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Study of a dynamic viscoelastic problems with short memory

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Abstract

This paper is devoted to the study of the asymptotic behavior of a viscoelastic problem with short memory in a three-dimensional thin domain Ω^{ε}. We prove some convergence results when the thickness tends to zero. The contact is modeled with the Tresca friction law. We derive a variational formulation of the problem and prove its unique weak solution. Then we prove some convergence results when the small parameter ε tends to zero. Finally, the specific Reynolds limit equation and the limit of Tresca-free boundary conditions are obtained.

Keywords: asymptotic approach, boundary value problem, displacement field, Reynolds equation, short memory
2020 MSC: 35R35, 76F10, 78M35, 35B40, 35J85, 49J40

1 Introduction

In recent decades, many authors have applied asymptotic methods to reduce the problems of three-dimensional frictional contact to the two-dimensional models of a thin domain. In the literature, the asymptotic behavior of partial differential equations in a thin domain, particularly those governed by elastic systems has been widely studied. Ciarlet and Destuynder in [5], studied the equilibrium states of a thin plate $\Omega \times (-\varepsilon, \varepsilon)$, to justify the two-dimensional model of the plates, under external forces, where Ω is a smooth domain in $\mathbb{R}^2$ and ε is a small parameter. The same method is used recently for problems of general elastic shells in frictionless unilateral contract with an obstacle (see for instance [2, 11]). Moreover, there are many problems have been studied before by using mathematical aspect and after, they used the simplified Tresca interface condition which can be found it in monographs (see [12, 15] and references therein). In our paper, we study the asymptotic behavior of a non linear problem in a dynamic regime with Tresca free boundary friction conditions occupying a bounded homogeneous domain Ω^{ε}. The boundary of this thin domain consists of three parts: the bottom, the lateral part, and the top surface. We model the friction with version of Tresca's law. Our attention is devoted to the appearance of the short memory term. The equations considered in this paper with this term are widely used in applications dealing with physical and biological systems. These are the assumptions of elasticity and visco-elasticity of tire. Also their uses is very important in quantum mechanics ([8]). Other applications are related to the mechanism of ball bearing. In the context of the linear thin elasticity and in the dynamic case, the authors in [1, 15] proved the asymptotic analysis of a dynamical problem of isothermal and non-isothermal elasticity with non linear friction of Tresca type but without the short memory term. Other similar problems can be found in [4, 7, 13, 14].

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The paper is organized as follows. In section 2 and 3, we present some notations and give the problem dynamic and variational formulation. We prove that the displacement field satisfies an evolutionary variational inequality with viscosity. In Section 4, we use the asymptotic analysis, in which the small parameter ε is the height of the domain. We establish some estimates, independent on the parameter ε, for displacement and the velocity. We obtain some convergence results. The main results concerning the limit problem with a specific weak form of the Reynolds equation, the uniqueness of the limit displacement when the thickness tends to zero are given in section 5.

2 Basic equations and assumptions

We consider a mathematical problem governed with non linear equations in a three dimensional thin domain Ω^ε. Let $\omega \subset \mathbb{R}^2$ be fixed region in the plane. We suppose that ω has a Lipschitz boundary and is the bottom of the fluid domain. Let $h : \omega \to \mathbb{R}$ be a sufficiently smooth function such that

$$0 < h_{\min} \leq h(x_1, x_2) \leq h_{\max},$$

for all $x = (x_1, x_2) \in \omega$, where $h_{\min}$ and $h_{\max}$ are constants. Let $0 < \varepsilon < 1$ is a small parameter that will tend to zero. We denote by Ω^ε the domain of the flow (as in [8])

$$\Omega^\varepsilon = \{(x, x_3) \in \mathbb{R}^3 : (x, 0) \in \omega, 0 < x_3 < \varepsilon h(x)\}.$$

We decompose the boundary of Ω^ε as $\Gamma^\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon \cup \omega$, with

$$\begin{aligned} \omega &= \{(x, x_3) \in \overline{\Omega}^\varepsilon : x_3 = 0\}, \\ \Gamma_1^\varepsilon &= \{(x, x_3) \in \overline{\Omega}^\varepsilon : (x, 0) \in \omega, \ \ x_3 = \varepsilon h\,(x')\}, \\ \Gamma_L^\varepsilon &= \{(x, x_3) \in \overline{\Omega}^\varepsilon : x \in \partial\omega, \, 0 < x_3 < \varepsilon h(x)\}, \end{aligned}$$

where ω is the bottom of the domain, Γ_1^ε is the upper surface and Γ_L^ε the lateral part of Γ^ε

The law of conservation of momentum is given by

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} - \operatorname{div}(\sigma^\varepsilon) - f^\varepsilon = 0 \text{ in } \Omega^\varepsilon \times (0, T),$$

with $div(\sigma^\varepsilon) = (\sigma_{ij,j}^\varepsilon)$ and $f^\varepsilon = (f_i^\varepsilon)_{1 \leq i \leq 3}$ denote the body forces.

The constitutive law for linearly viscoelastic isotropic material with short memory is given by

$$\sigma_{ij} = 2\mu d_{ij}(u(t)) + \lambda d_{kk}(u(t))\delta_{ij} + SM, \text{ in } \Omega^\varepsilon \times (0, T),$$

where u is the displacement field to the x point, $\dot{u} = \frac{du}{dt}(t)$ the velocity field, λ and μ are the Lamé coefficients. The short memory (SM) is represented by the formula ([6]):

$$SM = 2\theta d_{ij}(\dot{u}(t)) + \zeta d_{kk}(\dot{u}(t))\delta_{ij}, \text{ in } \Omega^\varepsilon \times (0, T),$$

where θ and ζ represent the viscosity coefficients that satisfy $\theta > 0, \zeta > 0$. Finally, the *Kronecker* symbol is denoted by δ_{ij} and $d_{ij}(u^\varepsilon) = \dfrac{1}{2}(\nabla u^\varepsilon + (\nabla u^\varepsilon)^T)$.
However, the tangential velocity on $\omega \times (0, T)$ is unknown and satisfies the Tresca boundary conditions, with friction coefficient κ^ε ([8]):

$$\begin{cases} |\sigma_\tau^\varepsilon| < k^\varepsilon \implies \dot{u}_\tau^\varepsilon(t) = 0, \\ |\sigma_\tau^\varepsilon| = k^\varepsilon \implies \exists\beta \geq 0, \dot{u}_\tau^\varepsilon(t) = -\beta\sigma_\tau^\varepsilon. \end{cases} \quad \text{on } \omega \times (0, T)$$

We consider now, the following mechanical problem:
Problem 1. Find displacement vector $u^\varepsilon : \Omega^\varepsilon \to \mathbb{R}^3$ such that

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} - \operatorname{div}(\sigma^\varepsilon) - f^\varepsilon = 0, \text{ in } \Omega^\varepsilon \times (0, T), \tag{2.1}$$

$$\sigma_{ij} = 2\mu d_{ij}(u(t)) + \lambda d_{kk}(u(t))\delta_{ij} + 2\theta d_{ij}(\dot{u}(t)) + \zeta d_{kk}(\dot{u}(t))\delta_{ij}, \text{ in } \Omega^\varepsilon \times (0,T), \tag{2.2}$$

$$u^\varepsilon(t) = 0, \text{ on } (\Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon) \times (0,T), \tag{2.3}$$

$$u^\varepsilon(t).n = 0, \text{ on } \omega \times (0,T), \tag{2.4}$$

$$\begin{cases} |\sigma_\tau^\varepsilon| < k^\varepsilon \implies \dot{u}_\tau^\varepsilon(t) = 0, \\ |\sigma_\tau^\varepsilon| = k^\varepsilon \implies \exists \beta \geq 0, \dot{u}_\tau^\varepsilon(t) = -\beta\sigma_\tau^\varepsilon, \end{cases} \quad \text{on } \omega \times (0,T), \tag{2.5}$$

$$u^\varepsilon(0) = u_0^\varepsilon \text{ and } \frac{\partial u^\varepsilon}{\partial t}(0) = u_1^\varepsilon. \tag{2.6}$$

