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On stability, boundedness and square integrability of solutions for fourth-order differential equations of neutral type with variable delay

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Abstract

This paper establishes sufficient conditions to ensure the stability and boundedness of zero solution and square integrability of solutions and their derivatives to neutral type nonlinear differential equations of fourth order by constructing Lyapunov functionals.

Keywords: stability, boundedness, square integrability, Lyapunov functional, neutral fourth-order differential equations, variable delay
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1 Introduction

Qalitative behaviour, Stability boundedness and squar integrability of solution of a real scalar fourth-order differential equations are challenging subjects to deal with, the most known and powerfull mathematical objects used in studying these concepts are Lyapunov functionals historically presented by the Russian mathematician Aleksandr M. Lyapunov.

It is very known that neutral delay differential equations play an important role in explaining large class of phenomenae in several sciences such as electrodynamics, control theory, biology, economy, echology,... they can be primordial when the application of ordinary differential equations fails.

The subject of qualitative behaviour of this kind of equations has also central position in applications, for the best of our knowledge The results on qualitative behaviour of solution for nonlinear delay differential equations of fourth order of neutral type are few when we compare them by nonlinear delay differential equations of fourth order which received considerable attention and has been subject of many articles in the literature we have for instance [3, 4, 6, 7].

Motivated by the facts above in this article, we investigate some asymptotic properties of solutions of the fourth order nonlinear neutral delay differential equation,

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$$\Big(x'''(t)+\rho x'''\big(t-r(t)\big)\Big)'+a(t)\Big(\varphi\big(x(t)\big)x''(t)\Big)'+b(t)\Big(g\big(x(t)\big)x'(t)\Big)'+c(t)f\big(x(t)\big)x'(t)$$
$$+d(t)h\Big(x\big(t-r(t)\big)\Big)=\psi(t,x(t),x'(t),x''(t),x'''(t)), \tag{1.1}$$

where r is a bounded delay, $0 \le r(t) \le r_1, -r_3 \le r'(t) \le r_2, 0 < r_2 < 1, r_3 > 0$. The functions $a, b, c, d,$ are continuously differentiable functions. The functions f, g, h and φ, are continuously differentiable.
Equation (1.1) is equivalent to the system

$$\begin{cases} x' = y, \\ y' = z, \\ z' = w, \\ W' = -a(t)\varphi(x)w - \Big(b(t)g(x) + a(t)\theta_1\Big)z - \Big(b(t)\theta_2 + c(t)f(x)\Big)y \\ \qquad -d(t)h(x) + d(t)\displaystyle\int_{t-r(t)}^{t} y(s)h'(x(s))\,ds, \end{cases} \tag{1.2}$$

where

$$W(t) = x'''(t) + \rho x'''(t-r(t)) = w(t) + \rho w(t-r(t)), \theta_1(t) = \varphi'(x(t))x'(t),$$

and $\theta_2(t) = g'(x(t))x'(t)$.

The continuity of the functions $a, b, c, d, f, g, g', h, p, \varphi', q,$ and g' guarantees the existence of the solutions of (1.1).

2 Assumptions and main results

Firstly, let us set some assumptions on the functions that appeared in (1.1), and suppose that there are positive constants $a_0, b_0, c_0, d_0, f_0, \varphi_0, g_0, a_1, b_1, c_1, d_1, f_1, \varphi_1, g_1, h_0, m, M, \delta, \delta_0, \eta_1$ and η_2 such that

i) $0 < a_0 \le a(t) \le a_1;\ 0 < b_0 \le b(t) \le b_1;\ \ 0 < c_0 \le c(t) \le c_1;\ 0 < d_0 \le d(t) \le d_1$ and $d'(t) \le 0$ for $t \ge t_0 + r_1,\ \ t_0 \ge 0$.

ii) $0 < f_0 \le f(x) \le f_1;\ 0 < \varphi_0 \le \varphi(x) \le \varphi_1;\ \ 0 < g_0 \le g(x) \le g_1$ for $x \in \mathbb{R}$ and $0 < m < \min\{f_0, \varphi_0, g_0, 1\},\ \ M > \max\{f_1, \varphi_1, g_1, 1\}$.

iii) $\dfrac{h(x)}{x} \ge \delta > 0$ (for $x \ne 0$); $h(0) = 0$.

iv) $\dfrac{h_0}{m} - \dfrac{a_0 m \delta_0}{d_1} \le h'(x) \le \dfrac{h_0}{2M}$ for $x \in \mathbb{R}$,

v) $b_0 g_0 > \dfrac{c_1 M + \delta_0}{a_0 m} + \dfrac{a_1 d_1 h_0 M}{c_0 m} = \delta_1$

vi) $\displaystyle\int_{t_1}^{+\infty} (|a'(t)| + |b'(t)| + |c'(t)| + |r'(t)| - d'(t))\,dt < \eta_1 < +\infty,$ where $t_1 = t_0 + r_1$

vii) $\displaystyle\int_{-\infty}^{+\infty} (|\varphi'(s)| + |g'(s)| + |f'(s)|)\,ds < \eta_2 < \infty.$

Lemma 2.1. [2] Let $h(0) = 0,\ xh(x) > 0\ (x \ne 0)$ and $\delta(t) - h'(x) \ge 0\ (\delta(t) > 0)$, then

$$2\delta(t)H(x) \ge h^2(x) \qquad \text{where} \quad H(x) = \int_0^x h(s)ds.$$

Our first main result of this paper is the following theorem where $\psi(t, x, y, z, w) = 0$.

