Int. J. Nonlinear Anal. Appl. 14 (2023) 8, 311–325
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.25549.3051

More on proper commuting and Lie mappings on generalized matrix algebras

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(Communicated by Ali Jabbari)

Abstract

This paper is devoted to proper linear mappings on generalized matrix algebras and by obtaining their general form, we could obtain good results for commuting mappings and Lie centralizer and Lie triple centralizers, which are clearly established for triangular algebras and nest algebras as well.

Keywords: Commuting mapping, Generalized matrix algebra, Lie centralizer, Lie triple centralizer
2020 MSC: Primary: 15B33; Secondary:16W25

1 Introduction

Let $\mathcal{R}$ be an unital commutative ring, $\mathfrak{A}$ be an unital algebra on $\mathcal{R}$ and $\mathcal{Z}(\mathfrak{A})$ be the center of $\mathfrak{A}$. We determine the Lie product of elements $x, y \in \mathfrak{A}$ by $[x, y] = xy - yx$. A linear mapping $\phi : \mathfrak{A} \to \mathfrak{A}$ is called commuting if the identity

$$[\phi(x), x] = 0$$

holds for all $x \in \mathfrak{A}$. Commuting mappings have been studied by many authors (see [5, 6, 16, 22]). Each mapping with the rang in $\mathcal{Z}(\mathfrak{A})$ and the identity map are two easy examples of commuting mappings. A Lie centralizer mapping $\phi : \mathfrak{A} \to \mathfrak{A}$ is a linear mapping such that the identity

$$\phi[x, y] = [\phi(x), y]$$

holds for all $x, y \in \mathfrak{A}$. It is well-known that Lie centralizers are very important both theoretically and practically and have therefore been studied intensively (see [2, 8, 9, 13, 14, 15, 17]). A linear mapping $\phi : \mathfrak{A} \to \mathfrak{A}$ is called Lie triple centralizer if the identity

$$\phi([[x, y], z]) = [[\phi(x), y], z]$$

holds for all $x, y, z \in \mathfrak{A}$. Obviously, each Lie centralizer is a Lie triple centralizer, but the converse is not generally correct (see [8, Example 1.2]). In Remark 1.1, we will give an example of Lie triple centralizer. A Lie triple derivation $\delta : \mathfrak{A} \to \mathfrak{A}$ is a linear mapping such that

$$\delta[[x, y], z] = [[\delta(x), y], z] + [[x, \delta(y)], z] + [[x, y], \delta(z)]$$

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Received: December 2021 *Accepted:* February 2022

for all $x, y, z \in \mathfrak{A}$. A linear mapping $\delta : \mathfrak{A} \to \mathfrak{A}$ is called to be a generalized Lie triple derivation related to a Lie triple derivation $\alpha : \mathfrak{A} \to \mathfrak{A}$ if

$$\delta([[x, y], z]) = [[\delta(x), y], z] + [[x, \alpha(y)], z] + [[x, y], \alpha(z)]$$

for all $x, y, z \in \mathfrak{A}$.

Remark 1.1. A linear mapping $\delta : \mathfrak{A} \to \mathfrak{A}$ is a generalized Lie triple derivation related to a Lie triple derivation $\alpha : \mathfrak{A} \to \mathfrak{A}$ if and only if

$$(\delta - \alpha)([[x, y], z]) = [[(\delta - \alpha)(x), y], z]$$

for all $x, y, z \in \mathfrak{A}$.

We now introduce the properness of linear mappings. A linear mapping $\phi : \mathfrak{A} \to \mathfrak{A}$ is called proper if there exist $\lambda \in Z(\mathfrak{A})$ and a linear mapping $\tau : \mathfrak{A} \to Z(\mathfrak{A})$ such that

$$\phi(x) = \lambda x + \tau(x)$$

for all $x \in \mathfrak{A}$. It is easy to investigate that every such a map is a commuting map, which is called a proper commuting map. The main question, however, is that when a commuting map is proper. Cheung [6] was the first who studied proper commuting mappings on triangular algebras. Furthermore, by specifying the exact form of the commuting mappings, Xiao and Wei [22] obtained sufficient conditions for these mappings to be proper on generalized matrix algebras. Jabeen [15] has also studied properness for Lie centralizers on the generalized matrix algebras.

2 Preliminaries and Tools

Let us start by introducing the generalized matrix algebras provided by a Morita context [20]. A Morita context includes two algebras $\mathcal{A}$ and $\mathcal{B}$, two bimodules ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{N}_{\mathcal{A}}$, and two bimodule homomorphisms $\zeta_{\mathcal{MN}} : \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} \to \mathcal{A}$ and $\psi_{\mathcal{NM}} : \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M} \to \mathcal{B}$ which satisfy by the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M} & \xrightarrow{\zeta_{\mathcal{MN}} \otimes \mathcal{I}_{\mathcal{M}}} & \mathcal{A} \underset{\mathcal{A}}{\otimes} \mathcal{M} \\ \Big\downarrow{\scriptstyle \mathcal{I}_{\mathcal{M}} \otimes \psi_{\mathcal{NM}}} & & \Big\downarrow{\scriptstyle \cong} \\ \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{B} & \xrightarrow{\cong} & \mathcal{M} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} & \xrightarrow{\psi_{\mathcal{NM}} \otimes \mathcal{I}_{\mathcal{N}}} & \mathcal{B} \underset{\mathcal{B}}{\otimes} \mathcal{N} \\ \Big\downarrow{\scriptstyle \mathcal{I}_{\mathcal{N}} \otimes \zeta_{\mathcal{MN}}} & & \Big\downarrow{\scriptstyle \cong} \\ \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{A} & \xrightarrow{\cong} & \mathcal{N} \end{array}$$

Consider a Morita context $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \zeta_{\mathcal{MN}}, \psi_{\mathcal{NM}})$, then the set

$$\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B}) = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}, b \in \mathcal{B} \right\}$$

by applying addition and multiplication of a matrix, it forms an algebra, which we call generalized matrix algebra.

These algebras were first introduced by Sands [21] to study radicals of rings in Morita contexts. When $\mathcal{N} = 0$, the algebra $\mathcal{G}(\mathcal{A}, \mathcal{M}, 0, \mathcal{B})$ is called triangular algebra.

Generalized matrix algebras contain a wide range of algebras, such as full matrix algebras and nest algebras, etc, so they are of great importance in mathematics. Therefore, we decided to investigate proper linear mappings on these algebras, to get good results for those algebras as well.

As mentioned in [4, 10], we introduce weaker conditions than faithfulness on a generalized matrix algebra. A generalized matrix algebra $\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ is called weakly faithful if

$$a\mathcal{M} = 0, \mathcal{N}a = 0 \Longrightarrow a = 0$$
$$\mathcal{M}b = 0, b\mathcal{N} = 0 \Longrightarrow b = 0$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. When a generalized matrix algebra is weakly faithful , its center is simplified as follows

Lemma 2.1. ([3, Proposition 2.1]) Consider a generalized matrix algebra $\mathcal{G}(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ is weakly faithful. The center of $\mathcal{G}$ is as follows

$$Z(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, am = mb, na = bn, \ \textit{for all } m \in \mathcal{M}, n \in \mathcal{N} \right\}$$

In addition, there is a unique algebra isomorphism $\eta:\pi_{\mathcal{A}}(Z(\mathcal{G}))\longrightarrow\pi_{\mathcal{B}}(Z(\mathcal{G}))$ such that $am = m\eta(a)$ and $na = \eta(a)n$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$.

Considering that many authors tried to show properness for different mappings, in section 3, we essentially characterize this type of mapping on generalized matrix algebras to obtain the necessary and sufficient conditions under which a mapping to be valid in this property. In Proposition 3.1, we obtain these conditions, and in the sequel, the properties are found that the elements of a mapping must have to be proper.

In section 4, we deal with commuting mappings and create the conditions of Theorem 3.4 for these maps on the generalized matrix algebras so that we can conclude [22, Theorem 3.6].

Section 5 is dedicated to examining the maps that apply to the property (★) as follows

$$xy = yx = 0 \Rightarrow [\Phi(x), y] \in Z(\mathcal{G}) \tag{★}$$

for all $x, y \in \mathfrak{A}$, and provides sufficient conditions for these maps to be proper. Characterization of maps through commutative zero products are recentally studied by some authors (see [1, 7, 11, 12] and references therein).