Here $|.|$ denotes the Euclidean norm on $\mathbb{R}^3$. Let $n = (n_1, n_2, n_3)$ be the unit outward normal to Γ^ε, we define the normal and the tangential components of displacement and stress tensors on the boundary, respectively, as follows

$$v_n^\varepsilon = v^\varepsilon.n, \quad v_\tau^\varepsilon = v^\varepsilon - v_n^\varepsilon.n, \quad \sigma_n^\varepsilon = (\sigma^\varepsilon.n).n, \quad \sigma_\tau^\varepsilon = \sigma^\varepsilon n - \sigma_n^\varepsilon.n.$$

3 Weak variational formulation

To get weak formulation, we consider the following spaces

$$H^1(\Omega^\varepsilon) = \{\varphi \in L^2(\Omega^\varepsilon), \frac{\partial \varphi_i}{\partial x_j} \in L^2(\Omega^\varepsilon) \quad \text{for all} \quad i,j = 1,2,3\},$$
$$K^\varepsilon = \left\{\varphi \in H^1(\Omega^\varepsilon)^3 : \varphi = 0 \text{ on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon, \ \varphi.n = 0 \text{ on } \omega\right\}.$$

Owing to the *Green*'s formula and by (2.3)–(2.6), we have to find the displacement vector $u^\varepsilon \in K^\varepsilon$ such that:

$$\begin{aligned}(\frac{\partial^2 u^\varepsilon}{\partial t^2}(t), \varphi - \dot{u}^\varepsilon(t)) + a(u^\varepsilon(t), \varphi - \dot{u}^\varepsilon(t)) + B\left(\dot{u}^\varepsilon(t), \varphi - \dot{u}^\varepsilon(t)\right) + j(\varphi) - \\ j\left(\dot{u}^\varepsilon(t)\right) \geq (f^\varepsilon(t), \varphi - \dot{u}^\varepsilon(t)),\end{aligned} \tag{3.1}$$

for all $\varphi \in K^\varepsilon$, where

$$\begin{aligned}
(\frac{\partial^2 u^\varepsilon}{\partial t^2}(t), \varphi - \dot{u}^\varepsilon(t)) &= \int_{\Omega^\varepsilon} \frac{\partial^2 u^\varepsilon}{\partial t^2}(t)(\varphi - \dot{u}^\varepsilon(t))dxdx_3, \\
a(u^\varepsilon(t), \varphi - \dot{u}^\varepsilon(t)) &= 2\mu \int_{\Omega^\varepsilon} d_{i,j}(u^\varepsilon(t))d_{i,j}(\varphi - \dot{u}^\varepsilon(t))dxdx_3 + \\
&\quad \lambda \int_{\Omega^\varepsilon} div(u^\varepsilon(t))div(\varphi - \dot{u}^\varepsilon(t))dxdx_3, \\
B\left(\dot{u}^\varepsilon(t), \varphi - \dot{u}^\varepsilon(t)\right) &= 2\theta \int_{\Omega^\varepsilon} d_{i,j}(\dot{u}^\varepsilon(t))d_{i,j}(\varphi - \dot{u}^\varepsilon(t))dxdx_3 + \\
&\quad \zeta \int_{\Omega^\varepsilon} div(\dot{u}^\varepsilon(t))div(\varphi - \dot{u}^\varepsilon(t))dxdx_3, \\
j(v) &= \int_\omega k^\varepsilon |v| \ dx, \\
(f^\varepsilon(t), v) &= \int_{\Omega^\varepsilon} f^\varepsilon(t) v dxdx_3 = \sum_{i=1}^3 \int_{\Omega^\varepsilon} f_i^\varepsilon(t) v_i dxdx_3.
\end{aligned}$$

We introduce some results which will be used in the next.

$$\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C \|D(u^\varepsilon)\|_{L^2(\Omega^\varepsilon)} \ (\textit{Korn inequality}, \text{ see. } [\textbf{?}\ ]), \tag{3.2}$$

$$\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq \varepsilon h_{\max} \left\|\frac{\partial u^\varepsilon}{\partial z}\right\|_{L^2(\Omega^\varepsilon)} \quad \text{for } i = 1,2 \ (\textit{Poincaré inequality}, \text{ see. } [3]), \tag{3.3}$$

$$\text{There exists} \quad C_1 > 0 \text{ such that } |B(u,v)| \leq C_1 \|u\| \|v\|, (\textit{see.}[16,\ 17]) \tag{3.4}$$

$$\text{There exists} \quad C_2 > 0 \text{ such that } B(u,u) \geq C_2 \|u\|^2, (\textit{see.}[16,\ 17]). \tag{3.5}$$

4 Change of the domain and study of convergence

In this section, we will use the technique of scaling in Ω^ε on the coordinate x_3. By introducing the change of the variables $z = \dfrac{x_3}{\varepsilon}$, we obtain a fixed domain Ω which is independent of ε.

$$\Omega = \left\{(x,\, z) \in \mathbb{R}^3 :\, (x, 0) \in \omega, \quad 0 < z < h(x)\right\}.$$

Next, we denote by $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_L \cup \overline{\omega}$ its boundary, then we define the following functions in Ω

$$\begin{array}{rcl} \hat{u}_i^\varepsilon(x,z,t) & = & u_i^\varepsilon(x,x_3,t), i = 1,2, \\ \hat{u}_3^\varepsilon(x,z,t) & = & \varepsilon^{-1} u_3^\varepsilon(x,x_3,t). \end{array} \tag{4.1}$$

Now, we assume that

$$\begin{array}{c} \hat{f}(x,z,t) = \varepsilon^2 f^\varepsilon(x,x_3,t), \hat{k} = \varepsilon k^\varepsilon, \\ (\hat{u}_0)_i = (u_0^\varepsilon)_i \ \text{ and } \ (\hat{u}_0)_3 = \varepsilon^{-1}(u_0^\varepsilon)_3, i = 1,2 \end{array} \tag{4.2}$$

and we consider the sets

$$\begin{aligned} K(\Omega) &= \left\{\hat{\varphi} \in \left(H^1(\Omega)\right)^3 : \hat{\varphi} = 0 \text{ on } \Gamma_1 \cup \Gamma_L;\ \hat{\varphi}.n = 0 \text{ on } \omega\right\}, \\ V_z &= \left\{\hat{\varphi} \in \left(L^2(\Omega)\right)^2 ; \frac{\partial \hat{\varphi}_i}{\partial z} \in L^2(\Omega) : \hat{\varphi} = 0 \ \text{ on } \Gamma_1\right\}, \end{aligned}$$