Theorem 2.1. Suppose that assumptions i) $\sim$ vii) hold. Then every solution $x(t)$ of (1.1) and their derivatives $x'(t), x''(t)$ and $x'''(t)$ are uniformly asymptotically stable, provided that

$$r_1 < \frac{1}{\lambda}\min\Big\{1,\ 2\varepsilon c_0 m - \alpha\rho c_1 M - \alpha\rho d_1\lambda_0, 2\varepsilon - \rho\left(\alpha a_1 M + 2 + \mu_2'\right),$$
$$2b_0 g_0 - 2\delta_1 - 2\varepsilon M\left(a_1 + c_1\right) - \rho\left(\alpha b_1 g_1 + \beta + \mu_1'\right)\Big\}, \tag{2.1}$$

where

$$\rho < \min\Big\{1,\ \frac{4\varepsilon}{\alpha h_0},\ \frac{2\varepsilon c_0 m}{\alpha c_1 M + \alpha d_1\lambda_0},\ 2\frac{b_0 g_0 - \delta_1 - \varepsilon M\left(a_1 + c_1\right)}{\alpha b_1 g_1 + \beta + \mu_1'},\ \frac{2\varepsilon}{\alpha a_1 M + 2 + \mu_2'}\Big\},$$

$$\mu_1' = \frac{\alpha d_1\lambda_0 + \alpha d_1(1 + r_3)}{1 - r_2}, \qquad \mu_2' = \frac{\alpha a_1 M + \alpha b_1 g_1 + \alpha c_1 M + 2\alpha d_1(1 + r_3) + \beta + 3}{1 - r_2},$$

$$\lambda = \frac{d_1\lambda_0(\alpha + \beta + 1)}{1 - r_2},\ \lambda_0 = \max\Big\{\frac{h_0}{2M},\ \left|\frac{h_0}{m} - \frac{a_0 m\delta_0}{d_1}\right|\Big\}, \tag{2.2}$$

$$\text{and} \qquad \varepsilon < \min\Big\{\frac{1}{a_0 m},\ \frac{d_1 h_0}{c_0 m},\ \frac{b_0 g_0 - \delta_1}{M\left(a_1 + c_1\right)}\Big\}. \tag{2.3}$$

Our second main result is the following theorem where $\psi(t, x, y, z, w) \neq 0$.

Theorem 2.2. Let all the conditions of Theorem 2.1 and the assumption

$$|\psi(t, x, y, z, w)| \leq |e(t)| \ \text{ and } \ \int_{t_1}^{+\infty} |e\left(s\right)| ds < \eta_3 < +\infty \tag{2.4}$$

where η_3 is positive constant. Then, there exists a finite positive constant K such that every solution $x(.)$ of (1.1) and their derivatives $x'(.),\ \ x''(.)$ and $x'''(.)$ satisfy

1. $|x(t)| \leq \sqrt{K},\ |\ x'(t)| \leq \sqrt{K},\ |x''(t)| \leq \sqrt{K},\ |x'''\left(t\right) + \rho x'''\left(t - r(t)\right)| \leq \sqrt{K},$ for all $t \geq t_0 + r_1$,

2. $\int_{t_1}^{\infty} \left(x'^2(s) + x''^2(s) + x'''^2(s)\right) ds < \infty.$

Proof . Proof of Theorem 2.1. The proof depend on some fundamental properties of a continuously differentiable Lyapunov functional $U = U(t, x_t, y_t, z_t, w_t)$ defined by

$$U = e^{-\frac{1}{\eta}\int_{t_1}^{t}\gamma\left(s\right) ds} V, \tag{2.5}$$

where

$$\gamma\left(t\right) = |a'\left(t\right)| + |b'\left(t\right)| + |c'\left(t\right)| + |r'\left(t\right)| - d'\left(t\right) + |\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)|,$$

and

$$2V = 2V_0(t, x_t, y_t, z_t, W) + \lambda\int_{-r(t)}^{0}\int_{t+s}^{t} y^2\left(\theta\right) d\theta ds + \mu_1\int_{t-r(t)}^{t} z^2\left(s\right) ds + \mu_2\int_{t-r(t)}^{t} w^2\left(s\right) ds,$$

such that

$$\begin{aligned}
\theta_3\left(t\right) &= f'\left(x\left(t\right)\right) x'\left(t\right),\\
2V_0 &= 2\beta d\left(t\right) H\left(x\right) + c\left(t\right) f\left(x\right) y^2 + \alpha b\left(t\right) g(x) z^2 + a\left(t\right)\varphi(x) z^2 + 2\beta a\left(t\right)\varphi(x) yz\\
&\quad + \left[\beta b\left(t\right) g(x) - \alpha h_0 d\left(t\right)\right] y^2 - \beta z^2 + \alpha W^2 + 2\alpha c\left(t\right) f\left(x\right) yz + 2\alpha d\left(t\right) h\left(x\right) z\\
&\quad + 2\alpha\rho d\left(t\right) h\left(x\right) z\left(t - r(t)\right) + \alpha\rho d\left(t\right)\left(z\left(t - r(t)\right)\right)^2 + 2d\left(t\right) h\left(x\right) y + 2\beta yW + 2zW,
\end{aligned}$$

with $\alpha = \dfrac{1}{a_0 m} + \varepsilon$, $\beta = \dfrac{d_1 h_0}{c_0 m} + \varepsilon$, ε, and η are positive constants to be determined later in the proof. We rewrite $2V_0$ as

$$2V_0 = a(t)\varphi(x)\left[\frac{W}{a(t)\varphi(x)} + z + \beta y\right]^2 + c(t)f(x)\left[\frac{d(t)h(x)}{c(t)f(x)} + y + \alpha z\right]^2 + \frac{d^2(t)h^2(x)}{c(t)f(x)} + V_1 + V_2 + V_3 + V_4,$$

where

$$\begin{aligned}
V_1 &= 2d(t)\int_0^x h(s)\left[\frac{d_1 h_0}{c_0 m} - 2\frac{d(t)}{c(t)f(x)}h'(s)\right]ds,\\
V_2 &= \left[\alpha b(t)g(x) - \beta - \alpha^2 c(t)f(x)\right]z^2,\\
V_3 &= \left[\beta b(t)g(x) - \alpha h_0 d(t) - \beta^2 a(t)\varphi(x)\right]y^2 + \left[\alpha - \frac{1}{a(t)\varphi(x)}\right]W^2, \text{ and}\\
V_4 &= 2\varepsilon d(t)H(x) + 2\alpha\rho d(t)h(x)z(t-r(t)) + \alpha\rho d(t)\left(z(t-r(t))\right)^2.
\end{aligned}$$

Now, we will prove that V is positive definite. Take

$$\varepsilon < \min\left\{\frac{1}{a_0 m}\ ,\ \frac{d_1 h_0}{c_0 m}\ ,\ \frac{b_0 g_0 - \delta_1}{M(a_1 + c_1)}\right\}, \tag{2.6}$$

then

$$\frac{1}{a_0 m} < \alpha < \frac{2}{a_0 m}, \qquad \frac{d_1 h_0}{c_0 m} < \beta < 2\frac{d_1 h_0}{c_0 m}. \tag{2.7}$$