At last, in section 6, we obtain some results about commuting maps, Lie centralizers and Lie triple centralizer which are the main results of [6, 8, 15].

3 Proper linear mappings on generalized matrix algebras

Let us begin this section by characterizing proper linear mappings on generalized matrix algebras.

Proposition 3.1. Let Φ be a linear mapping on generalized matrix algebra $\mathcal{G}$. Φ is proper if and only if Φ is of following form

$$\Phi\begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} f_1(a)+g_1(m)+h_1(n)+k_1(b) & cm \\ nc & f_4(a)+g_4(m)+h_4(n)+k_4(b) \end{bmatrix},$$

where f_i, g_i, h_i and k_i, $i \in \{1,4\}$, are linear mappings on $\mathcal{A},\mathcal{M},\mathcal{N}$ and $\mathcal{B}$, respectively, and there exists $c \in \pi_{\mathcal{A}}(Z(\mathcal{G}))$ such that

(i) $\begin{bmatrix} g_1(m)+h_1(n) & 0 \\ 0 & g_4(m)+h_4(n) \end{bmatrix} \in Z(\mathcal{G})$,

(ii) $\begin{bmatrix} f_1(a)-ca & 0 \\ 0 & f_4(a) \end{bmatrix} \in Z(\mathcal{G})$,

(iii) $\begin{bmatrix} k_1(b) & 0 \\ 0 & k_4(b)-\eta(c)b \end{bmatrix} \in Z(\mathcal{G})$,

for all $a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}$ and $b \in \mathcal{B}$.

Proof . Each linear mapping $\Phi : \mathcal{G} \to \mathcal{G}$ is of the followoing form,

$$\Phi\begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} f_1(a)+g_1(m)+h_1(n)+k_1(b) & f_2(a)+g_2(m)+h_2(n)+k_2(b) \\ f_3(a)+g_3(m)+h_3(n)+k_3(b) & f_4(a)+g_4(m)+h_4(n)+k_4(b) \end{bmatrix},$$

where f_i, g_i, h_i and k_i, $1 \le i \le 4$, are linear mappings on $\mathcal{A},\mathcal{M},\mathcal{N}$ and $\mathcal{B}$, respectively.

Let $\Phi(x) = \lambda x + \tau(x)$ for all $x \in \mathcal{G}$ where $\lambda \in Z(\mathcal{G})$ and $\tau(\mathcal{G}) \subseteq Z(\mathcal{G})$. λ and τ are of the following forms

$$\lambda = \begin{bmatrix} c & 0 \\ 0 & \eta(c) \end{bmatrix}$$

and

$$\tau\begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} F_1(a)+G_1(m)+H_1(n)+K_1(b) & 0 \\ 0 & F_4(a)+G_4(m)+H_4(n)+K_4(b) \end{bmatrix}$$

where F_i, G_i, H_i and K_i, $i = 1, 4$, are linear mappings on $\mathcal{A}, \mathcal{M}, \mathcal{N}$ and $\mathcal{B}$, respectively. so for each $a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}$ and $b \in \mathcal{B}$, we have

$$\Phi \begin{bmatrix} a & m \\ n & b \end{bmatrix} = \lambda \begin{bmatrix} a & m \\ n & b \end{bmatrix} + \tau \begin{bmatrix} a & m \\ n & b \end{bmatrix}$$
$$= \begin{bmatrix} ca & cm \\ \eta(c)n & \eta(c)b \end{bmatrix} + \begin{bmatrix} F_1(a) + G_1(m) + H_1(n) + K_1(b) & 0 \\ 0 & F_4(a) + G_4(m) + H_4(n) + K_4(b) \end{bmatrix}$$

therefore

$$f_3 = g_3 = k_3 = f_2 = h_2 = k_2 = 0,$$

and

$$\begin{bmatrix} g_1(m) + h_1(n) & 0 \\ 0 & g_4(m) + h_4(n) \end{bmatrix} \in Z(\mathcal{G})$$

and

$$g_2(m) = cm, h_3(n) = \eta(c)n = nc,$$

for all $a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}$ and $b \in \mathcal{B}$.

From $\begin{bmatrix} f_1(a) & 0 \\ 0 & f_4(a) \end{bmatrix} = \begin{bmatrix} F_1(a) + ca & 0 \\ 0 & F_4(a) \end{bmatrix}$ and $\begin{bmatrix} F_1(a) & 0 \\ 0 & F_4(a) \end{bmatrix} \in Z(\mathcal{G})$, we get $\begin{bmatrix} f_1(a) - ca & 0 \\ 0 & f_4(a) \end{bmatrix} \in Z(\mathcal{G})$ for all $a \in \mathcal{A}$.

From $\begin{bmatrix} k_1(b) & 0 \\ 0 & k_4(b) \end{bmatrix} = \begin{bmatrix} K_1(b) & 0 \\ 0 & K_4(b) + \eta(c)b \end{bmatrix}$ and $\begin{bmatrix} K_1(b) & 0 \\ 0 & K_4(b) \end{bmatrix} \in Z(\mathcal{G})$, we get $\begin{bmatrix} k_1(b) & 0 \\ 0 & k_4(b) - \eta(c)b \end{bmatrix} \in Z(\mathcal{G})$, for all $b \in \mathcal{B}$.

The converse is obviously achieved. □

In the following, we describe the properties of proper linear mappings on weakly faithful generalized matrix algebras.

Proposition 3.2. Let a generalized matrix algebra $\mathcal{G}$ be weakly faithful. A linear mapping $\Phi : \mathcal{G} \to \mathcal{G}$ is proper if and only if Φ is of the following form

$$\Phi\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} f_1(a) + g_1(m) + h_1(n) + k_1(b) & g_2(m) \\ h_3(n) & f_4(a) + g_4(m) + h_4(n) + k_4(b) \end{bmatrix}$$

for all $a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}$ and $b \in \mathcal{B}$ where $f_1 : A \to A, g_1 : \mathcal{M} \to Z(A), h_1 : \mathcal{N} \to Z(A), k_1 : \mathcal{B} \to Z(A), g_2 : \mathcal{M} \to \mathcal{M}, h_3 : \mathcal{N} \to \mathcal{N}, f_4 : A \to Z(\mathcal{B}), g_4 : \mathcal{M} \to Z(\mathcal{B}), h_4 : \mathcal{N} \to Z(\mathcal{B})$ and $k_4 : \mathcal{B} \to \mathcal{B}$ are linear mappings such that

(i) $g_1(m)m' = m'g_4(m)$, $\quad ng_1(m) = g_4(m)n$,
(ii) $h_1(n)m = mh_4(n)$, $\quad n'h_1(n) = h_4(n)n'$,
(iii) $f_4(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(Z(\mathcal{G}))$, $\quad k_1(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(Z(\mathcal{G}))$,
(iv) $h_3(bn) = k_4(b)n - nk_1(b)$, $\quad h_3(na) = nf_1(a) - f_4(a)n$,
(v) $g_2(am) = f_1(a)m - mf_4(a)$, $\quad g_2(mb) = mk_4(b) - k_1(b)m$,

for all $a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}$ and $b \in \mathcal{B}$.

Proof . $\Longrightarrow$ Let $\Phi(X) = \lambda X + \tau(X)$ for all $X \in \mathcal{G}$, where $\lambda \in Z(\mathcal{G})$ and $\tau(\mathcal{G}) \subseteq Z(\mathcal{G})$.

We know that (i, ii) are deduced by Proposition 3.1(i), and (iii) is concluded by Proposition 3.1(ii, iii).

From Proposition 3.1(ii), we have $f_1(a) - ca = \eta^{-1}(f_4(a))$ or $f_1(a) - \eta^{-1}(f_4(a)) = ca$, by puttying $a = 1_{\mathcal{A}}$, we get

$$f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}})) = c. \tag{3.1}$$

and so

$$f_1(a) - \eta^{-1}(f_4(a)) = (f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}})))a \tag{3.2}$$

for all $a \in \mathcal{A}$.