With these new data, unknown factors in (3.1) and multiplying by ε, to deduce

$$\begin{array}{r} \sum_{i=1}^2 \varepsilon^2 (\frac{\partial^2 \hat{u}_i^\varepsilon}{\partial t^2}(t), \hat{\varphi}_i - \hat{u}_i^\varepsilon(t)) + \varepsilon^4 (\frac{\partial^2 \hat{u}_3^\varepsilon}{\partial t^2}(t), \hat{\varphi}_3 - \hat{u}_3^\varepsilon(t)) + \\ \tilde{a}(\hat{u}^\varepsilon(t), \hat{\varphi} - \hat{u}^\varepsilon(t)) + \tilde{B}\left(\hat{u}^\varepsilon(t), \hat{\varphi} - \hat{u}^\varepsilon(t)\right) + j_0(\hat{\varphi}) - \\ j_0\left(\hat{u}^\varepsilon\right) \geq (\hat{f}, \hat{\varphi} - \hat{u}^\varepsilon),\ \forall \hat{\varphi} \in K(\Omega), \end{array} \tag{4.3}$$

where

$$\begin{aligned} \tilde{a}(\hat{u}^\varepsilon(t), \hat{\varphi} - \hat{u}^\varepsilon(t)) = \mu \sum_{i,j=1}^2 \int_\Omega \varepsilon^2 \left(\frac{\partial \hat{u}_i^\varepsilon(t)}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon(t)}{\partial x_i}\right) \frac{\partial(\hat{\varphi}_i - \hat{u}_i^\varepsilon(t))}{\partial x_j} dxdz \\ + \mu \sum_{i=1}^2 \int_\Omega \left(\frac{\partial \hat{u}_i^\varepsilon(t)}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon(t)}{\partial x_i}\right) \frac{\partial(\hat{\varphi}_i - \hat{u}_i^\varepsilon(t))}{\partial z} dxdz \\ + 2\mu \int_\Omega \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon(t)}{\partial z} \frac{\partial(\hat{\varphi}_3 - \hat{u}_3^\varepsilon(t))}{\partial z} dxdz + \\ \mu \sum_{j=1}^2 \int_\Omega \varepsilon^2 \left(\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon(t)}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon(t)}{\partial z}\right) \frac{\partial(\hat{\varphi}_3 - \hat{u}_3^\varepsilon(t))}{\partial x_j} dxdz \\ + \lambda \varepsilon^2 \int_\Omega div(\hat{u}^\varepsilon(t)) div(\hat{\varphi} - \hat{u}^\varepsilon(t)) dxdz, \end{aligned}$$

$$\begin{aligned} \tilde{B}(\hat{u}^\varepsilon(t), \hat{\varphi} - \hat{u}^\varepsilon(t)) = \theta \sum_{i,j=1}^2 \int_\Omega \varepsilon^2 \left(\frac{\partial \hat{\dot{u}}_i^\varepsilon(t)}{\partial x_j} + \frac{\partial \hat{\dot{u}}_j^\varepsilon(t)}{\partial x_i}\right) \frac{\partial(\hat{\varphi}_i - \hat{u}_i^\varepsilon(t))}{\partial x_j} dxdz \\ + \theta \sum_{i=1}^2 \int_\Omega \left(\frac{\partial \hat{\dot{u}}_i^\varepsilon(t)}{\partial z} + \varepsilon^2 \frac{\partial \hat{\dot{u}}_3^\varepsilon(t)}{\partial x_i}\right) \frac{\partial(\hat{\varphi}_i - \hat{u}_i^\varepsilon(t))}{\partial z} dxdz \\ + 2\theta \int_\Omega \varepsilon^2 \frac{\partial \hat{\dot{u}}_3^\varepsilon(t)}{\partial z} \frac{\partial(\hat{\varphi}_3 - \hat{u}_3^\varepsilon(t))}{\partial z} dxdz \\ + \theta \sum_{j=1}^2 \int_\Omega \varepsilon^2 \left(\varepsilon^2 \frac{\partial \hat{\dot{u}}_3^\varepsilon(t)}{\partial x_j} + \frac{\partial \hat{\dot{u}}_j^\varepsilon(t)}{\partial z}\right) \frac{\partial(\hat{\varphi}_3 - \hat{u}_3^\varepsilon(t))}{\partial x_j} dxdz \\ + \zeta \varepsilon^2 \int_\Omega div(\hat{\dot{u}}^\varepsilon(t)) div(\hat{\varphi} - \hat{u}^\varepsilon(t)) dxdz, \end{aligned}$$

$$j_0(\hat{\varphi}) = \int_\omega \hat{k}|\hat{\varphi}|dx$$

and

$$(\hat{f}^\varepsilon(t), \hat{\varphi} - \hat{u}^\varepsilon(t)) = \sum_{j=1}^{2} \int_\Omega \hat{f}_i(t)(\hat{\varphi}_i - \hat{u}_i^\varepsilon(t))dxdz + \int_\Omega \varepsilon \hat{f}_3(t)(\hat{\varphi}_3 - \hat{u}_3^\varepsilon(t))dxdz.$$

We now establish the estimates for the displacement $\hat{u}^\varepsilon$ and the velocity $\hat{\dot{u}}^\varepsilon$ in the domain Ω.

Theorem 4.1. Assume that $f^\varepsilon, \frac{\partial f^\varepsilon}{\partial t}, \frac{\partial^2 f^\varepsilon}{\partial t^2} \in L^2(0,T,L^2(\Omega^\varepsilon)^3)$ and $u_0 \in H^2(\Omega^\varepsilon), u_1 \in H^1(\Omega^\varepsilon)$, the friction coefficient $k > 0$ in $L^\infty(\omega)$, then there exists a constant $C > 0$ independent of ε such that :

$$\sum_{i,j=1}^{2} \left\| \varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j}(t) \right\|^2_{L^2(\Omega)} + \left\| \varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z}(t) \right\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \left(\left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z}(t) \right\|^2_{L^2(\Omega)} + \left\| \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}(t) \right\|^2_{L^2(\Omega)} \right) \leq C. \tag{4.4}$$

$$\sum_{i,j=1}^{2} \left\| \varepsilon \frac{\partial^2 \hat{u}_i^\varepsilon}{\partial x_j \partial t}(t) \right\|^2_{L^2(\Omega)} + \left\| \varepsilon \frac{\partial^2 \hat{u}_3^\varepsilon}{\partial z \partial t}(t) \right\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \left(\left\| \frac{\partial^2 \hat{u}_i^\varepsilon}{\partial z \partial t}(t) \right\|^2_{L^2(\Omega)} + \left\| \varepsilon^2 \frac{\partial^2 \hat{u}_3^\varepsilon}{\partial x_i \partial t}(t) \right\|^2_{L^2(\Omega)} \right) \leq C. \tag{4.5}$$

Proof . We take $\varphi = 0$ in (3.1), to obtain

$$\left(\frac{\partial^2 u^\varepsilon}{\partial t^2}, \dot{u}^\varepsilon\right) + a(u^\varepsilon(t), \dot{u}^\varepsilon(t)) + b\left(\dot{u}^\varepsilon(t), \dot{u}^\varepsilon(t)\right) + \int_\omega k^\varepsilon \left|\dot{u}^\varepsilon(t)\right| dx \leq (f^\varepsilon(t), \dot{u}^\varepsilon(t)). \tag{4.6}$$

Integrating (4.6) for 0 to T, from Korn's inequality, there exists a constant C_k independent of ε; such that

$$\left[\|\dot{u}^\varepsilon(t)\|^2 + 2\mu C_k \left\|\nabla u^\varepsilon(t)\right\|^2_{L^2(\Omega^\varepsilon)} \right] + 4\theta C_k \left\|\nabla \dot{u}^\varepsilon(t)\right\|^2_{L^2(0,T,\Omega^\varepsilon)} \leq$$
$$\|u_1\|^2_{L^2(\Omega^\varepsilon)} + (2\mu + 3\lambda) \left\|\nabla u_0^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} + 2\int_0^s (f^\varepsilon(t), \dot{u}^\varepsilon(t))dt.$$

But

$$\int_0^t (f^\varepsilon(s), \dot{u}^\varepsilon(s))ds = (f^\varepsilon(t), u^\varepsilon(t)) - (f^\varepsilon(0), u^\varepsilon(0)) - \int_0^t \left(\frac{\partial f^\varepsilon}{\partial t}(s), u^\varepsilon(s)\right)ds$$