Using conditions (i) $\sim$ (v) and inequalities (2.6)and (2.7) we get

$$\begin{aligned}
V_1 &\geq 4d(t)\frac{d_1}{c_0 m}\int_0^x h(s)\left[\frac{h_0}{2} - h'(s)\right]ds \geq 0,\\
V_2 &= \Big(\alpha\Big(b(t)g(x) - \beta a(t) - \alpha c(t)f(x)\Big) + \beta\big(\alpha a(t) - 1\big)\Big)z^2\\
&\geq \alpha\left(b_0 g_0 - \frac{d_1 h_0 a_1}{c_0 m} - \frac{c_1 M}{a_0 m} - \varepsilon(a_1 + c_1 M)\right)z^2 + \beta\big(\frac{1}{m} - 1\big)z^2\\
&\geq \alpha\big(b_0 g_0 - \delta_1 - \varepsilon M(a_1 + c_1)\big)z^2 \geq 0,\\
V_3 &\geq \beta\left(b_0 g_0 - \frac{\alpha}{\beta}h_0 d_1 - \beta a_1 M\right)y^2 + \left(\alpha - \frac{1}{a_0 m}\right)W^2\\
&\geq \beta\left(b_0 g_0 - \frac{c_0}{a_0} - a_1\frac{d_1 h_0 M}{c_0 m} - \varepsilon(c_0 m + a_1 M)\right)y^2 + \varepsilon W^2\\
&\geq \beta\big(b_0 g_0 - \delta_1 - \varepsilon M(c_1 + a_1)\big)y^2 + \varepsilon W^2 \geq 0,
\end{aligned}$$

and by choosing $\rho < \frac{4\varepsilon}{\alpha h_0}$

$$\begin{aligned}
V_4 &= 2\varepsilon d(t)\int_0^x h(\xi)d\xi + \alpha\rho d(t)\left[\left(z(t-r(t)) + h(x)\right)^2 - h^2(x)\right]\\
&\geq 2\varepsilon d(t)\int_0^x h(\xi)d\xi - \frac{\alpha\rho}{2}d(t)\int_0^x h'(\xi)h(\xi)d\xi\\
&\geq 2d(t)\int_0^x\left(\varepsilon - \frac{\alpha\rho h_0}{4}\right)h(\xi)d\xi\\
&\geq \delta d_0\left(\varepsilon - \frac{\alpha\rho h_0}{4}\right)x^2.
\end{aligned}$$

Using the fact that the integral $\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds$ is positive, we deduce that there exists positive number D_0 such that

$$2V \ge D_0\left(y^2+z^2+W^2+H(x)\right). \tag{2.8}$$

By Lemma 2.1 and conditions (iii) and (iv) it follows that there is a positive constant D_1 such that

$$2V \ge D_1\left(x^2+y^2+z^2+W^2\right); \tag{2.9}$$

thus V is positive definite. From (i)—(iii), it is not difficult to see that there is a positive constant Δ_1 such that

$$V \le \Delta_1\left(x^2+y^2+z^2+W^2\right).$$

By (vii), we have

$$\begin{aligned}\int_{t_1}^{t}\left(|\theta_1(s)|+|\theta_2(s)|+|\theta_3(s)|\right)ds &= \int_{\alpha_1(t)}^{\alpha_2(t)}\left(|\varphi'(u)|+|g'(u)|+|f'(u)|\right)du\\ &\le \int_{-\infty}^{+\infty}\left(|\varphi'(u)|+|g'(u)|+|f'(u)|\right)du < \eta_2 < \infty,\end{aligned} \tag{2.10}$$

where $\alpha_1(t)=\min\{x(t_1),x(t)\}$, and $\alpha_2(t)=\max\{x(t_1),x(t)\}$. From inequalities (2.5), (2.9), and (2.10), it follows that

$$U \ge D_2(x^2+y^2+z^2+W^2), \tag{2.11}$$

where $D_2=\dfrac{D_1}{2}e^{-\frac{\eta_1+\eta_2}{\eta}}$. Also, it is easy to see that there is a positive constant Δ_2 such that

$$U \le \Delta_2(x^2+y^2+z^2+W^2), \tag{2.12}$$

for all x, y, z and w, and all $t \ge t_0+r_1$.

Now we prove that $\dot{U}$ is a negative definite function. The derivative of the function V, along any solution $(x(t),y(t),z(t),w(t))$ of system (1.2), with respect to t is after simplifying

$$2\dot{V}_{(1.2)} = V_5+V_6+V_7+V_8+V_9+V_{10}+V_{11},$$

where

$$\begin{aligned}
V_5 =& -2\left(\frac{d_1h_0}{c_0m}c(t)f(x)-d(t)h'(x)\right)y^2-2\alpha d(t)\left(h_0-h'(x)\right)yz,\\
V_6 =& -2\big(b(t)g(x)-\alpha c(t)f(x)-\beta a(t)\varphi(x)\big)z^2,\\
V_7 =& -2\big(\alpha a(t)\varphi(x)-1\big)w^2,\\
V_8 =& -2\varepsilon c(t)f(x)y^2-2\alpha\rho a(t)\varphi(x)w_tw-2\alpha\rho b(t)g(x)zw_t-2\alpha\rho c(t)f(x)yw_t\\
&+2\alpha\rho d(t)h'(x)yz_t+\mu_1z^2+\mu_2w^2-\mu_1(1-r'(t))z_t^2-\mu_2(1-r'(t))w_t^2\\
&-2\alpha\rho d(t)r'(t)h(x)w_t+2\alpha\rho d(t)(1-r'(t))z_tw_t+2\rho ww_t+2\beta\rho zw_t\\
&-\alpha\rho d_1|r'(t)|h^2(x),\\
V_9 =&\lambda r(t)y^2(t)-\lambda(1-r'(t))\int_{t-r(t)}^{t}y^2(u)du+2\alpha Wd(t)\int_{t-r(t)}^{t}y(s)h'(x(s))ds\\
&+2\beta yd(t)\int_{t-r(t)}^{t}y(s)h'(x(s))ds+2zd(t)\int_{t-r(t)}^{t}y(s)h'(x(s))ds,
\end{aligned}$$

$$\begin{aligned}
V_{10} =& -a(t)\theta_1\Big(z^2+2\alpha zW\Big)-b(t)\theta_2\Big(\beta y^2+2\alpha yW+2yz-\alpha z^2\Big)\\
&+c(t)\theta_3\Big(y^2+2\alpha yz\Big),
\end{aligned}$$

and

$$\begin{aligned}V_{11} &= d'(t)\left[2\beta H(x) - \alpha h_0 y^2 + 2h(x)y + 2\alpha h(x)z\right] + c'(t)\left[f(x)y^2 + 2\alpha f(x)yz\right] \\ &\quad + b'(t)\left[\alpha g(x)z^2 + \beta g(x)y^2\right] + a'(t)\left[\varphi(x)z^2 + 2\beta\varphi(x)yz\right] \\ &\quad + \alpha\rho d'(t)\left[z(t-r(t)) + h(x)\right]^2 - \alpha\rho d'(t)h^2(x) + \alpha\rho d_1|r'(t)|h^2(x).\end{aligned}$$