From Proposition 3.1(iii), we have $k_4(b) - \eta(c)b = \eta(k_1(b))$ or $k_4(b) - \eta(k_1(b)) = \eta(c)b$, by puttying $b = 1_{\mathcal{B}}$, we get

$$k_4(1_{\mathcal{B}}) - \eta(k_1(1_{\mathcal{B}})) = \eta(c). \tag{3.3}$$

and so

$$k_4(b) - \eta(k_1(b)) = (k_4(1_{\mathcal{B}}) - \eta(k_1(1_{\mathcal{B}})))b \tag{3.4}$$

for all $b \in \mathcal{B}$.

Therefore the equalities 3.1 and 3.3 imply that

$$\begin{bmatrix} f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}})) & 0 \\ 0 & k_4(1_{\mathcal{B}}) - \eta(k_1(1_{\mathcal{B}})) \end{bmatrix} \in Z(\mathcal{G}), \tag{3.5}$$

therefore Proposition 3.1(1) and the relations 3.1, 3.4 and 3.5, induce that

$$\begin{aligned} h_3(bn) &= bnc = bn(f_1(a) - \eta^{-1}(f_4(a))) = (k_4(1_{\mathcal{B}}) - \eta(k_1(1_{\mathcal{B}})))bn \\ &= (k_4(b) - \eta(k_1(b)))n = k_4(b)n - nk_1(b) \end{aligned}$$

for all $n \in \mathcal{N}$ and $b \in \mathcal{B}$.

By Proposition 3.1(1) and the fact that $c \in Z(\mathcal{A})$ and relations 3.1 and 3.2, we have

$$\begin{aligned} h_3(na) &= nac = n(f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}})))a = n(f_1(a) - \eta^{-1}(f_4(a))) \\ &= nf_1(a) - f_4(a)n \end{aligned}$$

for all $a \in \mathcal{A}$ and $n \in \mathcal{N}$.

From Proposition 3.1(1) and the relations 3.1, 3.4 and 3.5 we have

$$\begin{aligned} g_2(mb) &= cmb = (f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}})))mb = m(k_4(1_{\mathcal{B}}) - \eta(k_1(1_{\mathcal{B}})))b \\ &= m(k_4(b) - \eta(k_1(b))) = mk_4(b) - k_1(b)m, \end{aligned}$$

for all $m \in \mathcal{M}$ and $b \in \mathcal{B}$.

By Proposition 3.1(1) and 3.1 and 3.2 we have

$$g_2(am) = (f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}})))am = (f_1(a) - \eta^{-1}(f_4(a)))m = f_1(a)m - mf_4(a)$$

for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$.

$\Longleftarrow$ At first, the equalities (i, ii) imply Proposition 3.1(2).

From (iii, iv) we have $h_3(bn) = (k_4(b) - \eta(k_1(b))n$ by putting $b = 1_{\mathcal{B}}$ we get $h_3(n) = (k_4(1_{\mathcal{B}}) - \eta(k_1(1_{\mathcal{B}}))n$, then replacing n by bn, we have $h_3(bn) = (k_4(1_{\mathcal{B}}) - \eta(k_1(1_{\mathcal{B}}))bn$ therefore we get

$$(k_4(1_{\mathcal{B}}) - \eta(k_1(1_{\mathcal{B}}))bn = (k_4(b) - \eta(k_1(b))n \tag{3.6}$$

From (iii, iv) we have $h_3(na) = n(f_1(a) - \eta^{-1}(f_4(a))$ by putting $a = 1_{\mathcal{A}}$ we get $h_3(n) = n(f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}})))$, then replacing n by na, we have $h_3(na) = na(f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}})))$ therefore we get

$$na(f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}}))) = n(f_1(a) - \eta^{-1}(f_4(a)) \tag{3.7}$$

From (iii, v) we have $g_2(am) = (f_1(a) - \eta^{-1}(f_4(a)))m$ by putting $a = 1_{\mathcal{A}}$ we get $g_2(m) = (f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}})))m$, then replacing m by am, we have $g_2(am) = (f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}})))am$ therefore we get

$$(f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}})))am = (f_1(a) - \eta^{-1}(f_4(a)))m \tag{3.8}$$

From (iii, v) we have $g_2(mb) = m(k_4(b) - \eta(k_1(b)))$ by putting $b = 1_{\mathcal{B}}$ we get $g_2(m) = m(k_4(1_{\mathcal{B}}) - \eta(k_1(1_{\mathcal{B}})))$, then replacing m by mb, we have $g_2(mb) = mb(k_4(1_{\mathcal{B}}) - \eta(k_1(1_{\mathcal{B}})))$ therefore we get

$$mb(k_4(1_{\mathcal{B}}) - \eta(k_1(1_{\mathcal{B}}))) = m(k_4(b) - \eta(k_1(b))) \tag{3.9}$$

By 3.7 and 3.8 and using weakly faithfulness of $\mathcal{G}$, we have

$$(f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}}))a = f_1(a) - \eta^{-1}(f_4(a))$$

so $\eta^{-1}(f_4(a)) = f_1(a) - (f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}}))a$ and by considering $c = f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}})$ we have

$$\begin{bmatrix} f_1(a) - (f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}}))a & 0 \\ 0 & f_4(a) \end{bmatrix} = \begin{bmatrix} f_1(a) - ca & 0 \\ 0 & f_4(a) \end{bmatrix} \in Z(\mathcal{G})$$

By 3.6 and 3.9 and using weakly faithfulness of $\mathcal{G}$ we have

$$b(k_4(1_{\mathcal{B}}) - \eta(k_1(1_{\mathcal{B}}))) = k_4(b) - \eta(k_1(b))$$

so $\eta(k_1(b)) = k_4(b) - b(k_4(1_{\mathcal{B}}) - \eta(k_1(1_{\mathcal{B}})))$ and by considering $\eta(c) = k_4(1_{\mathcal{B}}) - \eta(k_1(1_{\mathcal{B}}))$ we have

$$\begin{bmatrix} k_1(b) & 0 \\ 0 & k_4(b) - b(k_4(1_{\mathcal{B}}) - \eta(k_1(1_{\mathcal{B}}))) \end{bmatrix} = \begin{bmatrix} k_1(b) & 0 \\ 0 & k_4(b) - b\eta(c) \end{bmatrix} \in Z(\mathcal{G})$$

□

Remark 3.3. In Proposition 3.2, we have

(i) $h_3(n) = k_4(1_{\mathcal{B}})n - nk_1(1_{\mathcal{B}}) = nf_1(1_{\mathcal{A}}) - f_4(1_{\mathcal{A}})n$,
(ii) $h_3(bn) = bh_3(n), h_3(na) = h_3(n)a$,
(iii) $g_2(m) = f_1(1_{\mathcal{A}})m - mf_4(1_{\mathcal{A}}) = mk_4(1_{\mathcal{B}}) - k_1(1_{\mathcal{B}})m$,
(iv) $g_2(mb) = g_2(m)b, g_2(am) = ag_2(m)$,
(v) we can replace the condition Proposition 3.2(iii) by

$$f_4(1_{\mathcal{A}}) \subseteq \pi_{\mathcal{B}}(Z(\mathcal{G})), \qquad k_1(1_{\mathcal{B}}) \subseteq \pi_{\mathcal{A}}(Z(\mathcal{G}))$$

Proof . (i)-(iv) come from the following equalities, respectively.

$$\begin{aligned} h_3(bn) &= bnf_1(1_{\mathcal{A}}) - f_4(1_{\mathcal{A}})bn = b(nf_1(1_{\mathcal{A}}) - f_4(1_{\mathcal{A}})n) = bh_3(n), \\ h_3(na) &= k_4(1_{\mathcal{B}})na - nak_1(1_{\mathcal{B}}) = (k_4(1_{\mathcal{B}})n - nk_1(1_{\mathcal{B}}))a = h_3(n)a, \\ g_2(mb) &= f_1(1_{\mathcal{A}})mb - mbf_4(1_{\mathcal{A}}) = (f_1(1_{\mathcal{A}})m - mf_4(1_{\mathcal{A}}))b = g_2(m)b, \\ g_2(am) &= amk_4(1_{\mathcal{B}}) - k_1(1_{\mathcal{B}})am = a(mk_4(1_{\mathcal{B}}) - k_1(1_{\mathcal{B}})m) = ag_2(m), \end{aligned}$$

To prove (v), let $f_4(1_{\mathcal{A}}) \in \pi_{\mathcal{B}}(Z(\mathcal{G}))$. By Proposition 3.2(v), we have

$$g_2(am) = f_1(a)m - mf_4(a) = (f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}})))am,$$

so

$$(f_1(a) - (f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}})))a)m = mf_4(a). \tag{3.10}$$

for all $m \in \mathcal{M}$.