We apply the Cauchy-Schwarz, Poincaré and Young inequalities, we deduce

$$\left|2\int_0^t (f^\varepsilon(s), \dot{u}^\varepsilon(s))ds\right| \leq \mu C_k \|\nabla u^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + \|\nabla u^\varepsilon(0)\|^2_{L^2(\Omega^\varepsilon)} + \frac{(\varepsilon h_{\max})^2}{\mu C_k} \int_0^t \|\frac{\partial f^\varepsilon}{\partial t}(s)\|^2_{L^2(\Omega^\varepsilon)} ds +$$
$$\frac{(\varepsilon h_{\max})^2}{\mu C_k} \|f^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + (\varepsilon h_{\max})^2 \|f^\varepsilon(0)\|^2_{L^2(\Omega^\varepsilon)} + \frac{\mu C_k}{2} \int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds. \tag{4.7}$$

From (4.7), we get

$$\|\dot{u}^\varepsilon(t)\|^2 + \mu C_k \left\|\nabla u^\varepsilon(t)\right\|^2_{L^2(\Omega^\varepsilon)} + 4\theta C_k \left\|\nabla \dot{u}^\varepsilon(t)\right\|^2_{L^2(0,T,\Omega^\varepsilon)} \leq \|u_1\|^2_{L^2(\Omega^\varepsilon)} + (1 + 2\mu + 3\lambda) \left\|\nabla u_0^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} +$$
$$\frac{(\varepsilon h_{\max})^2}{\mu C_k} \|f^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + (\varepsilon h_{\max})^2 \|f^\varepsilon(0)\|^2_{L^2(\Omega^\varepsilon)} + \frac{(\varepsilon h_{\max})^2}{\mu C_k} \int_0^t \|\frac{\partial f^\varepsilon}{\partial t}(s)\|^2_{L^2(\Omega^\varepsilon)} ds +$$
$$\|\nabla u^\varepsilon(0)\|^2_{L^2(\Omega^\varepsilon)} + \frac{\mu C_k}{2} \int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds. \tag{4.8}$$

Multiplying (4.8) by ε, and since $\varepsilon^2 \left\| f^\varepsilon(t) \right\|^2_{L^2(0,T,\Omega^\varepsilon)} = \varepsilon^{-1} \left\| \hat{f}^\varepsilon(t) \right\|^2_{L^2(0,T,\Omega)}$, we obtain

$$\varepsilon \left\| \dot{u}^\varepsilon(t) \right\|^2 + \mu C_k \varepsilon \left\| \nabla u^\varepsilon(t) \right\|^2_{L^2(\Omega^\varepsilon)} + 4\theta C_k \varepsilon \left\| \nabla \dot{u}^\varepsilon(t) \right\|^2_{L^2(0,T,\Omega^\varepsilon)} \le C_1 + \varepsilon \mu C_k \int_0^t \left\| \nabla u^\varepsilon(s) \right\|^2_{L^2(\Omega^\varepsilon)} ds. \tag{4.9}$$

with

$$C_1 = \|\hat{u}_1\|^2_{L^2(\Omega)} + (1 + 2\mu + 3\lambda) \|\nabla \hat{u}_0^\varepsilon\|^2_{L^2(\Omega)} + \frac{(h_{\max})^2}{\mu C_k} \int_0^t \|\frac{\partial f^\varepsilon}{\partial t}(s)\|^2_{L^2(\Omega)} ds + \tag{4.10}$$

$$\frac{h_{\max}}{\mu C_k} \|\hat{f}^\varepsilon(t)\|^2_{L^2(\Omega)} + h_{\max} \|\hat{f}^\varepsilon(0)\|^2_{L^2(\Omega)} + \|\nabla \hat{u}^\varepsilon(0)\|^2_{L^2(\Omega)} \tag{4.11}$$

By Gronwall's inequality ([9]), it follows that

$$\varepsilon \left\| \dot{u}^\varepsilon(t) \right\|^2 + \varepsilon \left\| \nabla u^\varepsilon(t) \right\|^2_{L^2(\Omega^\varepsilon)} \le C. \tag{4.12}$$

So, from (4.12) we deduce (4.4).

To obtain a second estimate, let's regularize the function Φ using a family of functional $\varphi_\varsigma = j_\varsigma$ where $\varsigma > 0$ and j_ς is the following differentiable functional

$$j_\varsigma^\varepsilon(u) = \int_\omega k^\varepsilon(x) \varphi_\varsigma(|u_\tau|^2) dx, \quad \text{and} \quad \varphi_\varsigma(\lambda) = \frac{1}{1+\varsigma} |\lambda|^{(1+\varsigma)}.$$

We consider the following approximate equation

$$\begin{cases} (\frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t), \phi) + a(u_\varsigma^\varepsilon(t), \phi) + B\left(\dot{u}_\varsigma^\varepsilon(t), \phi\right) + \left((j_\varsigma^\varepsilon)'(\frac{\partial u_\varsigma^\varepsilon}{\partial t})(t), \phi\right) = (f^\varepsilon(t), \phi). \\ u_\varsigma^\varepsilon(0) = u_0, \quad \frac{\partial u_\varsigma^\varepsilon}{\partial t}(0) = u_1, \quad (u_1)_\tau = 0. \end{cases} \tag{4.13}$$

Now we derive equation (4.13) with respect to time and by substituting ϕ with $\frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}$, we find

$$(\frac{\partial^3 u_\varsigma^\varepsilon}{\partial t^3}(t), \frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t)) + a(\dot{u}_\varsigma^\varepsilon(t), \frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t)) + B\left(\frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t), \frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t)\right) + \left((j_\varsigma^\varepsilon)''(\frac{\partial u_\varsigma^\varepsilon}{\partial t}(t)), \frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t)\right) = (\frac{\partial f^\varepsilon}{\partial t}(t), \frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t)).$$

and because $\left((j_\varsigma^\varepsilon)''(\frac{\partial u_\varsigma^\varepsilon}{\partial t}(t)), \frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t)\right) \ge 0$ and $b\left(\frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t), \frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t)\right) \ge 0$, we obtain

$$\frac{1}{2}\frac{d}{dt} \|\frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t)\|^2_{L^2(\Omega^\varepsilon)} + a(\dot{u}_\varsigma^\varepsilon(t), \frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t)) \le (\frac{\partial f^\varepsilon}{\partial t}(t), \frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t)). \tag{4.14}$$

For $s \in [0, t]$; integrating (4.14) and using Korn's inequality, we get

$$\|\frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t)\|^2_{L^2(\Omega^\varepsilon)} + 2\mu C_k \|\nabla \frac{\partial u_\varsigma^\varepsilon}{\partial t}(t)\|^2_{L^2(\Omega^\varepsilon)} \le \|\frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(0)\|^2_{L^2(\Omega^\varepsilon)} + (2\mu + 3\lambda) \|\nabla \frac{\partial u_\varsigma^\varepsilon}{\partial t}(0)\|^2_{L^2(\Omega^\varepsilon)} + 2\int_0^s (\frac{\partial f^\varepsilon}{\partial t}(t), \frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t)) ds.$$

