = 1$, for any given integer k, we consider the approximation solution

$$u_k(t) = \sum_{l=1}^{k} u_{lk}(t)\, v_l,$$

where u_k is the solutions to the following Cauchy problem

$$\left(u_k''(t),\ v_l\right) + \left(\Delta^2 u_k(t),\ v_l\right) - \left(\Delta u_k(t),\ v_l\right) + \left(\left|u_k'(t)\right|^{m(x)-2} u_k'(t),\ v_l\right)$$

$$= \left(|u_k(t)|^{r(x)-2} u_k(t),\ v_l\right), \quad l = 1,\ 2, ..., k, \tag{3.1}$$

$$u_k(0) = u_{0k} = \sum_{i=1}^{k} \left(u_k(0),\ v_l\right) v_l \to u_0 \text{ in } H_0^2(\Omega) \cap H^4(\Omega), \tag{3.2}$$

$$u_k'(0) = u_{1k} = \sum_{l=1}^{k} \left(u_k'(0),\ v_l\right) v_l \to u_1 \text{ in } L^2(\Omega). \tag{3.3}$$

Note that, we can solve the system (3.1)-(3.3) by a Picard's iteration method in ordinary differential equations. Hence, there exists a solution in $[0,\ T_*)$ for some $T_* > 0$ and we can extend this solution to the whole interval $[0,\ T]$ for any given $T > 0$ by making use of the priori estimates below.

The first estimate. Multiplying equation (3.1) by $u_{lk}'(t)$ and summing over l from 1 to k,

$$\frac{d}{dt}\left(\frac{1}{2}\left\|u_k'\right\|_2^2 + \frac{1}{2}\|\nabla u_k\|_2^2 + \frac{1}{2}\|\Delta u_k\|_2^2 - \int_{\Omega} \frac{1}{r(x)} |u_k|^{r(x)}\right) dx = -\int_{\Omega} \left|u_k'\right|^{m(x)} dx. \tag{3.4}$$

Then

$$E'(u_k(t)) = -\int_{\Omega} \left|u_k'\right|^{m(x)} dx \le 0.$$

Integrating (3.4) over $(0,\ t)$, we obtain the estimate

$$\frac{1}{2}\left\|u_k^{'}\right\|_2^2+\frac{1}{2}\left\|\nabla u_k\right\|_2^2+\frac{1}{2}\left\|\Delta u_k\right\|_2^2$$
$$-\int_{\Omega}\frac{1}{r(x)}\left|u_k\right|^{r(x)}dx+\int_0^t\int_{\Omega}\left|u_k^{'}\right|^{m(x)}dxds\leq E(0). \tag{3.5}$$

Then, from (2.10), the inequality (3.5) becomes

$$\frac{1}{2}\sup_{t\in(0,\ T)}\left\|u_k^{'}\right\|_2^2+\frac{r_1-2}{2r_1}\sup_{t\in(0,\ T)}\left\|\nabla u_k\right\|_2^2+\frac{r_1-2}{2r_1}\sup_{t\in(0,\ T)}\left\|\Delta u_k\right\|_2^2$$
$$+\int_0^t\int_{\Omega}\left|u_k^{'}\right|^{m(x)}dxds\leq E(0). \tag{3.6}$$

From (3.6), we conclude that

$$\begin{cases}\{u_k\}\ \text{is uniformly bounded in}\ L^{\infty}\left([0,\ T],\ H_0^2(\Omega)\right),\\ \left\{u_k^{'}\right\}\ \text{is uniformly bounded in}\ L^{\infty}\left([0,\ T],\ L^2(\Omega)\right)\cap L^{m(x)}(\Omega\times[0,\ T]).\end{cases} \tag{3.7}$$

Since $\left\{u_k^{'}\right\}$ is uniformly bounded in $L^{m(x)}(\Omega\times[0,\ T])$, then $\left\{\left|u_k^{'}\right|^{m(x)-2}u_k^{'}\right\}$ is bounded in $L^{\frac{m(x)}{m(x)-1}}(\Omega\times[0,\ T])$; hence, up to a subsequence, $\left|u_k^{'}\right|^{m(x)-2}u_k^{'}\rightharpoonup\Phi$ weakly in $L^{\frac{m(x)}{m(x)-1}}(\Omega\times[0,\ T])$. As in **[11],** we have to show that $\Phi=\left|u^{'}\right|^{m(x)-2}u$.