By conditions (i), (ii), (iv), (v) and inequality (2.6), (2.7)and using Cauchy-Schwartz inequality we obtain

$$\begin{aligned}V_5 &\leq -2\left[d(t)h_0 - d(t)h'(x)\right]y^2 - 2\alpha d(t)\left[h_0 - h'(x)\right]yz \\ &\leq -2d(t)\left[h_0 - h'(x)\right]y^2 - 2\alpha d(t)\left[h_0 - h'(x)\right]yz \\ &\leq -2d(t)\left[h_0 - h'(x)\right]\left[\left(y + \frac{\alpha}{2}z\right)^2 - \left(\frac{\alpha}{2}z\right)^2\right] \\ &\leq \frac{\alpha^2}{2}d(t)\left[h_0 - h'(x)\right]z^2.\end{aligned}$$

Therefore,

$$\begin{aligned}V_5 + V_6 &\leq -2\left[b(t)g(x) - \alpha c(t)f(x) - \beta a(t)\varphi(x) - \frac{\alpha^2}{4}d(t)\left[h_0 - h'(x)\right]\right]z^2 \\ &\leq -2\left[b_0 g_0 - \left(\frac{1}{a_0 m} + \varepsilon\right)c_1 M - \left(\frac{d_1 h_0}{c_0 m} + \varepsilon\right)a_1 M - \frac{\alpha^2}{4}\left(a_0 m \delta_0\right)\right]z^2 \\ &\leq -2\left[b_0 g_0 - \frac{M}{a_0 m}c_1 - \frac{d_1 h_0 a_1 M}{c_0 m} - \frac{\delta_0}{a_0 m} - \varepsilon M\left(a_1 + c_1\right)\right]z^2 \\ &\leq -2\left[b_0 g_0 - \delta_1 - \varepsilon M\left(a_1 + c_1\right)\right]z^2 \leq 0,\end{aligned}$$

$$V_7 \leq -2\left[\alpha a_0 m - 1\right]w^2 = -2\varepsilon w^2 \leq 0,$$

$$\begin{aligned}V_8 &\leq -2\varepsilon c(t)f(x)y^2 + \alpha\rho a_1 M w_t^2 + \alpha\rho a_1 M w^2 + \alpha\rho b_1 g_1 z^2 + \alpha\rho b_1 g_1 w_t^2 + \alpha\rho c_1 M y^2 \\ &\quad + \alpha\rho c_1 M w_t^2 + \alpha\rho d_1\lambda_0 y^2 + \alpha\rho d_1\lambda_0 z_t^2 + \mu_1 z^2 + \mu_2 w^2 - \mu_1(1-r_2)z_t^2 - \mu_2(1-r_2)w_t^2 \\ &\quad + \alpha\rho d_1(1+r_3)z_t^2 + 2\alpha\rho d_1(1+r_3)w_t^2 + 2\rho w^2 + \beta\rho z^2 + 2\rho w_t^2 + \beta\rho w_t^2 \\ &\quad - 2\rho|ww_t| + (\rho - \rho^2)w_t^2 \\ &\leq -\left(2\varepsilon c_0 m - \alpha\rho c_1 M - \alpha\rho d_1\lambda_0\right)y^2 + \left(\alpha\rho b_1 g_1 + \beta\rho + \mu_1\right)z^2 + \left(\alpha\rho a_1 M + 2\rho + \mu_2\right)w^2 \\ &\quad + \left(\alpha\rho a_1 M + \alpha\rho b_1 g_1 + \alpha\rho c_1 M + 2\alpha\rho d_1(1+r_3) + \beta\rho + 3\rho - \mu_2(1-r_2)\right)w_t^2 \\ &\quad + \left(\alpha\rho d_1\lambda_0 + \alpha\rho d_1(1+r_3) - \mu_1(1-r_2)\right)z_t^2 - \rho^2 w_t^2 - 2\rho|ww_t|,\end{aligned}$$

where

$$\lambda_0 = \max\left\{\frac{h_0}{2M}, \left|\frac{h_0}{m} - \frac{a_0 m\delta_0}{d_1}\right|\right\} \text{ and } \rho < \min\left\{1, \frac{4\varepsilon}{\alpha h_0}, \frac{2\varepsilon c_0 m}{\alpha c_1 M + \alpha d_1\lambda_0}\right\}. \tag{2.13}$$

By taking

$$\begin{cases}\mu_1 = \rho\mu_1' \text{ where } \mu_1' = \dfrac{\alpha d_1\lambda_0 + \alpha d_1(1+r_3)}{1-r_2}, \\[2ex] \mu_2 = \rho\mu_2' \text{ where } \mu_2' = \dfrac{\alpha a_1 M + \alpha b_1 g_1 + \alpha c_1 M + 2\alpha d_1(1+r_3) + \beta + 3}{1-r_2},\end{cases}$$

we obtain

$$\begin{aligned}V_8 &\leq -\left(2\varepsilon c_0 m - \alpha\rho c_1 M - \alpha\rho d_1\lambda_0\right)y^2 + \left(\alpha\rho b_1 g_1 + \beta\rho + \mu_1\right)z^2 + \left(\alpha\rho a_1 M + 2\rho + \mu_2\right)w^2 \\ &\quad - \rho^2 w_t^2 - 2\rho|ww_t|\end{aligned}$$