AS

$$2(\frac{\partial f^\varepsilon}{\partial t}(t), \frac{\partial^2 u_\varsigma^\varepsilon}{\partial t^2}(t)) = 2(\frac{\partial f^\varepsilon}{\partial t}(t), \frac{\partial u_\varsigma^\varepsilon}{\partial t}(t)) - 2(\frac{\partial f^\varepsilon}{\partial t}(0), \frac{\partial u_\varsigma^\varepsilon}{\partial t}(0)) - 2\int_0^t (\frac{\partial^2 f^\varepsilon}{\partial t^2}(t), \frac{\partial u_\varsigma^\varepsilon}{\partial t}(t)) ds$$

and by the Cauchy-Schwarz, Poincaré and Young inequalities, we have

$$\|\frac{\partial^2 u^\varepsilon_\varsigma}{\partial t^2}(t)\|^2_{L^2(\Omega^\varepsilon)} + \mu C_k\|\nabla\frac{\partial u^\varepsilon_\varsigma}{\partial t}(t)\|^2_{L^2(\Omega^\varepsilon)} \le \|\frac{\partial^2 u^\varepsilon_\varsigma}{\partial t^2}(0)\|^2_{L^2(\Omega^\varepsilon)} + \mu C_k\|\nabla\frac{\partial u^\varepsilon_\varsigma}{\partial t}(0)\|^2_{L^2(\Omega^\varepsilon)} + (2\mu+3\lambda)\|\nabla\frac{\partial u^\varepsilon_\varsigma}{\partial t}(0)\|^2_{L^2(\Omega^\varepsilon)} + \frac{(\varepsilon h_{\max})^2}{\mu C_k}\int_0^t \|\frac{\partial^2 f^\varepsilon}{\partial t^2}(s)\|^2_{L^2(\Omega^\varepsilon)}ds + \frac{(\varepsilon h_{\max})^2}{\mu C_k}\|\frac{\partial f^\varepsilon}{\partial t}(0)\|^2_{L^2(\Omega^\varepsilon)} + \frac{(\varepsilon h_{\max})^2}{\mu C_k}\|\frac{\partial f^\varepsilon}{\partial t}(t)\|^2_{L^2(\Omega^\varepsilon)} + \frac{\mu C_k}{2}\int_0^t \|\nabla u^\varepsilon_\varsigma(s)\|^2_{L^2(\Omega^\varepsilon)}ds. \tag{4.15}$$

And from (4.13), we have

$$\left(\frac{\partial^2 u^\varepsilon_\varsigma}{\partial t^2}(0), \phi\right) = (f^\varepsilon(0), \phi) - a(u^\varepsilon_\varsigma(0), \phi)$$

also

$$|\left(\frac{\partial^2 u^\varepsilon_\varsigma}{\partial t^2}(0), \phi\right)| \le \left(\varepsilon h_{\max}\|f^\varepsilon(0)\|_{L^2(\Omega^\varepsilon)} + (2\mu+3\lambda)\|u^\varepsilon_\varsigma(0)\|_{H^1(\Omega^\varepsilon)}\right)\|\phi\|_{H^1(\Omega^\varepsilon)}. \tag{4.16}$$

We multiply (4.16) by $\sqrt{\varepsilon}$ and since $\varepsilon^{\frac{3}{2}}\|f^\varepsilon(0)\|_{L^2(\Omega^\varepsilon)} = \left\|\hat{f}^\varepsilon(0)\right\|_{L^2(\Omega)}$, $\sqrt{\varepsilon}\left\|u^\varepsilon_\varsigma(0)\right\|_{H^1(\Omega^\varepsilon)} = \left\|\hat{u}^\varepsilon_\varsigma(0)\right\|_{H^1(\Omega)}$, we obtain

$$\sqrt{\varepsilon}\|\frac{\partial^2 u^\varepsilon_\varsigma}{\partial t^2}(0)\|_{L^2(\Omega\varepsilon)} \le C_2. \tag{4.17}$$

with $C_2 = h_{\max}\|\hat{f}^\varepsilon(0)\|_{L^2(\Omega)} + (2\mu+3\lambda)\|\hat{u}^\varepsilon_\varsigma(0)\|_{H^1(\Omega)}$.
And for $\varsigma \to 0$ in (4.15) , we obtain

$$\|\frac{\partial^2 u^\varepsilon}{\partial t^2}(t)\|^2_{L^2(\Omega^\varepsilon)} + \mu C_k\|\nabla\frac{\partial u^\varepsilon}{\partial t}(t)\|^2_{L^2(\Omega^\varepsilon)} \le \|\frac{\partial^2 u^\varepsilon}{\partial t^2}(0)\|^2_{L^2(\Omega^\varepsilon)} + \|\nabla u^\varepsilon(0)\|^2_{L^2(\Omega^\varepsilon)} + (\mu C_k + 2\mu+3\lambda)\|\nabla\frac{\partial u^\varepsilon}{\partial t}(0)\|^2_{L^2(\Omega^\varepsilon)} + \frac{(\varepsilon h_{\max})^2}{\mu C_k}\int_0^t \|\frac{\partial^2 f^\varepsilon}{\partial t^2}(s)\|^2_{L^2(\Omega^\varepsilon)}ds + \frac{(\varepsilon h_{\max})^2}{\mu C_k}\|\frac{\partial f^\varepsilon}{\partial t}(0)\|^2_{L^2(\Omega^\varepsilon)} + \frac{(\varepsilon h_{\max})^2}{\mu C_k}\|\frac{\partial f^\varepsilon}{\partial t}(t)\|^2_{L^2(\Omega^\varepsilon)} + \frac{\mu C_k}{2}\int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}ds.$$

Multiplying the last inequality by ε, we deduce

$$\varepsilon\|\frac{\partial^2 u^\varepsilon}{\partial t^2}(t)\|^2_{L^2(\Omega^\varepsilon)} + \varepsilon\mu C_k\|\nabla\frac{\partial u^\varepsilon}{\partial t}(t)\|^2_{L^2(\Omega^\varepsilon)} \le C_3 + \varepsilon\mu C_k\int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)}ds. \tag{4.18}$$

with

$$C_3 = C_2^2 + (\mu C_k + 2\mu+3\lambda)\|\nabla\frac{\partial \hat{u}^\varepsilon}{\partial t}(0)\|^2_{L^2(\Omega)} + \frac{(h_{\max})^2}{\mu C_k}\int_0^t \|\frac{\partial^2 \hat{f}^\varepsilon}{\partial t^2}(s)\|^2_{L^2(\Omega)}ds + \frac{(h_{\max})^2}{\mu C_k}\|\frac{\partial \hat{f}^\varepsilon}{\partial t}(0)\|^2_{L^2(\Omega)} + \frac{(h_{\max})^2}{\mu C_k}\|\frac{\partial \hat{f}^\varepsilon}{\partial t}(t)\|^2_{L^2(\Omega)}$$

By Gronwall's inequality ([9]), it follows that

$$\varepsilon\|\frac{\partial^2 u^\varepsilon}{\partial t^2}(t)\|^2_{L^2(\Omega^\varepsilon)} + \varepsilon\|\nabla\frac{\partial u^\varepsilon}{\partial t}(t)\|^2_{L^2(\Omega^\varepsilon)} \le C$$

This completes the proof of (4.4) and (4.5). □

Theorem 4.2. Under the same assumptions as in Theorem 4.1, there exists

$$u^\star(t) = (u_1^\star(t), u_2^\star(t)) \in L^2(0,T,V_z)\bigcap L^\infty(0,T,V_z),$$

such that

$$\begin{cases} \hat{u}_i^\varepsilon(t) \rightharpoonup u_i^\star(t), \ i=1,2, \\ \dot{\hat{u}}_i^\varepsilon(t) \rightharpoonup \dot{u}_i^\star(t), \ i=1,2, \end{cases} \tag{4.19}$$

$$\begin{cases} \varepsilon \dfrac{\partial \hat{u}_i^\varepsilon}{\partial x_j}(t) \rightharpoonup 0, \ i,j=1,2, \\ \varepsilon \dfrac{\partial \hat{\dot{u}}_i^\varepsilon}{\partial x_j}(t) \rightharpoonup 0, \ i,j=1,2, \end{cases} \tag{4.20}$$

$$\begin{cases} \varepsilon \dfrac{\partial \hat{u}_3^\varepsilon}{\partial z}(t) \rightharpoonup 0, \\ \varepsilon \dfrac{\partial \hat{\dot{u}}_3^\varepsilon}{\partial z}(t) \rightharpoonup 0, \end{cases} \tag{4.21}$$

$$\begin{cases} \varepsilon^2 \dfrac{\partial \hat{u}_3^\varepsilon}{\partial x_i}(t) \rightharpoonup 0, \ i=1,2 \\ \varepsilon^2 \dfrac{\partial \hat{\dot{u}}_3^\varepsilon}{\partial x_i}(t) \rightharpoonup 0, \ i=1,2 \end{cases} \tag{4.22}$$

$$\begin{cases} \varepsilon \hat{u}_3^\varepsilon(t) \rightharpoonup 0, \\ \varepsilon \hat{\dot{u}}_3^\varepsilon(t) \rightharpoonup 0, \end{cases} \tag{4.23}$$