Furthemore, we have from Lemma 2.3 and (3.7) that

$$\left\{\left|u_k\right|^{r(x)-2}u_k\right\}\ \text{is uniformly bounded in}\ L^{\infty}\left([0,\ T],\ L^2(\Omega)\right). \tag{3.8}$$

By (3.7) and (3.8), we infer that there exists a subsequence of u_k (denote still by the same symbol) and a function u such that

$$\begin{cases}u_k\rightharpoonup u\ \text{weakly star in}\ L^{\infty}\left([0,\ T],\ H_0^2(\Omega)\right),\\ u_k^{'}\rightharpoonup u^{'}\ \text{weakly star in}\ L^{\infty}\left([0,\ T],\ L^2(\Omega)\right)\ \text{and weakly in}\ L^{m(x)}(\Omega\times[0,\ T]),\\ \left|u_k\right|^{r(x)-2}u_k\rightharpoonup\Psi\ \text{weakly in}\ L^{\infty}\left([0,\ T],\ L^2(\Omega)\right).\end{cases} \tag{3.9}$$

By the Aubin-Lions compactness Lemma **[7]**, we conclude from (3.9) that

$$u_k\rightharpoonup u\ \text{strongly in}\ C\left([0,\ T],\ H_0^2(\Omega)\right),$$

which implies

$$u_k\rightharpoonup u\ \text{everywhere in}\ \Omega\times[0,T]. \tag{3.10}$$

It follow from (3.9) and (3.10) that

$$\left|u_k\right|^{r(x)-2}u_k\rightharpoonup\left|u\right|^{r(x)-2}u\ \text{weakly in}\ L^{\infty}\left([0,\ T],\ L^2(\Omega)\right). \tag{3.11}$$

The second estimate. Now, we would like to get more estimates. In doing so, differentiating (3.1) with respect to t, we get

$$\left(u_k^{'''}(t),\ v_l\right)+\left(\Delta^2u_k^{'}(t),\ v_l\right)-\left(\Delta u_k^{'}(t),\ v_l\right)+\left((m(x)-1)\left|u_k^{'}(t)\right|^{m(x)-2}u_k^{''}(t),\ v_l\right)$$
$$=\left((r(x)-1)\left|u_k(t)\right|^{r(x)-2}u_k^{'}(t),\ v_l\right),\quad l=1,\ 2,...,k, \tag{3.12}$$

Next, multiplying the equation (3.12) by $u_{lk}^{''}(t)$ and summing over l from 1 to k, we get

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega}\left|u_k^{''}\right|^2 dx+\int_{\Omega}\left|\Delta u_k^{'}\right|^2 dx+\int_{\Omega}\left|\nabla u_k^{'}\right|^2 dx\right)+\int_{\Omega}(m(x)-1)\left|u_k^{'}\right|^{m(x)-2}u_k^{''2}dx$$
$$=\int_{\Omega}(r(x)-1)\left|u_k\right|^{r(x)-2}u_k^{'}u_k^{''}dx \tag{3.13}$$

We have from Hölder's inequality that

$$\left|\int_{\Omega}(r(x)-1)\left|u_k\right|^{r(x)-2}u_k^{'}u_k^{''}dx\right|\leq(r_2-1)\left\|u_k\right\|_{2(r(x)-1)}^{r(x)-2}\left\|u_k^{'}\right\|_{2(r(x)-1)}\left\|u_k^{''}\right\|_2 \tag{3.14}$$

We have $u_k\in L^{\infty}\left([0,\ T],\ H_0^2(\Omega)\right)$, then

$$\int_{\Omega}\left|u_k\right|^{2r(x)-2}dx\leq\int_{\Omega}\left|u_k\right|^{2r_1-2}dx+\int_{\Omega}\left|u_k\right|^{2r_2-2}dx<+\infty,$$

since, $2(r_1-1)\leq 2(r(x)-1)\leq 2(r_2-1)\leq 2\frac{n}{n-2}$.
The inequality (3.14), becomes

$$\left|\int_{\Omega}(r(x)-1)\left|u_k\right|^{r(x)-2}u_k^{'}u_k^{''}dx\right|\leq c_1\left\|u_k^{'}\right\|_{2(r(x)-1)}\left\|u_k^{''}\right\|_2 \tag{3.15}$$

We have from Young's inequality and Poincare's inequality that

$$\left|\int_{\Omega}(r(x)-1)\left|u_k\right|^{r(x)-2}u_k^{'}u_k^{''}dx\right|\leq c_{\delta}\left\|\nabla u_k^{'}\right\|_2^2+\delta\left\|u_k^{''}\right\|_2^2 \tag{3.16}$$

Substituting (3.16) into (3.13) and integrating over $(0,\ t)$ for all $t\in[0,\ T]$, we obtain

$$\int_{\Omega}\left|u_k^{''}\right|^2 dx+\int_{\Omega}\left|\Delta u_k^{'}\right|^2 dx+\int_{\Omega}\left|\nabla u_k^{'}\right|^2 dx$$
$$\leq\left\|u_k^{''}(0)\right\|_2^2+\left\|\Delta u_k^{'}(0)\right\|_2^2+\left\|\nabla u_k^{'}(0)\right\|_2^2+c_2\int_0^t\left(\left\|\nabla u_k^{'}\right\|_2^2+\left\|u_k^{''}\right\|_2^2\right)ds \tag{3.17}$$