In addition, from Proposition 3.2(iv), we have

$$h_3(na) = nf_1(a) - f_4(a)n = na(f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}}))),$$

so

$$n(f_1(a) - a(f_1(1_{\mathcal{A}}) - \eta^{-1}(f_4(1_{\mathcal{A}})))) = f_4(a)n. \tag{3.11}$$

for all $a \in \mathcal{A}, n \in \mathcal{N}$.

From 3.10 and 3.11, we get $f_4(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(Z(\mathcal{G}))$. Similarily, it is deduced that $k_1(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(Z(\mathcal{G}))$.

□

Now the proper mappings on weakly faithful generalized matrix algebras are presented in the simplest way.

Theorem 3.4. Let a generalized matrix algebra $\mathcal{G}$ be weakly faithful. A linear mapping $\Phi : \mathcal{G} \to \mathcal{G}$ is proper if and only if Φ is of the following form

$$\Phi\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} f_1(a) + g_1(m) + h_1(n) + k_1(b) & f_1(1_{\mathcal{A}})m - mf_4(1_{\mathcal{A}}) \\ k_4(1_{\mathcal{B}})n - nk_1(1_{\mathcal{B}}) & f_4(a) + g_4(m) + h_4(n) + k_4(b) \end{bmatrix}$$

for all $a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}$ and $b \in \mathcal{B}$ where $f_1 : A \to A, g_1 : \mathcal{M} \to Z(A), h_1 : \mathcal{N} \to Z(A), k_1 : \mathcal{B} \to Z(A), f_4 : A \to Z(\mathcal{B}), g_4 : \mathcal{M} \to Z(\mathcal{B}), h_4 : \mathcal{N} \to Z(\mathcal{B})$ and $k_4 : \mathcal{B} \to \mathcal{B}$ are linear mappings such that

(i) $g_1(m)m' = m'g_4(m), \qquad ng_1(m) = g_4(m)n,$
(ii) $h_1(n)m = mh_4(n), \qquad n'h_1(n) = h_4(n)n',$
(iii) $f_4(1_{\mathcal{A}}) \in \pi_{\mathcal{B}}(Z(\mathcal{G})), \qquad k_1(1_{\mathcal{B}}) \in \pi_{\mathcal{A}}(Z(\mathcal{G})),$
(iv) $k_4(b)n - nk_1(b) = b(k_4(1_{\mathcal{B}})n - nk_1(1_{\mathcal{B}})),$
$nf_1(a) - f_4(a)n = ((k_4(1_{\mathcal{B}})n - nk_1(1_{\mathcal{B}}))a,$
(v) $f_1(a)m - mf_4(a) = a(f_1(1_{\mathcal{A}})m - mf_4(1_{\mathcal{A}})),$
$mk_4(b) - k_1(b)m = (f_1(1_{\mathcal{A}})m - mf_4(1_{\mathcal{A}}))b,$

for all $a \in \mathcal{A}, m, m' \in \mathcal{M}, n \in \mathcal{N}$ and $b \in \mathcal{B}$.

4 Proper commuting mappings

As we know, proper commuting mappings has been studied in many articles, including [5], [6] and [22]. In this section, to show that Theorem 3.4 has many applications, we state it's result for the commuting mappings on the generalized matrix algebras. By considering the characterization [22, proposition 3.3] and imposing some conditions on the generalized matrix algebras and using the Lemma 4.1 and Corollaries 4.3 and 4.5, the conditions of Theorem 3.4 arise, and thus sufficient conditions are obtained for commuting mappins on generalized matrix algebras to be proper, therefore Theorem 3.4 conclude [22, Theorem 3.6].

Lemma 4.1. Let $\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a generalized matrix algebra and $m_0 \in \mathcal{M}, n_0 \in \mathcal{N}$ be available such that

$$Z(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} | a \in Z(\mathcal{A}), b \in Z(\mathcal{B}), am_0 = m_0b, n_0a = bn_0 \right\}.$$

Suppose that linear mappings $r : \mathcal{M} \to Z(\mathcal{A})$, $s : \mathcal{M} \to Z(\mathcal{B})$, $t : \mathcal{N} \to Z(\mathcal{A})$ and $u : \mathcal{N} \to Z(\mathcal{B})$ satisfying in the following equalities

$$r(m)m = ms(m),\ nr(m) = s(m)n,\ t(n)m = mu(n),\ nt(n) = u(n)n,$$

for all $m \in \mathcal{M}, n \in \mathcal{N}$, therefore we get

$$\begin{bmatrix} r(m) & 0 \\ 0 & s(m) \end{bmatrix} \in Z(\mathcal{G}), \qquad \begin{bmatrix} t(n) & 0 \\ 0 & u(n) \end{bmatrix} \in Z(\mathcal{G}),$$

for all $m \in \mathcal{M}, n \in \mathcal{N}$.

Proof . We only prove $\begin{bmatrix} r(m) & 0 \\ 0 & s(m) \end{bmatrix} \in Z(\mathcal{G})$ and the proof of $\begin{bmatrix} t(n) & 0 \\ 0 & u(n) \end{bmatrix} \in Z(\mathcal{G})$ is calculated similarly.Using the definition of $Z(\mathcal{G})$ and the given condition, it suffices to show that $r(m)m_0 = m_0s(m)$, for all $m \in \mathcal{M}$.

According to the assumptions, $r(m_0)m_0 = m_0s(m_0), n_0r(m_0) = s(m_0)n_0$, therefore $\begin{bmatrix} r(m_0) & 0 \\ 0 & s(m_0) \end{bmatrix} \in Z(\mathcal{G})$, so $r(m_0)m = ms(m_0)$, for all $m \in \mathcal{M}$. On the other hand,

$$\begin{aligned} r(m + m_0)(m + m_0) &= (m + m_0)s(m + m_0) \\ &= ms(m) + m_0s(m) + ms(m_0) + m_0s(m_0), \end{aligned}$$

and

$$\begin{aligned} r(m + m_0)(m + m_0) &= r(m)m + r(m)m_0 + r(m_0)m + r(m_0)m_0 \\ &= ms(m) + r(m)m_0 + r(m_0)m + m_0s(m_0), \end{aligned}$$

consequently, $r(m)m_0 = m_0s(m)$ for all $m \in \mathcal{M}$, that is our desired result.

□

Proposition 4.2. Consider $\mathcal{M}$ as a faithful left $\mathcal{A}-$module or $\mathcal{N}$ as a faithful right $\mathcal{A}-$module and we have

1) $a(f_1(1_{\mathcal{A}})m - mf_4(1_{\mathcal{A}})) = f_1(a)m - mf_4(a),$

2) $(nf_1(1_{\mathcal{A}}) - f_4(1_{\mathcal{A}})n)a = nf_1(a) - f_4(a)n$,

for all $a \in \mathcal{A}$ and $m \in \mathcal{M}, n \in \mathcal{N}$, then
$[a, f_1(1_{\mathcal{A}})] \in Z(\mathcal{A})$ for all $a \in \mathcal{A}$, if and only if $[\mathcal{A}, \mathcal{A}] \subseteq f_4^{-1}(\pi_{\mathcal{B}}(Z(\mathcal{G})))$

Proof .

we have
$$a'a(f_1(1_{\mathcal{A}})m - mf_4(1_{\mathcal{A}})) = f_1(a'a)m - mf_4(a'a) = a'(f_1(a)m - mf_4(a)),$$
for all $a, a' \in \mathcal{A}$ and $m \in \mathcal{M}$, then
$$f_1(a'a)m - mf_4(a'a) - a'(f_1(a)m - mf_4(a)) = 0. \tag{4.1}$$
for all $a, a' \in \mathcal{A}$ and $m \in \mathcal{M}$. On the other hand,
$$aa'(f_1(1_{\mathcal{A}})m - mf_4(1_{\mathcal{A}})) = f_1(aa')m - mf_4(aa'),$$
for all $a, a' \in \mathcal{A}$ and $m \in \mathcal{M}$, and
$$a(f_1(1_{\mathcal{A}})a'm - a'mf_4(1_{\mathcal{A}})) = f_1(a)a'm - a'mf_4(a),$$
for all $a, a' \in \mathcal{A}$ and $m \in \mathcal{M}$, imply that
$$a[a', f_1(1_{\mathcal{A}})]m = f_1(aa')m - mf_4(aa') - f_1(a)a'm + a'mf_4(a). \tag{4.2}$$
for all $a, a' \in \mathcal{A}$ and $m \in \mathcal{M}$. The equation 4.1 and 4.2 imply that
$$(a[a', f_1(1_{\mathcal{A}})] - f_1([a, a'] - [a', f_1(1_{\mathcal{A}})]))m = mf_4([a', a]). \tag{4.3}$$
for all $a, a' \in \mathcal{A}$ and $m \in \mathcal{M}$.