Proof . By Theorem 4.1, there exists a constant C independent of ε such that

$$\left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z}(t) \right\|_{L(0,T,L^2(\Omega))} \le C, \ i=1,$$

using this estimate and Poincaré's inequality

$$\begin{cases} \|\hat{u}_i^\varepsilon(t)\|_{L(0,T,L^2(\Omega))} \le h_{\max} \left\| \dfrac{\partial \hat{u}_i^\varepsilon}{\partial z}(t) \right\|_{L(0,T,L^2(\Omega))} \\ \left\| \dfrac{\partial \hat{u}_i^\varepsilon}{\partial t}(t) \right\|_{L(0,T,L^2(\Omega))} \le h_{\max} \left\| \dfrac{\partial^2 \hat{u}_i^\varepsilon}{\partial z \partial t}(t) \right\|_{L(0,T,L^2(\Omega))} \end{cases}, \ i=1,2$$

then, $\hat{u}_i^\varepsilon(t)$ and $\hat{\dot{u}}_i^\varepsilon(t)$ is bounded in $L^2(0,T,V_z) \bigcap L^\infty(0,T,V_z)$, $i=1,2$, this implies the existence of $u_i^*(t), \dot{u}_i^*(t)$ in $L^2(0,T,V_z) \bigcap L^\infty(0,T,V_z)$ such that $\hat{u}_i^\varepsilon(t)$ converges to $u_i^*(t)$ and $\hat{\dot{u}}_i^\varepsilon(t)$ converges to $\dot{u}_i^*(t)$. While (4.20)-(4.23) follow from (4.4) and (4.5). □

5 Study of the limit problem

In this section, we give both the equations satisfied by $u^*(t)$ and $\dot{u}*(t)$ in Ω and the inequalities for the trace of the velocity $u^*(x',0,t)$ and the stress $\frac{\partial u}{\partial z}(x',0,t)$ on ω.

Theorem 5.1. With the same assumptions of Theorem 4.1, the solution u^* satisfies

$$\left. \begin{array}{c} \mu \sum\limits_{i=1}^{2} \int_\Omega \dfrac{\partial(u_i^*(t))}{\partial z} \dfrac{\partial(\hat{\varphi}_i - \dot{u}_i^*(t))}{\partial z} dxdz + \theta \sum\limits_{i=1}^{2} \int_\Omega \dfrac{\partial(\dot{u}_i^*(t))}{\partial z} \dfrac{\partial(\hat{\varphi}_i - \dot{u}_i^*(t))}{\partial z} dxdz \\ +\hat{\jmath}(\hat{\varphi}) - \hat{\jmath}(\dot{u}^*(t)) \ge \sum\limits_{i=1}^{2} \int_\Omega \hat{f}_i(t)(\hat{\varphi}_i - \dot{u}_i^*(t)) dxdz, \ \forall \hat{\varphi} \in \Pi(K), \\ u_i^*(0) = u_{0,i}^*, \end{array} \right\} \tag{5.1}$$

$$-\mu \frac{\partial^2 u_i^*(t)}{\partial z^2} - \theta \frac{\partial^2 \dot{u}_i^*(t)}{\partial z^2} = \hat{f}_i(t) \in L^2(\Omega), \tag{5.2}$$

where,

$$\Pi(K) = \left\{ \hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2) \in H^1(\Omega)^2, \hat{\varphi} = 0 \text{ on } \Gamma_1 \cup \Gamma_L \right\}.$$

Proof . Using the variational inequality (4.3) and applying Theorem 4.2, we get

$$\lim_{\varepsilon \to 0} \sum_{i=1}^{2} \varepsilon^2 \left(\frac{\partial^2 \hat{u}_i^\varepsilon}{\partial t^2}(t), \hat{\varphi}_i - \hat{u}_i^\varepsilon(t)\right) = 0,$$

$$\lim_{\varepsilon\to 0} \tilde{a}(\hat{u}^{\varepsilon}(t), \hat{\varphi} - \hat{u}^{\varepsilon}(t)) = \sum_{i=1}^{2} \int_{\Omega} \mu \frac{\partial u_i^*}{\partial z} \frac{\partial \left(\hat{\varphi}_i - u_i^*(t)\right)}{\partial z} dxdz,$$

$$\lim_{\varepsilon\to 0} \tilde{B}(\hat{u}^{\varepsilon}(t), \hat{\varphi} - \hat{u}^{\varepsilon}(t)) = \sum_{i=1}^{2} \int_{\Omega} \theta \frac{\partial \dot{u}_i^*}{\partial z} \frac{\partial \left(\hat{\varphi}_i - \dot{u}_i^*(t)\right)}{\partial z} dxdz,$$

$$\lim_{\varepsilon\to 0} (\hat{f}, \hat{\varphi} - \hat{u}^{\varepsilon}(t)) = \sum_{i=1}^{2} \int_{\Omega} \hat{f}_i \left(\hat{\varphi}_i - u_i^*(t)\right).$$

By the fact j_0 is convex and lower semi-continuous $\left(\liminf j_0\left(\hat{u}^{\varepsilon}(t)\right) \geq j_0\left(\dot{u}^*(t)\right)\right)$, we obtain (5.1).
We choose in (5.1), $\varphi_i = \dot{u}_i^* \pm \phi_i \quad i = 1,2$, with $\phi \in H_0^1(\Omega)$, to get

$$\mu \sum_{i=1}^{2} \int_{\Omega} \frac{\partial u_i^*}{\partial z}(t) \frac{\partial \phi_i}{\partial z} dxdz + \theta \sum_{i=1}^{2} \int_{\Omega} \frac{\partial \dot{u}_i^*}{\partial z}(t) \frac{\partial \phi_i}{\partial z} dxdz = \sum_{i=1}^{2} \int_{\Omega} \hat{f}_i(t) \phi_i dxdz.$$

If $\phi_1 = 0$ and $\phi_2 \in H_0^1(\Omega)$ or $\phi_2 = 0$ and $\phi_1 \in H_0^1(\Omega)$, then

$$\mu \int_{\Omega} \frac{\partial u_i^*}{\partial z}(t) \frac{\partial \phi_i}{\partial z} dxdz + \theta \int_{\Omega} \frac{\partial \dot{u}_i^*}{\partial z}(t) \frac{\partial \phi_i}{\partial z} dxdz = \int_{\Omega} \hat{f}_i(t) \phi_i dxdz.$$

By Green's formula, we find

$$-\frac{\partial}{\partial z}\left(\mu \frac{\partial u_i^*(t)}{\partial z} + \theta \frac{\partial \dot{u}_i^*(t)}{\partial z}\right) = \hat{f}_i(t), \quad i = 1,2 \quad \in H^{-1}(\Omega), \tag{5.3}$$

and since $\hat{f}_i \in L^2(\Omega)$, $\forall t \in (0,T)$, we deduce that $-\frac{\partial}{\partial z}\left(\mu \frac{\partial u_i^*(t)}{\partial z} + \theta \frac{\partial \dot{u}_i^*(t)}{\partial z}\right) \in L^2(\Omega)$. Which implies (5.2). □