It follows from (3.3) and the fact $\left\|\nabla u_k^{'}(0)\right\|_2^2\leq c_3\left\|\Delta u_k^{'}(0)\right\|_2^2$ that

$$\left\|\nabla u_k^{'}(0)\right\|_2^2+\left\|\Delta u_k^{'}(0)\right\|_2^2\leq c_4 \tag{3.18}$$

where c_4 is a positive constant independent of k.
Multiplying both sides of (3.1) by $u_{lk}^{''}(t)$, and then summing over over l from 1 to k and putting $t=0$, we get

$$\left\|u_k^{''}(0)\right\|_2^2+\left(\Delta^2 u_k(0),\ u_k^{''}(0)\right)-\left(\Delta u_k(0),\ u_k^{''}(0)\right)+\left(\left|u_k^{'}(0)\right|^{m(x)-2}u_k^{'}(0),\ u_k^{''}(0)\right)$$
$$=\left(\left|u_k(0)\right|^{r(x)-2}u_k(0),\ u_k^{''}(0)\right)$$

We have from Young's inequality, (3.2) and (3.3) that

$$\left\|u_k^{''}(0)\right\|_2 \leq c_5 \tag{3.19}$$

where c_5 is a positive constant independent of k.

By (3.18) and (3.19), (3.17) becomes

$$\int_\Omega \left|u_k^{''}\right|^2 dx + \int_\Omega \left|\Delta u_k^{'}\right|^2 dx + \int_\Omega \left|\nabla u_k^{'}\right|^2 dx \leq c_6 + c_7 \int_0^t \left(\left\|u_k^{''}\right\|_2^2 + \left\|\Delta u_k^{'}\right\|_2^2 + \left\|\nabla u_k^{'}\right\|_2^2\right) ds. \tag{3.20}$$

We gain from (3.20) and Gronwall's lemma that

$$\left\|u_k^{''}\right\|_2^2 + \left\|\Delta u_k^{'}\right\|_2^2 + \left\|\nabla u_k^{'}\right\|_2^2 \leq c_8, \tag{3.21}$$

for all $t \in [0,\ T]$, and c_8 is a positive constant independent of k.

We conclude from (3.21) that

$$\begin{cases} \left\{u_k^{'}\right\} \text{ is uniformly bounded in } L^\infty\left([0,\ T],\ H_0^2(\Omega)\right), \\ \left\{u_k^{''}\right\} \text{ is uniformly bounded in } L^\infty\left([0,\ T],\ L^2(\Omega)\right). \end{cases} \tag{3.22}$$

Similarly, we have

$$\begin{cases} u_k^{'} \rightharpoonup u^{'} \text{ weakly star in } L^\infty\left([0,\ T],\ H_0^2(\Omega)\right), \\ u_k^{''} \rightharpoonup u^{''} \text{ weakly star in } L^\infty\left([0,\ T],\ L^2(\Omega)\right). \end{cases} \tag{3.23}$$

Setting up $k \longrightarrow \infty$ and passing to the limit in (3.1), we obtain

$$\left(u^{''}(t),\ v_l\right) + \left(\Delta^2 u(t),\ v_l\right) - \left(\Delta u(t),\ v_l\right) + \left(\left|u^{'}(t)\right|^{m(x)-2} u^{'}(t),\ v_l\right)$$

$$= \left(|u(t)|^{r(x)-2} u(t),\ v_l\right), \quad l = 1,\ 2, ..., k, \tag{3.24}$$

Since $\{v_l\}_{l=1}^\infty$ be a basis of $H_0^2(\Omega)$, we deduce that u satisfies the equation of (1.1). From (3.9), (3.23) and Lemma 3.1.7 in **[18]** with $B = H_0^2(\Omega)$ and $L^2(\Omega)$, respectively, we infer that

$$\begin{cases} u_k(0) \rightharpoonup u(0) \text{ weakly in } H_0^2(\Omega), \\ u_k^{'}(0) \rightharpoonup u^{'}(0) \text{ weakly in } L^2(\Omega). \end{cases} \tag{3.25}$$

We get from (3.2), (3.3) and (3.25) that $u(0) = u_0$, $u^{'}(0) = u_1$.

Thus, the proof of existence is complete.

Uniqueness of the solution. Now it remains to prove uniqueness. Let u^1, u^2 be two solutions in the class described in the statement of this theorem, and $w = u^1 - u^2$.