In the sequel, we have
$$(nf_1(1_{\mathcal{A}}) - f_4(1_{\mathcal{A}})n)aa' = nf_1(aa') - f_4(aa')n = (nf_1(a) - f_4(a)a'$$
for all $a, a' \in \mathcal{A}$ and $n \in \mathcal{N}$, then
$$nf_1(aa') - f_4(aa')n - (nf_1(a) - f_4(a)a') = 0. \tag{4.4}$$
for all $a, a' \in \mathcal{A}$ and $n \in \mathcal{N}$. On the other hand,
$$(nf_1(1_{\mathcal{A}}) - f_4(1_{\mathcal{A}})n)a'a = nf_1(a'a) - f_4(a'a)n,$$
for all $a, a' \in \mathcal{A}$ and $n \in \mathcal{N}$, and
$$(na'f_1(1_{\mathcal{A}}) - f_4(1_{\mathcal{A}})na')a = na'f_1(a) - f_4(a)na'$$
therefore
$$n[a', f_1(1_{\mathcal{A}})]a = -nf_1(a'a) + f_4(a'a)n + na'f_1(a) - f_4(a)na' \tag{4.5}$$
The equalities 4.4 and 4.5 deduce that
$$n([a', f_1(1_{\mathcal{A}})]a - f_1([a, a'] - [a', f_1(1_{\mathcal{A}})])) = f_4([a', a])n, \tag{4.6}$$

□

By substituting condition weakly faithfulness of $\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ in Proposition 4.2 instead of condition "faithfulness", we will have the following result.

Corollary 4.3. Suppose that a generalized matrix algebra $\mathcal{G}$ is weakly faithful and we have

1) $a(f_1(1_{\mathcal{A}})m - mf_4(1_{\mathcal{A}})) = f_1(a)m - mf_4(a)$,

2) $(nf_1(1_{\mathcal{A}}) - f_4(1_{\mathcal{A}})n)a = nf_1(a) - f_4(a)n$,

3) $[a, f_1(1_{\mathcal{A}})] \in Z(\mathcal{A})$,

for all $a \in \mathcal{A}, m \in \mathcal{M}$ and $n \in \mathcal{N}$, then $[\mathcal{A}, \mathcal{A}] \subseteq f_4^{-1}(\pi_{\mathcal{B}}(Z(\mathcal{G})))$

As Proposition 4.2, the following proposition is proved similarly.

Proposition 4.4. Consider $\mathcal{M}$ as a faithful right $\mathcal{B}-$module or $\mathcal{N}$ as a faithful left $B-$module and we have

1) $k_4(b)n - nk_1(b) = b(k_4(1_{\mathcal{B}})n - nk_1(1_{\mathcal{B}}))$,

2) $mk_4(b) - k_1(b)m = (mk_4(1_{\mathcal{B}}) - k_1(1_{\mathcal{B}})m)b$,

for all $a \in \mathcal{A}$ and $m \in \mathcal{M}, n \in \mathcal{N}$, then
$[b, k_4(1_{\mathcal{B}})] \in Z(\mathcal{A})$ for all $b \in \mathcal{B}$, if and only if $[\mathcal{B}, \mathcal{B}] \subseteq k_1^{-1}(\pi_{\mathcal{A}}(Z(\mathcal{G})))$.

When $\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be weakly faithful, the following corollary is deduced.

Corollary 4.5. Suppose that generalized matrix algebra $\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ is weakly faithful and we have

1) $k_4(b)n - nk_1(b) = b(k_4(1_{\mathcal{B}})n - nk_1(1_{\mathcal{B}}))$,

2) $mk_4(b) - k_1(b)m = (mk_4(1_{\mathcal{B}}) - k_1(1_{\mathcal{B}})m)b$,

3) $[b, k_4(1_{\mathcal{B}})] \in Z(\mathcal{B})$

for all $m \in \mathcal{M}, n \in \mathcal{N}$ and $b \in \mathcal{B}$, then $[\mathcal{B}, \mathcal{B}] \subseteq k_1^{-1}(\pi_{\mathcal{A}}(Z(\mathcal{G})))$.

Let us recall [22, proposition 3.3] in which commuting mappings on generalized matrix algebras are completely characterized, that was shown commuting mappings satisfy the conditions (iv) and (v) of Theorem 3.4.

Proposition 4.6. ([22, proposition 3.3]) Each commuting mapping Φ on generalized matrix algebra $\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ is of the following form

$$\Phi\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} f_1(a) + g_1(m) + h_1(n) + k_1(b) & f_1(1_{\mathcal{A}})m - mf_4(1_{\mathcal{A}}) \\ k_4(1_{\mathcal{B}})n - nk_1(1_{\mathcal{B}}) & f_4(a) + g_4(m) + h_4(n) + k_4(b) \end{bmatrix}$$

for all $a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}$ and $b \in \mathcal{B}$ where $f_1 : A \to A, g_1 : \mathcal{M} \to Z(A), h_1 : \mathcal{N} \to Z(A), k_1 : \mathcal{B} \to Z(A), f_4 : A \to Z(\mathcal{B}), g_4 : \mathcal{M} \to Z(\mathcal{B}), h_4 : \mathcal{N} \to Z(\mathcal{B})$ and $k_4 : \mathcal{B} \to \mathcal{B}$ are linear mappings such that

(i) f_1 and k_4 are commuting mapping on $\mathcal{A}$ and $\mathcal{B}$, repectively.
(ii) $f_1(a)m - mf_4(a) = a(f_1(1_{\mathcal{A}})m - mf_4(1_{\mathcal{A}}))$,
$f_4(a)n - nf_1(a) = (nk_1(1_{\mathcal{B}}) - k_4(1_{\mathcal{B}})n)a$.
(iii) $k_1(b)m - mk_4(b) = (mf_4(1_{\mathcal{A}}) - f_1(1_{\mathcal{A}})m)b$,
$k_4(b)n - nk_1(b) = b(k_4(1_{\mathcal{B}})n - nk_1(1_{\mathcal{B}}))$.
(iv) $g_1(m)m = mg_4(m), \ h_1(n)m = mh_4(n)$.
(v) $h_4(n)n = nh_1(n), \quad g_4(m)n = ng_1(m)$.

By using Lemma 4.1, the conditions (iv) and (v), imply the conditions (i) and (ii) of Theorem 3.4 and since in commuting mappings, $f_1(1_{\mathcal{A}}) \in Z(\mathcal{A})$ and $k_4(1_{\mathcal{B}}) \in Z(\mathcal{B})$, from Corollaries 4.3, 4.5 , we can conclude the following result.

Theorem 4.7. ([22, Theorem 3.6]) Let generalized matrix algebra $\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ is weakly faithful and the following three conditions are met

1. $Z(\mathcal{A}) = \pi_{\mathcal{A}}(Z(\mathcal{G}))$, or $[\mathcal{A}, \mathcal{A}] = \mathcal{A}$.
2. $Z(\mathcal{B}) = \pi_{\mathcal{B}}(Z(\mathcal{G}))$, or $[\mathcal{B}, \mathcal{B}] = \mathcal{B}$.
3. $m_0 \in \mathcal{M}$ and $n_0 \in \mathcal{N}$ are available such that

$$Z(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a \in Z(\mathcal{A}), b \in Z(\mathcal{B}), am_0 = m_0b, n_0a = bn_0 \right\},$$

then each commuting map Φ on $\mathcal{G}$ is proper.