and

$$
\begin{aligned}
V_9 &\le \alpha d_1\lambda_0\left(W^2 r(t) + \int_{t-r(t)}^{t} y^2(s)\,ds\right) + \beta d_1\lambda_0\left(y^2 r(t) + \int_{t-r(t)}^{t} y^2(s)\,ds\right) \\
&\quad + d_1\lambda_0\left(z^2 r(t) + \int_{t-r(t)}^{t} y^2(s)\,ds\right) + \lambda r(t) y^2(t) - \lambda(1-r_2)\int_{t-r(t)}^{t} y^2(u)\,du \\
&\le d_1\lambda_0 r(t)\left(\beta y^2 + z^2 + \alpha W^2\right) + \left(d_1\lambda_0(\alpha+\beta+1) - \lambda(1-r_2)\right)\int_{t-r(t)}^{t} y^2(s)\,ds \\
&\le d_1\lambda_0(\alpha+\beta+1) r_1\left(y^2 + z^2 + W^2\right). \qquad (2.14)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\sum_{i=5}^{9} V_i \;&\le\; -2\left[b_0 g_0 - \delta_1 - \varepsilon M\,(a_1 + c_1) - \frac{1}{2}\rho\,(\alpha b_1 g_1 + \beta + \mu_1')\right] z^2 \\
&\quad -\Big(2\varepsilon c_0 m - \alpha\rho c_1 M - \alpha\rho d_1\lambda_0\Big) y^2 - \Big(2\varepsilon - \rho\,(\alpha a_1 M + 2 + \mu_2')\Big) w^2 \\
&\quad -\rho^2 w_t^2 - 2\rho|w w_t| + d_1\lambda_0(\alpha+\beta+1) r_1\left(y^2 + z^2 + W^2\right) \\
&\le\; -2\varrho\left(y^2 + z^2 + w^2 + \rho^2 w_t^2 + 2\rho|w w_t|\right) + d_1\lambda_0(\alpha+\beta+1) r_1\left(y^2 + z^2 + W^2\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\varrho = \frac{1}{2}\min\Big\{1,\ 2\varepsilon c_0 m - \alpha\rho c_1 M - \alpha\rho d_1\lambda_0, 2\varepsilon - \rho\,(\alpha a_1 M + 2 + \mu_2'), \\
2b_0 g_0 - 2\delta_1 - 2\varepsilon M\,(a_1 + c_1) - \rho\,(\alpha b_1 g_1 + \beta + \mu_1')\Big\},
\end{aligned}
$$

and

$$
\rho < \min\Big\{1,\ \frac{4\varepsilon}{\alpha h_0},\ \frac{2\varepsilon c_0 m}{\alpha c_1 M + \alpha d_1\lambda_0},\ 2\frac{b_0 g_0 - \delta_1 - \varepsilon M\,(a_1 + c_1)}{\alpha b_1 g_1 + \beta + \mu_1'},\ \frac{2\varepsilon}{\alpha a_1 M + 2 + \mu_2'}\Big\}. \qquad (2.15)
$$

Hence, there exists a positive constant D_3 such that,

$$
\begin{aligned}
\sum_{i=5}^{9} V_i &\le -2\varrho\left(y^2 + z^2 + w^2 + \rho^2 w_t^2 + 2\rho|w w_t|\right) + \frac{d_1\lambda_0(\alpha+\beta+1)}{1-r_2} r_1\left(y^2 + z^2 + W^2\right) \qquad (2.16) \\
&\le -2\varrho\left(y^2 + z^2 + w^2 + \rho^2 w_t^2 + 2\rho w w_t\right) + \lambda r_1\left(y^2 + z^2 + W^2\right) \\
&\le -2D_3\left(y^2 + z^2 + W^2\right),
\end{aligned}
$$

where $D_3 = \varrho - \lambda\dfrac{r_1}{2}$. It can be seen that if $r_1 < \dfrac{2\varrho}{\lambda}$, then $D_3 > 0$. From (2.8), and the Cauchy-Schwartz inequality, we get

$$
\begin{aligned}
V_{10} &\le a(t)|\theta_1|\Big(z^2 + \alpha(z^2 + W^2)\Big) + b(t)|\theta_2|\Big(\alpha z^2 + \alpha(y^2 + W^2) + \beta y^2 + (y^2 + z^2)\Big) \\
&\quad + c(t)|\theta_3|\Big(y^2 + \alpha(y^2 + z^2)\Big) \\
&\le \lambda_1(|\theta_1| + |\theta_2| + |\theta_3|)\left(y^2 + z^2 + W^2 + H(x)\right) \\
&\le 2\frac{\lambda_1}{D_0}\Big(|\theta_1| + |\theta_2| + |\theta_3|\Big)V,
\end{aligned}
$$

where $\lambda_1 = \max\left\{a_1(1+\alpha), b_1(1+2\alpha+\beta), c_1(1+\alpha)\right\}$. Using condition (iv) and Lemma 2.1, we obtain

$$
h^2(x) \le h_0 H(x),
$$

consequently

$$\begin{aligned}|V_{11}| &\leq -d'(t)\left[2\beta H(x)+\alpha h_0 y^2+\left(h^2(x)+y^2\right)+\alpha\left(h^2(x)+z^2\right)+\alpha\rho h^2(x)\right]\\ &\quad+|c'(t)|\left[y^2+\alpha\left(y^2+z^2\right)\right]+|b'(t)|\left[\alpha z^2+\beta y^2\right]\\ &\quad+|a'(t)|\left[z^2+2\beta\left(y^2+z^2\right)\right]+\alpha\rho d_1|r'(t)|h^2(x)\\ &\leq \lambda_2\left[|a'(t)|+|b'(t)|+|c'(t)|+|r'(t)|-d'(t)\right]\left(y^2+z^2+W^2+H(x)\right)\\ &\leq 2\frac{\lambda_2}{D_0}\left[|a'(t)|+|b'(t)|+|c'(t)|+|r'(t)|-d'(t)\right]V,\end{aligned}$$

such that $\lambda_2=\max\left\{2\beta+(\alpha\rho+\alpha+2\beta+1)h_0+1,\alpha h_0(1+\rho d_1)+\alpha+2\beta+1\right\}$. By taking $\frac{1}{\eta}=\frac{1}{D_0}\max\left\{\lambda_1,\lambda_2\right\}$, we obtain

$$\dot{V}_{(1.2)}\leq -D_3(y^2+z^2+W^2)+\frac{1}{\eta}\gamma(t)V. \tag{2.17}$$

By conditions vi), vii) and inequalitie (2.17) we have

$$\begin{aligned}\dot{U}_{(1.2)} &= \left(\dot{V}_{(1.2)}-\frac{1}{\eta}\gamma(t)V\right)e^{-\frac{1}{\eta}\int_{t_1}^{t}\gamma(s)\,ds}\\ &\leq -D_3\left(y^2+z^2+W^2\right)e^{-\frac{1}{\eta}\int_{t_1}^{t}\gamma(s)\,ds}\\ &\leq -D_4\left(y^2+z^2+W^2\right),\end{aligned}$$

where $D_4=D_3e^{-\frac{\eta_1+\eta_2}{\eta}}$. Now take $W_3(\|X\|)=D_4(x^2+y^2+z^2)$. From (1.2) it easy to see that W_3 is positive definite function. Thus, we conclude that the solution of system (1.2) are uniformly asymptotically stable.
Now, it is evident from (1.2) that

$$\begin{cases}|x'(t)|=|y(t)|,\\ |x''(t)|=|z(t)|,\\ |x'''(t)|=|w(t)|,.\end{cases}$$