Then w satisfies

$$w_{tt} + \Delta^2 w - \Delta w + \left|u_t^1\right|^{m(x)-2} u_t^1 - \left|u_t^2\right|^{m(x)-2} u_t^2 = \left|u^1\right|^{r(x)-2} u^1 - \left|u^2\right|^{r(x)-2} u^2 \tag{3.26}$$

and

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x)$$

Multiplying (3.26) by w_t, then integrating with respect to x, we get

$$\frac{1}{2}\int_{\Omega}|w_t|^2\,dx+\frac{1}{2}\int_{\Omega}|\Delta w|^2\,dx+\frac{1}{2}\int_{\Omega}|\nabla w|^2\,dx$$
$$+\int_0^t\int_{\Omega}\left(\left|u_t^1\right|^{m(x)-2}u_t^1-\left|u_t^2\right|^{m(x)-2}u_t^2\right)w_t dxds=\int_0^t\int_{\Omega}\left(\left|u^1\right|^{r(x)-2}u^1-\left|u^2\right|^{r(x)-2}u^2\right)w_t dxds$$

By using the inequality

$$\left(|a|^{m(x)-2}a-|b|^{m(x)-2}b\right)(a-b)\geq 0,$$

for all $a,b\in\mathbb{R}$ and a.e. $x\in\Omega$.

This implies

$$\|w_t\|_2^2\,dx+\|\Delta w\|_2^2+\|\nabla w\|_2^2$$
$$\leq C\int_0^t\int_{\Omega}\left(\left|u^1\right|^{r(x)-2}u^1-\left|u^2\right|^{r(x)-2}u^2\right)w_t dxds \tag{3.27}$$

By repeating the estimate as in **[11]**, we arive at

$$\int_{\Omega}|w_t|^2\,dx+\|\Delta w\|_2^2+\|\nabla w\|_2^2\leq C\int_0^t\left(\int_{\Omega}|w_t|^2\,dx+\|\nabla w\|_2^2\right)ds \tag{3.28}$$

Then

$$\int_{\Omega}|w_t|^2\,dx+\|\Delta w\|_2^2+\|\nabla w\|_2^2\leq C\int_0^t\left(\int_{\Omega}|w_t|^2\,dx+\|\Delta w\|_2^2+\|\nabla w\|_2^2\right)ds \tag{3.29}$$

Gronwall's inequality yields

$$\|w_t\|_2^2+\|\Delta w\|_2^2+\|\nabla w\|_2^2=0.$$

Thus, $w=0$. The shows the uniqueness. □

4 Global existence and stability result

In this section our main result is based a Komornik's inequality **[2]**.

Now, we state our main results:

Theorem 4.1. Under the assumptions of lemma 2.7, the local solution of (1.1) is global.

Proof . We have

$$E(u(t))=\frac{1}{2}\left(\|u_t(t)\|_2^2+\|\nabla u(t)\|_2^2+\|\Delta u(t)\|_2^2\right)-\int_{\Omega}\frac{1}{r(x)}|u(t)|^{r(x)}\,dx.$$
$$\geq\frac{1}{2}\|u_t(t)\|_2^2+\frac{r_1-2}{2r_1}\|\nabla u(t)\|_2^2+\frac{r_1-2}{2r_1}\|\Delta u(t)\|_2^2.$$

So that

$$\|u_t(t)\|_2^2+\|\nabla u(t)\|_2^2\leq C\;E(t). \tag{4.1}$$

By Lemma 2.6, we obtain

$$\|u_t(t)\|_2^2+\|\nabla u(t)\|_2^2\leq C\;E(0). \tag{4.2}$$

This implies that the local solution is global in time. □

Lemma 4.2. Suppose that the assumptions of Lemma 2.7 hold, then, there exists a positive constant c such that

$$\int_\Omega |u(t)|^{m(x)}\,dx \le cE(t).$$

Proof .

$$\begin{aligned}\int_\Omega |u(t)|^{m(x)}\,dx \le\ & Max\left\{\|u(t)\|_{m(.)}^{m_1},\ \|u(t)\|_{m(.)}^{m_2}\right\}\\ &= \alpha\ Max\left\{\|u(t)\|_{r(.)}^{m_1},\ \|u(t)\|_{r(.)}^{m_2}\right\}\\ &+(1-\alpha)\,Max\left\{\|u(t)\|_{m(.)}^{m_1},\ \|u(t)\|_{m(.)}^{m_2}\right\}.\end{aligned}$$