5 Linear mappings satisfying (★)

In this section, we study linear mappings on generalized matrix algebras satisfying (★) and sufficient condition are given under which these mappings are proper. Before proving the next proposition, we present two orthogonal elements of a genealized matrix algebra $\mathcal{G}(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$.

$$\begin{bmatrix} a & am \\ -na & -nam \end{bmatrix}\begin{bmatrix} mbn & mb \\ -bn & -b \end{bmatrix} = \begin{bmatrix} mbn & mb \\ -bn & -b \end{bmatrix}\begin{bmatrix} a & am \\ -na & -nam \end{bmatrix} = 0 \tag{5.1}$$

for all $a \in \mathcal{A}, m \in M, n \in \mathcal{N}$ and $b \in \mathcal{B}$.

In order to examine the properties of the mappings satistying in (★), we will first charaterize the structure of these mappingss in the following statement.

Proposition 5.1. Each linear mapping $\Phi : \mathcal{G} \to \mathcal{G}$ satisfying

$$XY = YX = 0 \Longrightarrow [\Phi(X), Y] \in Z(\mathcal{G}) \qquad (\bigstar)$$

for all $X, Y \in \mathcal{G}$, is of the following form

$$\Phi\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} f_1(a) + g_1(m) + h_1(n) + k_1(b) & g_2(m) \\ h_3(n) & f_4(a) + g_4(m) + h_4(n) + k_4(b) \end{bmatrix}$$

for all $a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}$ and $b \in \mathcal{B}$ where $f_1 : A \to A, g_1 : \mathcal{M} \to Z(A), h_1 : \mathcal{N} \to Z(A), k_1 : \mathcal{B} \to Z(A), g_2 : \mathcal{M} \to \mathcal{M}, h_3 : \mathcal{N} \to \mathcal{N}, f_4 : A \to Z(\mathcal{B}), g_4 : \mathcal{M} \to Z(\mathcal{B}), h_4 : \mathcal{N} \to Z(\mathcal{B})$ and $k_4 : \mathcal{B} \to \mathcal{B}$ are linear mapping such that

(i) $aa' = a'a = 0 \Rightarrow [f_1(a), a'] = 0, \quad bb' = b'b = 0 \Rightarrow [k_4(b), b'] = 0$,

(ii) $g_1(m)m' = m'g_4(m), ng_1(m) = g_4(m)n$.

(iii) $h_1(n)m = mh_4(n), n'h_1(n) = h_4(n)n'$,

(iv) $h_3(bn) = k_4(b)n - nk_1(b), h_3(na) = nf_1(a) - f_4(a)n$,

(v) $g_2(am) = f_1(a)m - mf_4(a), g_2(mb) = mk_4(b) - k_1(b)m$,

(vi) $naf_1(mbn) = namh_3(bn) + f_4(mbn)na, f_1(mbn)am = g_2(mb)nam + amf_4(mbn)$,

(vii) $k_4(nam)bn = bnk_1(nam) + h_3(na)mbn, mbk_4(nam) = k_1(nam)mb + mbng_2(am)$,

(viii) $\begin{bmatrix} f_1(a)mbn - g_2(am)bn - mbnf_1(a) + mbh_3(na) & 0 \\ 0 & -h_3(na)mb + k_4(nam)b + bng_2(am) - bk_4(nam) \end{bmatrix} \in Z(\mathcal{G})$,

(ix) $\begin{bmatrix} f_1(mbn)a - g_2(mb)na - af_1(mbn) + amh_3(bn) & 0 \\ 0 & -h_3(bn)am + k_4(b)nam + nag_2(mb) - namk_4(b) \end{bmatrix} \in Z(\mathcal{G})$,

for all $a, a' \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}$ and $b, b' \in \mathcal{B}$.

Proof . Each linear mapping $\Phi : \mathcal{G} \to \mathcal{G}$ is of the form

$$\Phi\begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} f_1(a) + g_1(m) + h_1(n) + k_1(b) & f_2(a) + g_2(m) + h_2(n) + k_2(b) \\ f_3(a) + g_3(m) + h_3(n) + k_3(b) & f_4(a) + g_4(m) + h_4(n) + k_4(b) \end{bmatrix},$$

where f_i, g_i, h_i and k_i, $1 \le i \le 4$, are linear mappings on $\mathcal{A}, \mathcal{M}, \mathcal{N}$ and $\mathcal{B}$, respectively.

Since $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & m' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & m' \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} = 0$, for all $m, m' \in M$ and the relation (★), we get

$$\left[\begin{bmatrix} g_1(m) & g_2(m) \\ g_3(m) & g_4(m) \end{bmatrix}, \begin{bmatrix} 0 & m' \\ 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} -m'g_3(m) & g_1(m)m' - m'g_4(m) \\ 0 & g_3(m)m' \end{bmatrix} \in Z(\mathcal{G}),$$

therefore $g_1(m)m' = m'g_4(m)$ for all $m, m' \in \mathcal{M}$.

Since $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = 0$, for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, and by (★), we get

$$\left[\begin{bmatrix} f_1(a) & f_2(a) \\ f_3(a) & f_4(a) \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right] = \begin{bmatrix} 0 & f_2(a)b \\ -bf_3(a) & f_4(a)b - bf_4(a) \end{bmatrix} \in Z(\mathcal{G}),$$

$$\left[\begin{bmatrix} k_1(b) & k_2(b) \\ k_3(b) & k_4(b) \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} k_1(b)a - ak_1(b) & -ak_2(b) \\ k_3(b)a & 0 \end{bmatrix} \in Z(\mathcal{G}),$$

therefore $f_2 = f_3 = 0, f_4(\mathcal{A}) \subseteq Z(\mathcal{B})$, and $k_2 = k_3 = 0, k_1(\mathcal{B}) \subseteq Z(\mathcal{A})$, respectively.

Since $\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ n' & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ n' & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} = 0$, for all $n, n' \in \mathcal{N}$, and the relation ($\bigstar$), we get

$$\left[\begin{bmatrix} h_1(n) & h_2(n) \\ h_3(n) & h_4(n) \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ n' & 0 \end{bmatrix}\right] = \begin{bmatrix} h_2(n)n' & 0 \\ h_4(n)n' - n'h_1(n) & n'h_2(n) \end{bmatrix} \in Z(\mathcal{G}),$$

therefore $h_4(n)n' = n'h_1(n)$ for all $n, n' \in \mathcal{N}$.

From 5.1 and ($\bigstar$), we get

$$H = \left[\begin{bmatrix} f_1(a) + g_1(am) + h_1(-na) + k_1(-nam) & g_2(am) + h_2(-na) \\ g_3(am) + h_3(-na) & f_4(a) + g_4(am) + h_4(-na) + k_4(-nam) \end{bmatrix}, \begin{bmatrix} mbn & mb \\ -bn & -b \end{bmatrix}\right] \in Z(\mathcal{G}),$$

thus

$$\begin{aligned} H_{11} &= (f_1(a) + g_1(am) + h_1(-na) + k_1(-nam))mbn - (g_2(am) + h_2(-na))bn \\ &\quad - mbn(f_1(a) + g_1(am) + h_1(-na) + k_1(-nam)) - mb(g_3(am) + h_3(-na)), \\ H_{21} &= (g_3(am) + h_3(-na))mbn - (f_4(a) + g_4(am) + h_4(-na) + k_4(-nam))bn \\ &\quad + bn(f_1(a) + g_1(am) + h_1(-na) + k_1(-nam)) + b(g_3(am) + h_3(-na)), \\ H_{12} &= (f_1(a) + g_1(am) + h_1(-na) + k_1(-nam))mb - (g_2(am) + h_2(-na))b \\ &\quad - mbn(g_2(am) + h_2(-na)) - mb(f_4(a) + g_4(am) + h_4(-na) + k_4(-nam)), \\ H_{22} &= (g_3(am) + h_3(-na))mb - (f_4(a) + g_4(am) + h_4(-na) + k_4(-nam))b \\ &\quad + bn(g_2(am) + h_2(-na)) + b(f_4(a) + g_4(am) + h_4(-na) + k_4(-nam)), \end{aligned}$$

such that $\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \in Z(\mathcal{G})$ which implies that $H_{12} = H_{21} = 0$ and $\begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix} \in Z(\mathcal{G})$.