Clearly, from the above discussion

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}x'(t)=\lim_{t\to\infty}x''(t)=\lim_{t\to\infty}x'''(t)=0.$$

This fact completes the proof of the Theorem 2.1. □

Proof . Proof of Theorem 2.2 From (2.11), we have

$$U\geq D_2(x^2+y^2+z^2+W^2),\ \text{where}\ \ D_2=\frac{D_1}{2}e^{-\frac{\eta_1+\eta_2}{\eta}}. \tag{2.18}$$

Taking the time derivative of V with respect to t along the trajectory of system (1.2), we obtain

$$\begin{aligned}\dot{V}_{(1.2)}\leq&-D_3(y^2+z^2+W^2)+\frac{1}{\eta}\gamma(t)V\\ &+\left(\beta y+z+\alpha W\right)\psi(t,x,y,z,w).\end{aligned} \tag{2.19}$$

By condition iv) and inequalities (2.4), (2.18) and (2.19) and the Cauchy-Schwartz inequality we have

$$\begin{aligned}
\dot{U}_{(1.2)} &= \left(\dot{V}_{(1.2)} - \frac{1}{\eta}\gamma(t)V\right)e^{-\frac{1}{\eta}\int_{t_1}^{t}\gamma(s)\,ds} \\
&\le \Big(-D_3\left(y^2+z^2+W^2\right) + \left(\beta y + z + \alpha W\right)\psi(t,x,y,z,w)\Big)e^{-\frac{1}{\eta}\int_{t_1}^{t}\gamma(s)\,ds} \\
&\le \left(\beta|y| + |z| + \alpha|W|\right)|\psi(t,x,y,z,w)| \\
&\le D_5\left(|y|+|z|+|W|\right)|e(t)| \\
&\le D_5\left(3+y^2+z^2+W^2\right)|e(t)| \\
&\le D_5\left(3+\frac{1}{D_2}W\right)|e(t)| \\
&\le 3D_5|e(t)| + \frac{D_5}{D_2}W|e(t)|, \qquad (2.20)
\end{aligned}$$

where $D_5 = \max\{\alpha,\beta,1\}$.

Integrating (2.20) from 0 to t, using the inequality (2.4) and the Gronwall inequality, we have

$$\begin{aligned}
U(t,x,y,z,w) &\le U\big(0,x(0),y(0),z(0),w(0)\big) + 3D_5\eta_3 \\
&\quad + \frac{D_5}{D_2}\int_{t_1}^{t} U\big(s,x(s),y(s),z(s),w(s)\big)|e(s)|ds \\
&\le \Big(U\big(0,x(0),y(0),z(0),w(0)\big) + 3D_5\eta_3\Big)e^{\frac{D_5}{D_2}\int_{t_1}^{t}|e(s)|ds} \\
&\le \Big(U\big(0,x(0),y(0),z(0),w(0)\big) + 3D_5\eta_3\Big)e^{\frac{D_5}{D_2}\eta_3} = K_1 < \infty. \qquad (2.21)
\end{aligned}$$

In view of inequalities (2.18) and (2.21),

$$(x^2+y^2+z^2+W^2) \le \frac{1}{D_2}U \le K$$

where $K = \frac{K_1}{D_2}$. Aforementioned inequality implies that

$$|x(t)| \le \sqrt{K},\ |y(t)| \le \sqrt{K},\ |z(t)| \le \sqrt{K},\ |W(t)| \le \sqrt{K} \quad \text{for all} \quad t \ge t_0 + r_1.$$

Hence,

$$|x(t)| \le \sqrt{K},\ |x'(t)| \le \sqrt{K},\ |x''(t)| \le \sqrt{K},\ |x'''(t) + \rho x'''(t-r(t))| \le \sqrt{K} \quad \text{for all} \quad t \ge t_0 + r_1. \qquad (2.22)$$

for all $t \ge t_0 + r_1$. First from (2.16) we obtain

$$\begin{aligned}
\sum_{i=5}^{9} V_i &\le -2\varrho\left(y^2+z^2+w^2+\rho^2 w_t^2 + 2\rho|ww_t|\right) + \lambda r_1\left(y^2+z^2+w^2+\rho^2 w_t^2 + 2\rho|ww_t|\right) \\
&\le -2D_3\left(y^2+z^2+w^2+\rho^2 w_t^2 + 2\rho|ww_t|\right) \\
&\le -2D_3\left(y^2+z^2+w^2\right),
\end{aligned}$$

and we get,

$$\dot{V}_{(1.2)} \le -D_3(y^2+z^2+w^2) + \frac{1}{\eta}\gamma(t)V + \big(\beta y + z + \alpha W\big)\psi(t,x,y,z,w). \qquad (2.23)$$

From vi), (2.11) and (2.23) and the Cauchy-Schwartz inequality, we get

$$\begin{aligned}\dot{U}_{(1.2)} &= \left(\dot{V}_{(1.2)} - \frac{1}{\eta}\gamma(t)V\right)e^{-\frac{1}{\eta}\int_{t_1}^{t}\gamma(s)\,ds} \\ &\leq \Big(-D_3\left(y^2+z^2+w^2\right) + \left(\beta y + z + \alpha W\right)\psi(t,x,y,z,w)\Big)e^{-\frac{1}{\eta}\int_{t_1}^{t}\gamma(s)\,ds} \qquad (2.24)\end{aligned}$$