By the embedding of $H_0^1(\Omega)\hookrightarrow L^{m(.)}(\Omega)$ and $H_0^2(\Omega)\hookrightarrow L^{m(.)}(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega |u(t)|^{m(x)}\,dx &\le \alpha\ Max\left\{\lambda_{1,*}^{m_1}\|\nabla u(t)\|_2^{m_1},\ \lambda_{1,*}^{m_2}\|\nabla u(t)\|_2^{m_2}\right\}\\ &+(1-\alpha)\,Max\left\{\lambda_{2,*}^{m_1}\|\Delta u(t)\|_2^{m_1},\ \lambda_{2,*}^{m_2}\|\Delta u(t)\|_2^{m_2}\right\}\\ &\le \alpha\ Max\left\{\lambda_{1,*}^{m_1}\|\nabla u(t)\|_2^{m_1-2},\ \lambda_{1,*}^{m_2}\|\nabla u(t)\|_2^{m_2-2}\right\}\times\|\nabla u(t)\|_2^2\\ &+(1-\alpha)\,Max\left\{\lambda_{2,*}^{m_1}\|\Delta u(t)\|_2^{m_1-2},\ \lambda_{2,*}^{m_2}\|\Delta u(t)\|_2^{m_2-2}\right\}\times\|\Delta u(t)\|_2^2\\ &\le \alpha\ Max\left\{\lambda_{1,*}^{m_1}\left(\frac{2m_1}{m_1-2}E(0)\right)^{\frac{m_1-2}{2}},\ \lambda_{1,*}^{m_2}\left(\frac{2m_1}{m_1-2}E(0)\right)^{\frac{m_2-2}{2}}\right\}\\ &\times\|\nabla u(t)\|_2^2\\ &+(1-\alpha)\,Max\left\{\lambda_{2,*}^{m_1}\left(\frac{2m_1}{m_1-2}E(0)\right)^{\frac{m_1-2}{2}},\ \lambda_{2,*}^{m_2}\left(\frac{2m_1}{m_1-2}E(0)\right)^{\frac{m_2-2}{2}}\right\}\\ &\times\|\Delta u(t)\|_2^2\\ &= c_1\|\nabla u(t)\|_2^2+c_2\|\Delta u(t)\|_2^2.\end{aligned}$$

By using (2.8), we obtain

$$\int_\Omega |u(t)|^{m(x)}\,dx \le cE(t).$$

□

Theorem 4.3. Let the assumptions of Lemma 2.7, then, there exists a positive constant $C>0$, such that

$$E(t)\le \frac{C}{(1+t)^{\frac{2}{m_2-2}}},\quad \text{for all } t\ge 0.$$

Proof . Multiplying first equation of (1.1) by $u(t)\,E^{\frac{m_2-2}{2}}(t)$ and integrating over $\Omega\times(S,\ T)$, we obtain

$$\begin{aligned}&\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega u(t)\left[u_{tt}(t)+\Delta^2u(t)-\Delta u(t)+|u_t(t)|^{m(x)-2}u_t(t)\right]dxdt\\ &=\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |u(t)|^{r(x)}\,dxdt.\end{aligned}$$

So that

$$\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[(u(t)\,u_t(t))_t-|u_t(t)|^2+|\Delta u(t)|^2+|\nabla u(t)|^2\right.$$

$$+u(t)|u_t(t)|^{m(x)-2}u_t\Big]dxdt=\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |u(t)|^{r(x)}dxdt.$$

We add and substract the term

$$\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[\theta_1|\nabla u(t)|^2+\theta_2|\Delta u(t)|^2+(2+\theta_1+\theta_2)|u_t(t)|^2\right]dxdt,$$

and use (2.9), to get

$$\begin{aligned}&(1-\theta_1)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[|\nabla u(t)|^2+|u_t(t)|^2\right]dxdt\\&+(1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[|\Delta u(t)|^2+|u_t(t)|^2\right]dxdt\\&+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[(u(t)u_t(t))_t-(3-\theta_1-\theta_2)|u_t(t)|^2\right]dxdt\\&+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega u(t)|u_t(t)|^{m(x)-2}u_t(t)dxdt\\&=-\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[\theta_1|\nabla u(t)|^2+\theta_2|\Delta u(t)|^2-|u(t)|^{r(x)}\right]dxdt\le 0.\end{aligned}\tag{4.3}$$

It is clear that

$$\begin{aligned}&\gamma\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[\frac{1}{2}|\nabla u(t)|^2+\frac{1}{2}|\Delta u(t)|^2+\frac{|u_t(t)|^2}{2}-\frac{|u(t)|^{r(x)}}{r(x)}\right]dxdt\\&\le(1-\theta_1)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[\frac{1}{2}|\nabla u(t)|^2+\frac{|u_t(t)|^2}{2}\right]dxdt\\&+(1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \left[\frac{1}{2}|\Delta u(t)|^2+\frac{|u_t(t)|^2}{2}\right]dxdt,\end{aligned}\tag{4.4}$$

where $\gamma=Min\left((1-\theta_1),\ (1-\theta_2)\right)$. By (4.3), (4.4) and definition of $E(t)$, we get

$$\begin{aligned}\gamma\int_S^T E^{\frac{m_2}{2}}(t)dt\le&-\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega (u(t)u_t(t))_t dxdt\\&-\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega u(t)|u_t(t)|^{m(x)-2}u_t(t)dxdt\\&+(3-\theta_1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |u_t(t)|^2dxdt.\end{aligned}\tag{4.5}$$

