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Global existence, decay and blow up of solutions for a quasilinear hyperbolic equations with source terms

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Abstract

In this paper, we consider the boundary value problem related to the quasilinear hyperbolic equation with nonlinear source terms. We start by showing the local existence theorem. Then, we prove the global existence and the decay of the energy by using Nakao's inequality, finally we get the finite time blow up of solutions.

Keywords: blow up, energy function, quasilinear hyperbolic equation, source terms
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1 Introduction

In this work, we study the following value problem related to the quasilinear hyperbolic equation with nonlinear source terms:

$$\begin{cases} u_{tt} - |\nabla u| div\big(\alpha(x)\dfrac{\nabla u}{|\nabla u|}\big) - \Delta u_t + |u_t|^{q-1}u_t = |u|^{p-1}u, & x \in \Omega, t > 0, \\ u(t,x) = 0, & x \in \partial\Omega, t > 0, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Where Ω is bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$
$(n \geq 1)$, $p, q \geq 1$ and $\alpha(x)$ is is a stricty positive and continuously differentiable function in Ω.
Problems of this type arise in physics (for example, we represents the purely longitudinal motion of a viscoelastic configuration. (see[1])) and in image processing.
The following equation

$$u_{tt} - \Delta u + |u_t|^{m-2}u_t = |u|^{p-2}u. \tag{1.2}$$

Has been extensively studied by many authors. It is well known that in the absence of the damping mechanism $|u_t|^{m-2}u_t$, the source term $|u|^{p-2}u$ causes finite time blow up of solutions with the negative initial energy (see [2, 7]). On the other hand, in [4, 8] the damping assures global existence for arbitrary initial data in the absence of the source terms.
The interaction between the damping term $|u_t|^{m-2}u_t$ and the source terms $|u|^{p-2}u$ was showed in [10, 9] by Levine for linear damping u_t. Levine studied the global nonexistence solutions when the initial energy is negative. In [6] Georgiev

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and Todorova developed the result of Levine with nonlinear damping ($m > 2$). Many authors have introduced a new method and related between m and p in their work, for which there is the global existence and the other relations between m and p for which there is finite time blow up. Particulary, they showed that if $m \geq p$ then, the solutions with negative energy continue to exist globally and blow-up finite-time if $p > m$ and the initial energy is sufficiently negative. This result has been provesly generalized to unbounded domains and to an abstract setting in [11, 12]. In these works, the authors showed that no solution can be extended on $[0, \infty)$ with negative initial energy if $p > m$ and proved with noncontinuation theorems. This generalization allowed them to apply their noncontinuation results to quasilinear situations in the particular case of which the result of [6].
In the presence of the strong damping term Δu_t the following semilinear wave equation

$$u_{tt} - \Delta u - \omega \Delta u_t + \mu |u_t|^{q-1} u_t = |u|^{p-1} u, \tag{1.3}$$

studied by Yu in [22]. In the case when ($q = 1$) Gazzola and Squassina in [5] studied the global existence and blow up of solutions. Then when $\omega = 1$ and $\mu = 1$, Chen and Liu was studied in [3] the global existence, decay and exponential growth of solutions.
Many researchers have investigated to show the existence, blow up and asymptotic behavior of solutions of the (1.3) (see [14, 18, 20, 17, 19]).
Messaoudi extended the blow up of solutions results of [6] in [14] with the negative initial energy. Then he studies decay of solutions by using the techniques combination of the perturbed energy and potential well methods in [15]. In [19], Wu and Xue proved the uniform energy decay rates of solutions by utilizing the multiplier method.
In [13], the authors showed the blow up of solutions under some restriction for a system of semilinear wave equation and they gave an estimate for the blow up time T, this result allowed them also to apply their theorem to quasilinear situations, of which problem (1.1) is a particular case. So, we prove the same result of [13] in the case of a quasilinear hyperbolic equation with strong damping and source terms.

In this work, firstly we show the local and we study the global existance of solutions of the problem (1.1). Secondly, we show the finite time blow up of solutions with negative and positive initial energy by using the same techniques in [13] and [17].
This paper is organized as follows: In the section 2, we present some lemmas and we show the local existence theorem. In the section 3, we prove the global existence and decay of solutions and we give example. In the section 4, we show and prove the blow up of solutions with negative and positive initial energy in the case : ($q = 1$).