Now, we define $F_t = F(t, x(t), y(t), z(t), w(t))$ as

$$F_t = U + \sigma\int_{t_1}^{t}\left(y^2(s) + z^2(s) + w^2(s)\right)ds,$$

where $\sigma > 0$. It is easy to see that F_t is positive definite, since $W = W(t, x, y, z, w)$ is already positive definite. Using the estimate $e^{-\frac{\eta_1+\eta_2}{\eta}} \leq e^{-\frac{1}{\eta}\int_{t_1}^{t}\gamma(s)\,ds} \leq 1$ and (2.24) implies

$$\begin{aligned}\dot{F}_{t(1.2)} &\leq -D_3\Big(y^2(t)+z^2(t)+w^2(t)\Big)e^{-\frac{\eta_1+\eta_2}{\eta}} + D_4\Big(|y(t)|+|z(t)|+|W(t)|\Big)|\psi(t,x,y,z,w)| \\ &\quad +\sigma\Big(y^2(t)+z^2(t)+w^2(t)\Big) \\ &\leq -D_3\Big(y^2(t)+z^2(t)+w^2(t)\Big)e^{-\frac{\eta_1+\eta_2}{\eta}} + D_4\Big(|y(t)|+|z(t)|+|W(t)|\Big)|e(t)| \\ &\quad +\sigma\Big(y^2(t)+z^2(t)+w^2(t)\Big). \qquad (2.25)\end{aligned}$$

where D_4 is positive constant. By choosing $\sigma = D_3 e^{-\frac{\eta_1+\eta_2}{\eta}}$ we obtain

$$\begin{aligned}\dot{F}_{t(1.2)} &\leq D_4\Big(3 + y^2(t) + z^2(t) + W^2(t)\Big)|e(t)| \\ &\leq D_4\Big(3 + \frac{1}{D_2}U\Big)|e(t)| \\ &\leq 3D_4|e(t)| + \frac{D_4}{D_2}F_t|e(t)|. \qquad (2.26)\end{aligned}$$

Integrating the last inequality (2.26) from 0 to t, and again using the Gronwall inequality and the condition (2.4), we get

$$\begin{aligned}F_t &\leq F_0 + 3D_4\eta_3 + \frac{D_4}{D_2}\int_{t_1}^{t}F_s|e(s)|ds \\ &\leq \Big(F_0 + 3D_4\eta_3\Big)e^{\frac{D_4}{D_2}\int_{t_1}^{t}|e(s)|ds} \\ &\leq \Big(F_0 + 3D_4\eta_3\Big)e^{\frac{D_4}{D_2}\eta_3} = K_2 < \infty. \qquad (2.27)\end{aligned}$$

Therefore

$$\int_{t_1}^{\infty} y^2(s)ds < K_2 \quad , \quad \int_{t_1}^{\infty} z^2(s) < K_2 \text{ and } \int_{t_1}^{\infty} w^2(s)ds < K_2,$$

which implies that

$$\int_{t_1}^{\infty} x'^2(s)ds \leq K_2 \quad , \quad \int_{t_1}^{\infty} x''^2(s)ds \leq K_2 \quad , \quad \int_{t_1}^{\infty} x'''^2(s)ds \leq K_2. \qquad (2.28)$$

The proof of Theorem 2.2 is completed. □

3 Example

We consider the following fourth order non-autonomous delay differential equation of neutral type

$$\begin{aligned}&\left(x'''(t)+\frac{1}{170}x'''\left(t-\frac{t^2}{20(t^2+1)}\right)\right)'+\left(e^{-t}\sin t+2\right)\left(\left(\frac{x(t)+4e^{x(t)}+4e^{-x(t)}}{4\left(e^{x(t)}+e^{-x(t)}\right)}\right)x''(t)\right)'\\&+\left(\frac{\cos t+3t^2+3}{1+t^2}\right)\left(\left(\frac{\sin x(t)+6e^{x(t)}+6e^{-x(t)}}{e^{x(t)}+e^{-x(t)}}\right)x'(t)\right)'\\&+\left(e^{-2t}\sin^3 t+2\right)\left(\frac{x(t)\cos x(t)+5x^4(t)+5}{5\left(1+x^4(t)\right)}\right)x'(t)\\&+\left(\frac{1}{20\cosh t}+\frac{1+2\left(1+t^2\right)}{20\left(1+t^2\right)}\right)\left(\frac{x(t-\frac{t^2}{20(t^2+1)})}{x^2(t-\frac{t^2}{20(t^2+1)})+1}\right)\\&=\frac{2\sin t}{t^2+\left(x(t)+x'''(t)\right)^2+\left(x'(t)x''(t)\right)^2+1}.\end{aligned}\tag{3.1}$$

By taking $\varphi(x)=\dfrac{x+4e^x+4e^{-x}}{4\left(e^x+e^{-x}\right)}$, $g(x)=\dfrac{\sin x+6e^x+6e^{-x}}{e^x+e^{-x}}$,
$f(x)=\dfrac{x\cos x+5x^4+5}{5\left(1+x^4\right)}$, $h(x)=\dfrac{x}{x^2+1}$,
$\psi(t,x(t),x'(t),x''(t),x'''(t))=\dfrac{2\sin t}{t^2+\left(x(t)+x'''(t)\right)^2+\left(x'(t)x''(t)\right)^2+1}$
$a(t)=e^{-t}\sin t+2$, $b(t)=\dfrac{\cos t+3t^2+3}{1+t^2}$, $c(t)=e^{-2t}\sin^3 t+2$,
$d(t)=\dfrac{1}{20\cosh t}+\dfrac{1+2\left(1+t^2\right)}{20\left(1+t^2\right)}$, $e(t)=\dfrac{2\sin t}{t^2+1}$ and $r(t)=\dfrac{t^2}{20\left(t^2+1\right)}$ We have $m=\dfrac{9}{10}$, $M=\dfrac{11}{10}$,
$g_0=\dfrac{11}{2}$, $g_1=\dfrac{13}{2}$, $h_0=\dfrac{5}{2}$, $\delta_0=\dfrac{5}{3}$, $a_0=1$, $a_1=3$, $b_0=2$, $b_1=4$, $c_0=1$, $c_1=3$,
$d_0=\dfrac{1}{10}$, $d_1=\dfrac{1}{5}$, $\dfrac{h_0}{m}-\dfrac{a_0m\delta_0}{d_1}\leq-4.\ 55\leq h'(x)\leq 1.1\leq\dfrac{h_0}{2M}$, then $\delta_0=\frac{5}{3}$ and

$$\begin{aligned}b_0g_0&=11>\frac{397}{54}=\frac{d_1h_0a_1M}{c_0m}+\frac{c_1M+\delta_0}{a_0m}=\delta_1,\\ \epsilon&<\min\left\{\frac{1}{a_0m},\frac{d_1h_0}{c_0m},\frac{b_0g_0-\delta_1}{M(a_1+c_1)}\right\}=\frac{985}{1782},\quad |r'(t)|=\left|\frac{t}{10\left(t^2+1\right)^2}\right|\leq\frac{1}{20}=r_2=r_3.\end{aligned}$$