Using the definition of $E(t)$ and the following expression

$$\frac{d}{dt}\left(E^{\frac{m_2-2}{2}}(t)\int_\Omega u(t)u_t(t)dx\right)=E^{\frac{m_2-2}{2}}(t)\int_\Omega (u(t)u_t(t))_t dx$$

$$+\frac{m_2-2}{2}\int_S^T E^{\frac{m_2-2}{2}-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dxdt. \tag{4.6}$$

Then, inequality (4.5), becomes

$$\begin{aligned}\gamma\int_S^T E^{\frac{m_2}{2}}(t)\,dt \le & -\int_S^T \frac{d}{dt}\left(E^{\frac{m_2-2}{2}}(t)\int_\Omega u(t)\,u_t(t)\,dx\right)dt \\ & -\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega u(t)\,|u_t(t)|^{m(x)-2}\,u_t(t)\,dxdt \\ & +\frac{m_2-2}{2}\int_S^T E^{\frac{m_2-2}{2}-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dxdt \\ & +(3-\theta_1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |u_t(t)|^2\,dxdt. \end{aligned}\tag{4.7}$$

We estimate the terms in the right-hand side of (4.7) as follow:

For the first term, we have

$$\begin{aligned} & -\int_S^T \frac{d}{dt}\left(E^{\frac{m_2-2}{2}}(t)\int_\Omega u(t)\,u_t(t)\,dx\right)dxdt \\ & \le \left|E^{\frac{m_2-2}{2}}(t)\int_\Omega u(S)\,u_t(S)\,dx - E^{\frac{m_2-2}{2}}(t)\int_\Omega u(T)\,u_t(T)\,dx\right| \\ & \le E^{\frac{m_2-2}{2}}(t)\left|\int_\Omega u(x,S)\,u_t(x,S)\,dx\right| + E^{\frac{m_2-2}{2}}(t)\left|\int_\Omega u(x,T)\,u_t(x,T)\,dx\right| \\ & \le cE^{\frac{m_2}{2}}(S)+cE^{\frac{m_2}{2}}(T)\le cE^{\frac{m_2-2}{2}}(0)\,E(S) \\ & \le cE(S). \end{aligned}\tag{4.8}$$

For the second term, we use the following Young inequality:

$$XY\le \frac{\varepsilon}{\lambda_1}X^{\lambda_1}+\frac{1}{\lambda_2\varepsilon^{\frac{\lambda_2}{\lambda_1}}}Y^{\lambda_2},\ X,\ Y\ge 0,\ \varepsilon>0 \text{ and } \frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1.$$

with $\lambda_1(x)=m(x)$, $\lambda_2(x)=\frac{m(x)}{m(x)-1}$. By Lemma 2.6 and Lemma 4.2, we obtain

$$\begin{aligned} & -\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega u(t)\,|u_t(t)|^{m(x)-2}\,u_t(t)\,dxdt \\ & \le \int_S^T E^{\frac{m_2-2}{2}}(t)\left(\varepsilon c\int_\Omega |u(t)|^{m(x)}\,dx + c_\varepsilon\int_\Omega |u_t(t)|^{m(x)}\,dx\right)dt \\ & \le \varepsilon c\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |u(t)|^{m(x)}\,dxdt + c_\varepsilon\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)dt \\ & \le \varepsilon c\int_S^T E^{\frac{m_2}{2}}(t)\,dt + c_\varepsilon E(S). \end{aligned}\tag{4.9}$$

By Young's, Poincare's inequalities and (5.1), we obtain

$$\begin{aligned}
&\frac{m_2-2}{2}\int_S^T E^{\frac{m_2-2}{2}-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dxdt \\
&\le \frac{m_2-2}{2}\int_S^T E^{\frac{m_2-2}{2}-1}(t)\left(-E'(t)\right)\int_\Omega\left(\frac{1}{2}|u(t)|^2+\frac{1}{2}|u_t(t)|^2\right)dxdt \\
&\le c\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)dt \\
&\le cE^{\frac{m_2}{2}}(S)-E^{\frac{m_2}{2}}(T) \\
&\le cE^{\frac{m_2}{2}}(0)\,E(S)\le cE(S)
\end{aligned}\tag{4.10}$$

For the last term of (4.7), we have

$$\begin{aligned}
&(3-\theta_1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |u_t(t)|^2\,dxdt \\
&\le (3-\theta_1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\left[\int_{\Omega^-}|u_t(t)|^2\,dx+\int_{\Omega^+}|u_t(t)|^2\,dx\right]dt \\
&\le c\int_S^T E^{\frac{m_2-2}{2}}(t)\left[\left(\int_{\Omega^-}|u_t(t)|^{m_2}\,dx\right)^{\frac{2}{m_2}}+\left(\int_{\Omega^+}|u_t(t)|^{m_1}\,dx\right)^{\frac{2}{m_1}}\right]dt \\
&\le c\int_S^T E^{\frac{m_2-2}{2}}(t)\left[\left(\int_{\Omega}|u_t(t)|^{m(x)}\,dx\right)^{\frac{2}{m_2}}+\left(\int_{\Omega}|u_t(t)|^{m(x)}\,dx\right)^{\frac{2}{m_1}}\right]dt.
\end{aligned}$$