From $\begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix} \in Z(\mathcal{G})$, we get

$$g_4(\mathcal{M}) \subseteq Z(\mathcal{B}), \qquad h_4(\mathcal{N}) \subseteq Z(\mathcal{B}),$$

$$\begin{bmatrix} f_1(a)mbn - g_2(am)bn - mbnf_1(a) + mbh_3(na) & 0 \\ 0 & -h_3(na)mb + k_4(nam)b + bng_2(am) - bk_4(nam) \end{bmatrix} \in Z(\mathcal{G}).$$

for all $a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}$ and $b \in \mathcal{B}$.

Since $H_{21} = 0$, we have

$$\begin{aligned} k_4(nam)bn &= bnk_1(nam) + h_3(na)mbn, \\ h_3(na) &= nf_1(a) - f_4(a)n, \\ ng_1(m) &= g_4(m)n, \\ g_3 &= 0. \end{aligned}$$

Due to $H_{12} = 0$, we get

$$\begin{aligned} mbk_4(nam) &= k_1(nam)mb + mbng_2(am), \\ g_2(am) &= f_1(a)m - mf_4(a), \\ h_1(n)m &= mh_4(n), \\ h_2 &= 0, \end{aligned}$$

From 5.1 and ($\bigstar$), we get

$$K = \left[\begin{bmatrix} f_1(mbn) + g_1(mb) + h_1(-bn) + k_1(-b) & g_2(mb) \\ h_3(-bn) & f_4(mbn) + g_4(mb) + h_4(-bn) + k_4(-b) \end{bmatrix}, \begin{bmatrix} a & am \\ -na & -nam \end{bmatrix}\right] \in Z(\mathcal{G}),$$

thus

$$\begin{aligned}K_{11} =&(f_1(mbn)+g_1(mb)+h_1(-bn)+k_1(-b))a-g_2(mb)na\\&-a(f_1(mbn)+g_1(mb)+h_1(-bn)+k_1(-b))+amh_3(bn),\\K_{21} =&h_3(-bn)a-(f_4(mbn)+g_4(mb)+h_4(-bn)+k_4(-b))na\\&+na(f_1(mbn)+g_1(mb)+h_1(-bn)+k_1(-b))+namh_3(-bn),\\K_{12} =&(f_1(mbn)+g_1(mb)+h_1(-bn)+k_1(-b))am-g_2(mb)nam\\&-ag_2(mb)-am(f_4(mbn)+g_4(mb)+h_4(-bn)+k_4(-b)),\\K_{22} =&h_3(-bn)am-(f_4(mbn)+g_4(mb)+h_4(-bn)+k_4(-b))nam\\&+nag_2(mb)+nam(f_4(mbn)+g_4(mb)+h_4(-bn)+k_4(-b)),\end{aligned}$$

such that $\begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix}\in Z(\mathcal{G})$ which implies that $K_{12}=K_{21}=0$ and $\begin{bmatrix}K_{11} & 0\\ 0 & K_{22}\end{bmatrix}\in Z(\mathcal{G})$.

From $\begin{bmatrix}K_{11} & 0\\ 0 & K_{22}\end{bmatrix}\in Z(\mathcal{G})$, we get

$$g_1(\mathcal{M})\subseteq Z(\mathcal{A}),\qquad h_1(\mathcal{N})\subseteq Z(\mathcal{A}),$$

$$\begin{bmatrix}f_1(mbn)a-g_2(mb)na-af_1(mbn)+amh_3(bn) & 0\\ 0 & -h_3(bn)am+k_4(b)nam+nag_2(mb)-namk_4(b)\end{bmatrix}\in Z(\mathcal{G}),$$

for all $a\in\mathcal{A}, m\in\mathcal{M}, n\in\mathcal{N}$ and $b\in\mathcal{B}$.

Since $K_{21}=0$, we have

$$\begin{aligned}naf_1(mbn)&=f_4(mbn)na+namh_3(bn),\\h_3(bn)&=k_4(b)n-nk_1(b),\end{aligned}$$

for all $a\in\mathcal{A}, m\in\mathcal{M}, n\in\mathcal{N}$ and $b\in\mathcal{B}$.

Due to $K_{12}=0$, we get

$$\begin{aligned}f_1(mbn)am&=g_2(mb)nam+amf_4(mbn),\\g_2(mb)&=mk_4(b)-k_1(b)m,\end{aligned}$$

for all $a\in\mathcal{A}, m\in\mathcal{M}, n\in\mathcal{N}$ and $b\in\mathcal{B}$. □

From Proposition 3.2 and Proposition 5.1, the following theorem is deduced for each linear mapping $\Phi:\mathcal{G}\to\mathcal{G}$ satisfying in (★).

Theorem 5.2. Let generalized matrix algebra $\mathcal{G}(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ is weakly faithful. Let $\Phi:\mathcal{G}\to\mathcal{G}$ be a linear mapping which satisfies in (★) and is as follows

$$\Phi\left(\begin{bmatrix}a & m\\ n & b\end{bmatrix}\right)=\begin{bmatrix}f_1(a)+g_1(m)+h_1(n)+k_1(b) & f_1(1_{\mathcal{A}})m-mf_4(1_{\mathcal{A}})\\ nf_1(1_{\mathcal{A}})-f_4(1_{\mathcal{A}})n & f_4(a)+g_4(m)+h_4(n)+k_4(b)\end{bmatrix},$$

for all $a\in\mathcal{A}, m\in\mathcal{M}, n\in\mathcal{N}$ and $b\in\mathcal{B}$. Then Φ is proper if and only if $k_1(1_{\mathcal{B}})\in\pi_{\mathcal{A}}(Z(\mathcal{G})), f_4(1_{\mathcal{A}})\in\pi_{\mathcal{B}}(Z(\mathcal{G}))$.

Now, we are ready to present some conditions on generalized matrix algebra $\mathcal{G}(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ under which each linear mapping satisfying (★) is proper.

Theorem 5.3. Let generalized matrix algebra $\mathcal{G}(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ be weakly faithful and $\pi_{\mathcal{A}}(Z(\mathcal{G}))=Z(\mathcal{G})$ and $\pi_{\mathcal{B}}(Z(\mathcal{G}))=Z(\mathcal{B})$. Then each linear mapping $\Phi:\mathcal{G}\to\mathcal{G}$ satisfying (★) is proper.

6 Some applications

Each linear mapping sitisfying $XY=YX=0\Longrightarrow[\Phi(X),Y]=0$ for all $x,y\in\mathcal{G}$, holds in (★), so Theorem 5.1 can get the following result.

Theorem 6.1. Suppose that generalized matrix algebra $\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ is weakly faithful. Let $\Phi : \mathcal{G} \to \mathcal{G}$ be a linear mapping which satisfies

$$XY = YX = 0 \Longrightarrow [\Phi(X), Y] = 0$$

and is as follows

$$\Phi\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} f_1(a) + g_1(m) + h_1(n) + k_1(b) & f_1(1_{\mathcal{A}})m - mf_4(1_{\mathcal{A}}) \\ nf_1(1_{\mathcal{A}}) - f_4(1_{\mathcal{A}})n & f_4(a) + g_4(m) + h_4(n) + k_4(b) \end{bmatrix},$$

for all $a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}$ and $b \in \mathcal{B}$. Then the followings are equivalent:

1) Φ is proper.
2) $k_1(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(Z(\mathcal{G})), f_4(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(Z(\mathcal{G}))$;
3) $k_1(1_{\mathcal{B}}) \in \pi_{\mathcal{A}}(Z(\mathcal{G})), f_4(1_{\mathcal{A}}) \in \pi_{\mathcal{B}}(Z(\mathcal{G}))$.

Consequently, the following result is deduced.