Theorem 5.2. Under the same assumptions of Theorem 4.1, we have

$$\int_{\omega} k(|\psi + s^*| - |s^*|)dx - \int_{\omega} \mu \hat{\tau}^* \psi dx - \int_{\omega} \theta \frac{\partial \hat{\tau}^*}{\partial t} \psi \geq 0, \qquad \forall \psi \in L^2(\omega)^2, \tag{5.4}$$

$$\begin{cases} \left|\mu\tau^* + \theta \frac{\partial \tau^*}{\partial t}\right| < \hat{k} \Rightarrow s^* = 0 \\ \left|\mu\tau^* + \theta \frac{\partial \tau^*}{\partial t}\right| = \hat{k} \Rightarrow \exists \beta > 0 \text{ such that } s^* = \beta\left(\mu\tau^* + \theta \frac{\partial \tau^*}{\partial t}\right), \end{cases} \tag{5.5}$$

where

$$\hat{\tau}^* = \frac{\partial \hat{u}^*}{\partial z}(x,0,t) \text{ and } s^*(x) = u^*(x,0,t).$$

Also the limit function u^* and s^* satisfy the following weak form of the Reynolds equation

$$\int_{\omega} \left(\int_0^h \left[\mu u^*(x,z,t) + \theta \dot{u}^*(x,z,t)\right] dz + \tilde{F} - \frac{h}{2}\mu s^* - \frac{h}{2}\theta \frac{\partial s^*}{\partial t} \right) \nabla \psi(x)\, dx = 0, \tag{5.6}$$

for all $\psi \in H^1(\omega)$, where

$$\tilde{F}(x) = \int_0^h F(x,z,t)\, dz - \frac{h}{2} F(x,h,t),$$

$$F(x,z,t) = \int_0^z \int_0^{\xi} \hat{f}_i(x,\alpha)\, d\xi d\alpha.$$

Proof . Equation (4.3) can be written as

$$\begin{aligned}
&\textstyle\sum_{i=1}^{2}\varepsilon^2(\frac{\partial^2 \hat{u}_i^\varepsilon}{\partial t^2}(t),\hat{\varphi}_i-\hat{u}_i^\varepsilon(t)\\
&+\mu\varepsilon^2\sum_{i,j=1}^{2}\int_\Omega\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j}(t)+\frac{\partial \hat{u}_j^\varepsilon}{\partial x_i}(t)\right)\frac{\partial \psi}{\partial x_j}dxdz\\
&+\mu\textstyle\sum_{i=1}^{2}\int_\Omega\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z}(t)+\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}(t)\right)\left[\frac{\partial \psi_i}{\partial z}+\varepsilon^2\frac{\partial \psi_3}{\partial x_i}\right]dxdz\\
&+2\mu\varepsilon^2\int_\Omega\frac{\partial \hat{u}_3^\varepsilon}{\partial z}(t)\frac{\partial \psi_3}{\partial z}dxdz+\lambda\varepsilon^2\int_\Omega div\left(\hat{u}^\varepsilon(t)\right)div\left(\psi\right)dxdz\\
&+\theta\varepsilon^2\sum_{i,j=1}^{2}\int_\Omega\left(\frac{\partial \hat{\dot{u}}_i^\varepsilon}{\partial x_j}(t)+\frac{\partial \hat{\dot{u}}_j^\varepsilon}{\partial x_i}(t)\right)\frac{\partial \psi}{\partial x_j}dxdz+\\
&\theta\textstyle\sum_{i=1}^{2}\int_\Omega\left(\frac{\partial \hat{\dot{u}}_i^\varepsilon}{\partial z}(t)+\varepsilon^2\frac{\partial \hat{\dot{u}}_3^\varepsilon}{\partial x_i}(t)\right)\left[\frac{\partial \psi_i}{\partial z}+\varepsilon^2\frac{\partial \psi_3}{\partial x_i}\right]dxdz\\
&+2\theta\varepsilon^2\int_\Omega\frac{\partial \hat{\dot{u}}_3^\varepsilon}{\partial z}(t)\frac{\partial \psi_3}{\partial z}dxdz+\zeta\varepsilon^2\int_\Omega div\left(\hat{\dot{u}}^\varepsilon(t)\right)div\left(\psi\right)dxdz\\
&+\int_\omega\hat{k}\left|\psi+\dot{u}^\varepsilon(t)\right|dx-\int_\omega\hat{k}\left|\dot{u}^\varepsilon(t)\right|dx\geq\sum_{i=1}^{2}\left(\hat{f}_i,\psi_i\right)+\varepsilon\left(\hat{f}_3(t),\psi_3\right).
\end{aligned}$$

Passing to the limit and thanks to the Green formula, we deduce

$$\begin{aligned}
&-\sum_{i=1}^{2}\int_\Omega\mu\frac{\partial^2 u_i^*}{\partial z^2}(t)\psi_i dxdz+\int_\Gamma\mu\frac{\partial u_i^*}{\partial z}(t)n\psi_i d\sigma-\sum_{i=1}^{2}\int_\Omega\theta\frac{\partial^2 \dot{u}_i^*}{\partial z^2}(t)\psi_i dxdz\\
&+\int_\Gamma\theta\frac{\partial \dot{u}_i^*}{\partial z}(t)n\psi_i d\sigma+\int_\omega\hat{k}\left|\psi+s^*\right|dx-\int_\omega\hat{k}\left|s^*\right|dx\geq\sum_{i=1}^{2}\int_\Omega\hat{f}_i(t)\psi_i dxdz.
\end{aligned}$$

But

$$\begin{aligned}
\mu\int_\Gamma\frac{\partial u_i^*}{\partial z}(t)n\psi_i d\sigma&=\mu\int_\omega\frac{\partial u_i^*}{\partial z}(x,0,t)\psi_i dx,\\
\theta\int_\Gamma\frac{\partial \dot{u}_i^*}{\partial z}(t)n\psi_i d\sigma&=\theta\int_\omega\frac{\partial \dot{u}_i^*}{\partial z}(x,0,t)\psi_i dx,
\end{aligned}$$

then

$$\begin{aligned}
&-\sum_{i=1}^{2}\int_\Omega\mu\frac{\partial^2 u_i^*}{\partial z^2}(t)\psi_i dxdz-\int_\omega\mu\tau_i^*\psi_i dx-\sum_{i=1}^{2}\int_\Omega\theta\frac{\partial^2 \dot{u}_i^*}{\partial z^2}(t)\psi_i dxdz\\
&-\int_\omega\theta\frac{\partial \tau_i^*}{\partial t}\psi_i dx+\int_\omega\hat{k}\left|\psi+s^*\right|dx-\int_\omega\hat{k}\left|s^*\right|dx\geq\sum_{i=1}^{2}\int_\Omega\hat{f}_i(t)\psi_i dxdz,
\end{aligned}$$

for all $\psi\in H_{\Gamma_L\cup\Gamma_1}(\Omega)^2$, where

$$H_{\Gamma_L\cup\Gamma_1}(\Omega)=\left\{\psi\in H^1(\Omega),\psi=0\quad\text{on}\quad\Gamma_L\cup\Gamma_1\right\}.$$

On the other hand thanks to (5.2), we find

$$\int_\omega\hat{k}\left|\psi+s^*\right|dx-\int_\omega\hat{k}\left|s^*\right|dx-\int_\omega\mu\tau^*\psi dx-\int_\omega\theta\frac{\partial \tau^*}{\partial t}\psi dx\geq 0. \tag{5.7}$$

The inequality (5.7) is also valid for any $\psi\in D(\omega)^2$ and by density of $D(\omega)$ in $L^2(\omega)$, we obtain

$$\int_\omega\hat{k}\left|\psi+s^*\right|dx-\int_\omega\hat{k}\left|s^*\right|dx-\int_\omega\mu\tau^*\psi dx-\int_\omega\theta\frac{\partial \tau^*}{\partial t}\psi dx\geq 0,$$

for all $\psi\in L^2(\omega)^2$. Which gives (5.4).