This implies

$$\begin{aligned}
&(3-\theta_1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |u_t(t)|^2\,dxdt \\
&\le c\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)^{\frac{2}{m_2}}dt+c\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)^{\frac{2}{m_1}}dt.
\end{aligned}\tag{4.11}$$

First, we use Young's inequality with $\lambda_1=m_2/(m_2-2)$ and $\lambda_2=m_2/2$, we have

$$\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)^{\frac{2}{m_2}}dt\le\varepsilon c\int_S^T E^{\frac{m_2}{2}}(t)\,dt+c_\varepsilon\int_S^T\left(-E'(t)\right)dt.$$

This implies

$$\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)^{\frac{2}{m_2}}dt\le\varepsilon c\int_S^T E^{\frac{m_2}{2}}(t)\,dt+c_\varepsilon E(S).\tag{4.12}$$

On the other hand, we use the Young's inequality $\lambda_1=\frac{m_1}{m_1-2}$ and $\lambda_2=\frac{m_1}{2}$, to obtain

$$\begin{aligned}
\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)^{\frac{2}{m_1}}dt&\le\varepsilon c\int_S^T E^{\frac{m_1(m_2-2)}{2(m_1-2)}}(t)\,dt+c_\varepsilon\int_S^T\left(-E'(t)\right)dt \\
&\le\varepsilon c\int_S^T E^{\frac{m_1(m_2-2)}{2(m_1-2)}}(t)\,dt+c_\varepsilon E(S).
\end{aligned}$$

We notice that $\frac{m_1(m_2-2)}{2(m_1-2)} = \frac{m_2}{2} + \frac{m_2-m_1}{m_1-2}$, then

$$\begin{aligned}\int_S^T E^{\frac{m_2-2}{2}}(t)\left(-E'(t)\right)^{\frac{2}{m_1}} dt &\le \varepsilon c\,(E(S))^{\frac{m_2-m_1}{m_1-2}} \int_S^T E^{\frac{m_2}{2}}(t)\,dt + c_\varepsilon E(S) \\ &\le \varepsilon c\,(E(0))^{\frac{m_2-m_1}{m_1-2}} \int_S^T E^{\frac{m_2}{2}}(t)\,dt + c_\varepsilon E(S) \\ &\le \varepsilon c \int_S^T E^{\frac{m_2}{2}}(t)\,dt + c_\varepsilon E(S). \qquad (4.13)\end{aligned}$$

We substituting (4.12) and (4.13) in (4.11), we obtain

$$(3-\theta_1-\theta_2)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |u_t(t)|^2\,dxdt \le \varepsilon c\int_S^T E^{\frac{m_2}{2}}(t)\,dt + c_\varepsilon E(S). \qquad (4.14)$$

By insert (4.8), (4.9), (4.10) and (4.14) in (4.7), we arrive at

$$\gamma\int_S^T E^{\frac{m_2}{2}}(t)\,dt \le \varepsilon c\int_S^T E^{\frac{m_2}{2}}(t)\,dt + c_\varepsilon E(S).$$

Choosing ε small enough for that

$$\int_S^T E^{\frac{m_2}{2}}(t)\,dt \le cE(S).$$

By taking T goes to ∞, we get

$$\int_S^\infty E^{\frac{m_2}{2}}(t)\,dt \le cE(S).$$

By Komornik's integral inequality 2.4 yields the result. □

5 Numerical example

In this section, we present an application to illustrate numerically the stability result of Theorem 4.3. For this purpose, we numerically solve problem (1.1), for $n = 2$ where the domain is taken to be $\Omega = [-1,\ 1]^2$. We chosen $u_0(x_1,x_2) = (x_1+1)(x_1-1)(x_2+1)(x_2-1)$ and $u_1(x_1,x_2) = 0$, where will be chosen such that $E(0) > 0$, we take the exponent function $r(x_1,x_2) = 4$, and $m(x_1,x_2) = {x_1}^2 + {x_2}^2 + 2.5$ which satisfy condition (1.2), where $m_2 = 4.5$. We numerically verify that

$$E(t) \le C(1+t)^{-0.8}.$$

5.1 Numerical method

We first introduce a suitable numerical scheme to discretize (1.1) using finite differences for the time variable $t \in [0,T]$ and the space variable $x = (x_1,x_2) \in \Omega$. We subdivide the time interval $[0,\ T]$ into N equal subintervals $[t_{n-1},\ t_n]$, $\quad t_n = n\ \delta t, \quad n = 1, 2, ..., N+1$, where δt is the time step.