Theorem 6.2. Let generalized matrix algebra $\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be weakly faithful and $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$. Then each linear mapping $\Phi : \mathcal{G} \to \mathcal{G}$ satisfies

$$XY = YX = 0 \Longrightarrow [\Phi(X), Y] = 0$$

if and only if Φ is proper.

Since each Lie centralizer is true in (★) and also has the conditions of Corollaries 4.3 and 4.5, the following result can be obtained which is an extension of [15, Theorem 3.4].

Corollary 6.3. Let generalized matrix algebra $\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be weakly faithful and we have

1. $Z(\mathcal{A}) = \pi_{\mathcal{A}}(Z(\mathcal{G}))$, or $[\mathcal{A}, \mathcal{A}] = \mathcal{A}$,
2. $Z(\mathcal{B}) = \pi_{\mathcal{B}}(Z(\mathcal{G}))$, or $[\mathcal{B}, \mathcal{B}] = \mathcal{B}$,

then each Lie centralizer mapping $\Phi : \mathcal{G} \to \mathcal{G}$ is proper.

In the sequel, we deal with the proper form of Lie triple centralizer mappings. To present the condition that a proper mapping should have to be a Lie triple centralizer, the following lemma is given which is easily proved.

Lemma 6.4. Consider a proper linear mapping $\Phi : G \to G$, i.e. there exist $\lambda \in Z(G)$ and linear mapping $\tau : G \to Z(G)$ such that $\Phi(x) = \lambda x + \tau(x)$ for all $x \in G$. Φ is Lie triple centralizer if and only if $\tau([[x, y], z]) = 0$, for each $x, y, z \in G$.

Each Lie triple centralizer satisfies in (★), therefore the next corollary is concluded from Theorem 5.2 as follows.

Corollary 6.5. ([8, Theorem 3.3]) Suppose that generalized matrix algebra $\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ is weakly faithful. Each Lie triple centralizer $\Phi : \mathcal{G} \to \mathcal{G}$ is as follows

$$\Phi\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} f_1(a) + g_1(m) + h_1(n) + k_1(b) & f_1(1_{\mathcal{A}})m - mf_4(1_{\mathcal{A}}) \\ nf_1(1_{\mathcal{A}}) - f_4(1_{\mathcal{A}})n & f_4(a) + g_4(m) + h_4(n) + k_4(b) \end{bmatrix},$$

for all $a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}$ and $b \in \mathcal{B}$. Then the followings are equivalent:

1) Φ is proper.
2) $k_1(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(Z(\mathcal{G})), f_4(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(Z(\mathcal{G}))$;
3) $k_1(1_{\mathcal{B}}) \in \pi_{\mathcal{A}}(Z(\mathcal{G})), f_4(1_{\mathcal{A}}) \in \pi_{\mathcal{B}}(Z(\mathcal{G}))$.

Since Lie triple centralizers satisfy in $[a, f_1(1_{\mathcal{A}})] \in Z(\mathcal{A})$ and $[b, k_4(1_{\mathcal{B}})] \in Z(\mathcal{B})$, for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, from Corollaries 4.3, 4.5 and 6.5, we can conclude the following result.

Corollary 6.6. ([8, Corollary 3.4]) Let a generalized matrix algebra $\mathcal{G}(\mathcal{A}, \mathcal{M},$
$\mathcal{N}, \mathcal{B})$ is weakly faithful and we have

1. $Z(\mathcal{A}) = \pi_{\mathcal{A}}(Z(G))$, or $[\mathcal{A}, \mathcal{A}] = \mathcal{A}$.
2. $Z(\mathcal{B}) = \pi_{\mathcal{B}}(Z(G))$, or $[\mathcal{B}, \mathcal{B}] = \mathcal{B}$.

Then each Lie triple centralizer $\Phi : \mathcal{G} \to \mathcal{G}$ is proper Lie triple centralizer.

One knows that generalized matrix algebras contain the $n \times n$ upper triangular matrix algebras and nest algebras, according to what Cheung has given in [6], we can use Theorem 4.7 for this kind of algebras.

Corollary 6.7. ([6, Corollary 6]) Each commuting mapping on the $n \times n$ upper triangular matrix algebras is proper.

Corollary 6.8. ([6, Corollary 7]) Each commuting mapping on the nest algebras is proper.

Similarly, by using Corollary 6.6, we can obtain the following corollaries.

Corollary 6.9. Each Lie triple centralizer on the $n \times n$ upper triangular matrix algebras is proper.

Corollary 6.10. Each Lie triple centralizer on the nest algebras is proper.

Acknowledgment

The authors would like to acknowledge the financial support of University of Birjand for this research under contract number 1399/D/15693.

References

[1] A. Barari, B. Fadaee and H. Ghahramani, *Linear maps on standard operator algebras characterized by action on zero products*, Bull. Iran. Math. Soc. **45** (2019), 1573–1583.

[2] B. Behfar and H. Ghahramani, *Lie maps on triangular algebras without assuming unity*, Mediterr. J. Math. **18** (2021), 215.

[3] D. Benkovič, *Lie triple derivations of unital algebras with idempotents*, Linear Multilinear Algebra **63** (2015), 141–165.

[4] D. Benkovič and N. Širovnik, *Jordan derivations of unital algebras with idempotents*, Linear Algebra Appl. **437** (2012), 2271–2284.

[5] M. Brešar, *Commuting maps: A survey*, Taiwanese J. Math. **8** (2004), 361–397.

[6] W.S. Cheung, *Commuting maps of triangular algebras*, J. London Math. Soc. **63** (2001), 117–127.

[7] B. Fadaee and H. Ghahramani, *Linear maps on C^*-algebras behaving like (anti-)derivations at orthogonal elements*, Bull. Malays. Math. Sci. Soc. **43** (2020), 2851–2859.

[8] B. Fadaee and H. Ghahramani, *Lie centralizers at the zero products on generalized matrix algebras*, J. Alg. Appl. **21** (2022), no. 8.

[9] A. Fošner and W. Jing, *Lie centralizers on triangular rings and nest algebras*, Adv. Oper. Theory **4** (2019), no. 2, 342–350.

[10] H. Ghahramani, *Additive mappings derivable at nontrivial idempotents on Banach algebras*, Linear Multilinear Algebra **60** (2012), 725–742.

[11] H. Ghahramani, *Linear maps on group algebras determined by the action of the derivations or anti-derivations on a set of orthogonal elements*, Results Math. **73** (2018), 132–146.

[12] H. Ghahramani, *Characterizing Jordan maps on triangular rings through commutative zero products*, Mediterr. J. Math. **15** (2018), no. 2, 1–10.

[13] H. Ghahramani and W. Jing, *Lie centralizers at zero products on a class of operator algebras*, Ann. Funct. Anal. **12** (2021), 1–12.

[14] F. Ghomanjani and M.A. Bahmani, *A note on Lie centralizer maps*, Palest. J. Math. **7** (2018), no. 2, 468–471.

[15] A. Jabeen, *Lie (Jordan) centralizers on generalized matrix algebras*, Commun. Algebra 49 (2021), no. 1, 278–291.

[16] Y. Li, F. Wei and A. Fošner, *k-commuting mappings of generalized matrix algebras*, Period. Math. Hungar. **79** (2019), no. 1, 50–77.

[17] L. Liu, *On nonlinear Lie centralizers of generalized matrix algebras*, Linear Multilinear Algebra **70** (2020), 2693–2705.

[18] C.R. Miers, *Lie triple derivations of von Neumann algebras*, Proc. Amer. Math. Soc. **71** (1978), 57–61.

[19] A.H. Mokhtari and H.R. Ebrahimi Vishki, *More on Lie derivations of generalized matrix algebras*, Miskolc Math. Notes, **1** (2018), 385–396.

[20] K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*, Rep. Tokyo Kyoiku Diagaku Sect. A **6** (1958), 83–142.

[21] A.D. Sands, *Radicals and Morita contexts*, J. Algebra **24** (1973), no. 2, 335–345.

[22] Z.K. Xiao and F. Wei, *Commuting mappings of generalized matrix algebras*, Linear Algebra Appl. **433** (2010), 2178–2197.

[23] Z. Xiao and F. Wei, *Lie triple derivations of triangular algebras*, Linear Algebra Appl. **437** (2012), no. 5, 1234–1249.