To prove (5.5), it suffices to follow the same techniques of [3] for the Stokes fluid. In effect, we choose $\psi=\pm s^*$ in (5.4), we obtain

$$\int_\omega\hat{k}\left|s^*\right|dx-\int_\omega\left(\mu\tau^*+\theta\frac{\partial \tau^*}{\partial t}\right)\left|s^*\right|dx=0. \tag{5.8}$$

Taking $\psi = \phi - s^*$, with $\phi \in L^2(\omega)$ in (5.4), we get

$$\int_\omega \hat{k}\,|\phi|\,dx - \int_\omega \left(\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right)|\phi|\,dx \geq \int_\omega \hat{k}\,|s^*|\,dx - \int_\omega \left(\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right)|s^*|\,dx.$$

From (5.8), we deduce

$$\int_\omega \hat{k}\,|\phi|\,dx - \int_\omega \left(\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right)\phi dx \geq 0, \tag{5.9}$$

then by choosing $\phi = (\phi_1, \phi_2)$ such that $\phi_i \geq 0$ for $i = 1, 2$, in (5.9), we get

$$\int_\omega \left|\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right| \cos(\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}, \phi)dx \leq \hat{k}.$$

In (5.9), we take $-\phi$ such that $\phi_i \geq 0$ for $i = 1, 2$, we obtain

$$\int_\omega \left|\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right| \cos(\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}, \phi)dx \leq -\hat{k}.$$

Thus, we have

$$\int_\omega \left|\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right| dx \leq \hat{k}.$$

Then

$$\hat{k}\,|s^*| \geq \left|\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right| |s^*| \geq \left(\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right) s^*.$$

Moreover, we have

$$\hat{k}\,|s^*| = \left(\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right) s^*. \tag{5.10}$$

If $\hat{k} = \left|\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right|$, then from (5.10) we have

$$\left|\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right| |s^*| = \left(\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right) s^*.$$

Then, there exists β such that $s^* = \beta\left(\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right)$.
If $\hat{k} > \left|\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right|$, then from (5.10) we have

$$\hat{k}\,|s^*| - \left(\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right) s^* = 0 > \left[\hat{k} - \left(\mu\tau^* + \theta\frac{\partial\tau^*}{\partial t}\right)\right] |s^*|,$$

whence $s^* = 0$ on ω.
To prove (5.6), we integrate twice (5.2) for 0 to z, we obtain

$$\begin{aligned} -\mu u^*(x,z,t) - \theta\dot{u}^*(x,z,t) + \mu u^*(x,0,t) + \theta\dot{u}^*(x,0,t) \\ +\mu z\tau^* + \theta z\frac{\partial\tau^*}{\partial t} = \int_0^z \int_0^\xi f(x,\alpha,t)d\alpha d\xi. \end{aligned} \tag{5.11}$$

In particular for $z = h$, we obtain

$$\mu s^* + \theta\frac{\partial s^*}{\partial t} + \mu h\tau^* + \theta h\frac{\partial\tau^*}{\partial t} = \int_0^h \int_0^\xi f(x,\alpha,t)d\alpha d\xi. \tag{5.12}$$

Integrating (5.11) between 0 and h, we get

$$-\int_0^h \left(\mu u^*\left(x,z,t\right)+\theta\dot{u}^*\left(x,z,t\right)\right)dz+\mu h s^*+\theta h\frac{\partial s^*}{\partial t} \\ +\mu\frac{h^2}{2}\tau^*+\theta\frac{h^2}{2}\frac{\partial\tau^*}{\partial t}=\int_0^h F(x,z,t)dz. \tag{5.13}$$

By (5.12)-(5.13), we get (5.1). $\square$

Theorem 5.3. The solution $u^*(t)\in L^2(0,T,V_z)\bigcap L^\infty(0,T,V_z)$ of limit problem (5.1) is unique.

Proof . Let $u^1(t)$ and $u^2(t)$ be two solutions of (5.1). Taking $\varphi=\dot{u}^2(t)$ and $\varphi=\dot{u}^1(t)$ respectively, as test functions in (5.1) we get

$$\mu\sum_{i=1}^2\int_\Omega\frac{\partial u_i^1(t)}{\partial z}\frac{\partial(\dot{u}_i^2(t)-\dot{u}_i^1(t))}{\partial z}dxdz+\theta\sum_{i=1}^2\int_\Omega\frac{\partial\dot{u}_i^1(t)}{\partial z}\frac{\partial(\dot{u}_i^2(t)-\dot{u}_i^1(t))}{\partial z}dxdz \\ +\hat{j}(\dot{u}^2(t))-\hat{j}(\dot{u}^1(t))\geq\sum_{i=1}^2\int_\Omega\hat{f}_i(t)(\dot{u}_i^2(t)-\dot{u}_i^1(t))dxdz, \tag{5.14}$$

$$\mu\sum_{i=1}^2\int_\Omega\frac{\partial u_i^2(t)}{\partial z}\frac{\partial(\dot{u}_i^1(t)-\dot{u}_i^2(t))}{\partial z}dxdz+\theta\sum_{i=1}^2\int_\Omega\frac{\partial\dot{u}_i^2(t)}{\partial z}\frac{\partial(\dot{u}_i^1(t)-\dot{u}_i^2(t))}{\partial z}dxdz \\ +\hat{j}(\dot{u}^1(t))-\hat{j}(\dot{u}^2(t))\geq\sum_{i=1}^2\int_\Omega\hat{f}_i(t)(\dot{u}_i^1(t)-\dot{u}_i^2(t))dxdz. \tag{5.15}$$

By subtracting (5.14) from (5.15), we obtain

$$\theta\sum_{i=1}^2\int_\Omega\frac{\partial(\dot{u}_i^1(t)-\dot{u}_i^2(t))}{\partial z}\frac{\partial(\dot{u}_i^1(t)-\dot{u}_i^2(t))}{\partial z}dxdz\leq\mu\sum_{i=1}^2\int_\Omega\frac{\partial(u_i^1(t)-u_i^2(t))}{\partial z}\frac{\partial(\dot{u}_i^2(t)-\dot{u}_i^1(t))}{\partial z}dxdz.$$

We use now assumptions (3.4) and (3.5) to find

$$\left\|\frac{\partial}{\partial z}(\dot{u}^1(t)-\dot{u}^2(t))\right\|_{L^2(V_z)}\leq C\left\|\frac{\partial}{\partial z}(u^1(t)-u^2(t))\right\|_{L^2(V_z)}. \tag{5.16}$$

Moreover, since $u_i(0)=u_0$, it follows that

$$\frac{\partial u_i}{\partial z}(t)=\int_0^t\frac{\partial\dot{u}_i}{\partial z}(s)ds+\frac{\partial u_i}{\partial z}(0), \tag{5.17}$$

and, therefore

$$\left\|\frac{\partial}{\partial z}(u^1(t)-u^2(t))\right\|_{L^2(V_z)}\leq\int_0^t\left\|\frac{\partial}{\partial z}(\dot{u}^1(s)-\dot{u}^2(s))\right\|_{L^2(V_z)}. \tag{5.18}$$

By (5.16) and (5.18), we obtain

$$\left\|\frac{\partial}{\partial z}(\dot{u}^1(t)-\dot{u}^2(t))\right\|_{L^2(V_z)}\leq C\int_0^t\left\|\frac{\partial}{\partial z}(\dot{u}^1(s)-\dot{u}^2(s))\right\|_{L^2(V_z)}ds.$$

By Gronwall Lemma and Poincaré's inequality, we deduce

$$\dot{u}^1(s)=\dot{u}^2(s),$$

for all $s\in[0,T]$. Using then (5.17) to obtain that $u^1(t)=u^2(t)$, which completes the proof. $\square$

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