Let $U^n(x_1,x_2) = u(x_1,x_2,\ t_n)$, and use the finite-difference formulas: the first-order backward difference for

$$\partial_t U^n(x_1,x_2) = \frac{U^n(x_1,x_2) - U^{n-1}(x_1,x_2)}{\delta t}.$$

and the second-order center difference for

$$\partial_{tt} U^n(x_1,x_2) = \frac{U^{n+1}(x_1,x_2) - 2U^n(x_1,x_2) + U^{n-1}(x_1,x_2)}{(\delta t)^2}.$$

Then the time discrete problem of (1.1) reads: Given u_0 and u_1, find $\{U^2,\ U^3, ..., U^{n+1}\}$ such that

$$\begin{cases} \frac{U^{n+1}}{(\delta t)^2} - \Delta U^{n+1} = \frac{2U^n - U^{n-1}}{(\delta t)^2} - |\partial_t U^n|^{m(x_1,x_2)-2}\,\partial_t U^n & \\ \quad -\Delta^2 U^n + |U^n|^{r(x_1,x_2)-2}\,U^n, & \text{in } \Omega \\ U^{n+1} = 0, & \text{on } \partial\Omega \\ U^0 = u_0(x_1,x_2), \quad U^1 = U^0 + (\delta t)\,u_1(x_1,x_2), & \text{in } \Omega \end{cases} \tag{5.1}$$

Note that the above problem is linear in U^{n+1}, which is achieved by using the history data U^n and U^{n-1} in the second side of the equation. Problem(5.1) is solved iteratively as for given regular U^n, the solution U^{n+1} satisfies the boundary-value problem:

$$\begin{cases} \frac{U^{n+1}}{(\delta t)^2} - \Delta U^{n+1} = F\left(U^n,\ U^{n-1}\right), & \text{in } \Omega_h \\ U^{n+1} = 0, & \text{on } \partial\Omega_h \end{cases} \tag{5.2}$$

where $F\left(U^n,\ U^{n-1}\right) = \frac{2U^n - U^{n-1}}{(\delta t)^2} - \Delta^2 U^n - |\partial_t U^n|^{m(x_1,x_2)-2}\,\partial_t U^n + |U^n|^{r(x_1,x_2)-2}\,U^n$.

5.2 Numerical results

In this subsection, we present and discuss the stability results of the numerical scheme(5.1). The numerical results are obtained using the Matlab codes.

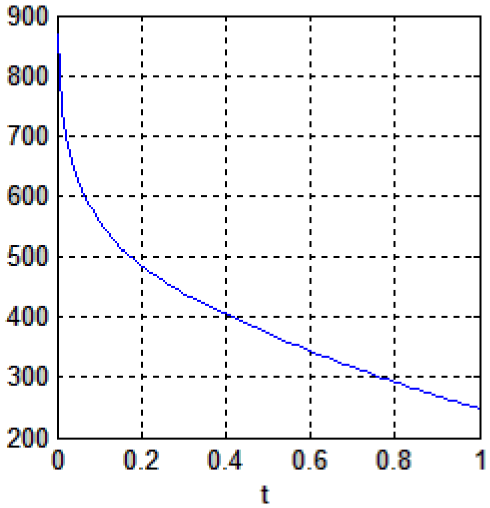

Figure 1: Energy: E(t)

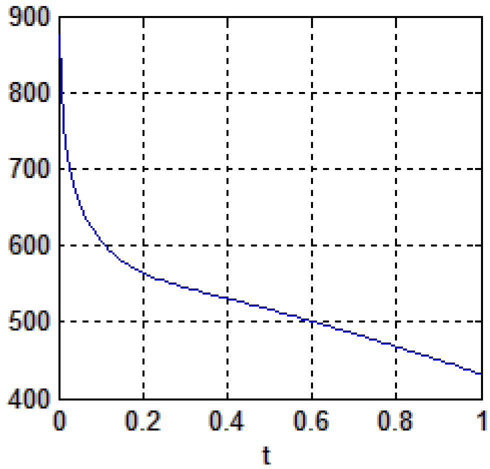

Figure 2: Polynomial decay:z=E(t)(1+t)$^{0.8}$

The parameters that have been set up for numerical experiments are:

- Number of discretisation points is: 100;
- Time step is: $\delta t = 0.01$;

Figures. 1 and 2 presents the energy $E(t)$ and $E(t)(1+t)^{0.8}$ respectively for the times $t_n \in \{1,\ 2, ..., 100\}$. The numerical solutions of problem (1.1) make the energy function $E(t)$ satisfy

$$E(t)(1+t)^{0.8} \leq 9 \times 10^2.$$

In conclusion, the above numerical application verifies and agrees with the stability results of Theorem 4.3.

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