By choosing $\epsilon=\dfrac{1}{4}$ we get $\alpha=\dfrac{M}{a_0m}+\epsilon=\dfrac{49}{36}$, $\beta=\dfrac{d_1h_0}{c_0m}+\epsilon=\dfrac{29}{36}$,

$$\begin{aligned}\lambda_0&=\max\left\{\frac{h_0}{2M},\left|\frac{h_0}{m}-\frac{a_0m\delta_0}{d_1}\right|\right\}=\frac{85}{18},\qquad\lambda=\frac{d_1\lambda_0(\alpha+\beta+1)}{1-r_2}=\frac{85}{27},\\ \mu_1'&=\frac{\alpha d_1\lambda_0+\alpha d_1\left(1+r_3\right)}{1-r_2}=\frac{50911}{30780},\\ \mu_2'&=\frac{\alpha a_1M+\alpha b_1g_1+\alpha c_1M+2\alpha d_1\left(1+r_3\right)+\beta+3}{1-r_2}=51\\ \rho&=\frac{1}{170}<\min\left\{1,\frac{4\varepsilon}{\alpha h_0},\frac{2\varepsilon c_0m}{\alpha c_1M+\alpha d_1\lambda_0},\right.\\&\left.\quad 2\frac{b_0g_0-\delta_1-\varepsilon M\left(a_1+c_1\right)}{\alpha b_1g_1+\beta+\mu_1'},\frac{2\varepsilon}{\alpha a_1M+2+\mu_2'}\right\}=\frac{60}{6899}\end{aligned}$$

$$\begin{aligned}r_1=\frac{1}{20}&<\frac{1}{\lambda}\min\left\{1,2\varepsilon c_0m-\alpha\rho c_1M-\alpha\rho d_1\lambda_0,\quad 2\varepsilon-\rho\left(\alpha a_1M+2+\mu_2'\right),\right.\\&\left.2b_0g_0-2\delta_1-2\varepsilon M\left(a_1+c_1\right)-\rho\left(\alpha b_1g_1+\beta+\mu_1'\right)\right\}=\frac{29709}{578\,000}\end{aligned}$$

On the other hand,

$$\int_{-\infty}^{+\infty}|\varphi'(x)|\,dx = \frac{1}{4}\int_{-\infty}^{+\infty}\left|\frac{1}{e^x+e^{-x}}+x\frac{e^{-x}-e^x}{(e^x+e^{-x})^2}\right|dx \le \frac{\pi}{4},$$

$$\begin{aligned}\int_{-\infty}^{+\infty}|g'(x)|\,dx &= \frac{1}{5}\int_{-\infty}^{+\infty}\left|\frac{(e^x+e^{-x})\cos x-(e^x-e^{-x})\sin x}{(e^x+e^{-x})^2}\right|dx \le \frac{\pi}{5},\\ \int_{-\infty}^{+\infty}|f'(x)|\,dx &= \frac{1}{5}\int_{-\infty}^{+\infty}\left|\frac{(\cos x - x\sin x)(x^4+1)-4x^4\cos x}{(x^4+1)^2}\right|dx\\ &\le \frac{1}{5}\int_{-\infty}^{+\infty}\frac{5+x^2}{x^4+1}dx = \frac{9}{10}\sqrt{2}\pi,\end{aligned}$$

then

$$\int_{-\infty}^{+\infty}(|\varphi'(s)|+|g'(s)|+|f'(s)|)\,ds < \infty.$$

We have also

$$\begin{aligned}\int_{r_1}^{+\infty}|a'(t)|\,dt &\le \int_0^{+\infty}|a'(t)|\,dt = \int_0^{+\infty}\left|\,(\cos t)\,e^{-t}-(\sin t)\,e^{-t}\right|dt \le \int_0^{+\infty} 2e^{-t}dt = 2,\\ \int_{r_1}^{+\infty}|b'(t)|\,dt &\le \int_0^{+\infty}|b'(t)|\,dt = \int_0^{+\infty}\left|-\frac{\sin t}{t^2+1}-2t\frac{\cos t}{(t^2+1)^2}\right|dt \le \int_0^{+\infty}\left(\frac{1}{t^2+1}+\frac{2t}{(t^2+1)^2}\right)dt\\ &\le \int_0^{+\infty}\frac{2}{t^2+1}dt = \pi,\\ \int_{r_1}^{+\infty}|c'(t)|\,dt &\le \int_0^{+\infty}|c'(t)|\,dt = \int_0^{+\infty}\left|\,3\left(\cos t\sin^2 t\right)e^{-2t}-2\left(\sin^3 t\right)e^{-2t}\right|dt \le \int_0^{+\infty} 5e^{-2t}dt = \frac{5}{2},\end{aligned}$$

and

$$\begin{aligned}\int_{r_1}^{+\infty}(-d'(t))\,dt &\le \int_0^{+\infty}\left(\frac{x+\sinh x+x\cosh 2x+2x^2\sinh x+x^4\sinh x}{10\cosh 2x+20x^2\cosh 2x+10x^4\cosh 2x+20x^2+10x^4+10}\right)dt\\ &= \frac{1}{10},\\ \int_{r_1}^{+\infty}|r'(t)|\,dt &\le \int_0^{+\infty}|r'(t)|\,dt = \int_0^{+\infty}\left|\frac{t}{10\,(t^2+1)^2}\right|dt = \frac{1}{20},\end{aligned}$$

$$\int_{r_1}^{+\infty}|e(t)|\,dt \le \int_0^{+\infty}|e(t)|\,dt = \int_0^{+\infty}\left|\frac{2\sin t}{t^2+1}\right|dt \le \int_0^{+\infty}\frac{2}{t^2+1}dt = \pi.$$

Hence

$$\int_{r_1}^{+\infty}(|a'(t)|+|b'(t)|+|c'(t)|-d'(t)+|r'(t)|)\,dt < +\infty.$$

Thus all the assumptions of Theorem 2.1 hold, this shows that every solution $x(t)$ of (3.1) and their derivatives $x'(t), x''(t)$ and $x'''(t)$ are uniformly asymptotically stable.

4 Conclusion

A class of fourth-order differential equations of neutral type with variable delay inspired from some previous works in the literature has been considered. Some new results are obtained on the stability, the boundedness and the square integrability of the solutions of this class using the powerful tool of Lyapunov's second method. A Lyapunov functional is defined to achieve our proofs.

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