2 Preliminaries

In this section, we give some lemmas and assumptions which we will be used in this paper.
Let $||.||_2$ and $||.||_p$ denote the usual $L^2(\Omega)$ and $L^p(\Omega)$ norm, respectively, and $||.||_{\mathcal{D}_0^{1,2}}$ denote the $\mathcal{D}_0^{1,2}(\Omega, \alpha)$ norm.
We introduce the weighted Sobolev space $\mathcal{D}_0^{1,2}(\Omega, \alpha)$ defined as the closure of C_0^∞ in the norm

$$||u||_{\mathcal{D}_0^{1,2}(\Omega,\alpha)} = \left(\int_\Omega \alpha(x) |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

Lemma 2.1. [16] Let $\phi(t)$ be a nonincreasing and nonnegative function defined on $[0, T]$, $T > 1$, satisfying

$$\phi^{1+\gamma}(t) \leq w_0(\phi(t) - \phi(t+1)), \quad t \in [0, T].$$

For w_0 is a positive constant and γ is a nonnegative constant. Then we have, for each $t \in [0, T]$,

$$\begin{cases} \phi(t) \leq \phi(0) e^{-w_1[t-1]^+}, & \gamma = 0, \\ \phi(t) \leq (\phi(0)^{-\gamma} + w_0^{-1}\gamma[t-1]^+)^{-\frac{1}{\gamma}}, & \gamma > 0, \end{cases}$$

where $[t-1]^+ = \max\{t-1, 0\}$ and $w_1 = \ln(\frac{w_0}{w_0 - 1})$.

Lemma 2.2. [13] Let us have $\delta > 0$ and let $B(t) \in C^2(0, \infty)$ be a nonnegative function satisfying

$$B''(t) - 4(\delta+1)B'(t) + 4(\delta+1)B(t) \geq 0. \tag{2.1}$$

If

$$B'(0) > r_2 B(0) + K_0, \tag{2.2}$$

with $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$, then $B'(t) > K_0$ for $t > 0$, where K_0 is a constant.

Lemma 2.3. [13] If $H(t)$ is a nonincreasing function on $[t_0,\infty)$ and satisfies the differential inequality

$$[H'(t)]^2 \geq a + b[H(t)]^{2+\frac{1}{b}} \quad \text{for } t \geq t_0, \tag{2.3}$$

where $a>0, b\in R$, then there exists a finite time T^* such that

$$\lim_{t\to T^{*-}} H(t) = 0.$$

Upper bounds for T^* are estimated as follows:

1. If $b<0$ and $H(t_0) < \min\left\{1, \sqrt{-\frac{a}{b}}\right\}$ then
$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{-\frac{a}{b}}}{\sqrt{-\frac{a}{b}} - H(t_0)}.$$
2. If $b=0$, then
$$T^* \leq t_0 + \frac{H(t_0)}{H'(t_0)}.$$
3. If $b>0$, then
$$T^* \leq \frac{H(t_0)}{\sqrt{a}} \quad \text{or} \quad T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}}\left[1-(1+cH(t_0))^{-\frac{1}{2\delta}}\right],$$
where $c = (\frac{a}{b})^{2+\frac{1}{\delta}}$.

Next, we show the local existence theorem which can be estableshed by [6] and [21].

Theorem 2.4 (Local existence). We suppose that: $2 < p+1 < \dfrac{2n}{n-2}$, $u_0 \in \mathcal{D}_0^{1,2}(\Omega)$ and $u_1 \in L^2(\Omega)$ then, the problem (1.1) has a unique local solution:

$$u \in C\big([0,T); \mathcal{D}_0^{1,2}(\Omega)\big) \quad \text{and} \quad u_t \in C\big([0,T); L^2(\Omega)\big) \cap L^{q+1}\big([0,T)\times\Omega\big).$$

Furthermore, one of the following statements holds true:

1. $T=\infty$,
2. $||u_t||_2^2 + ||u||^2_{\mathcal{D}_0^{1,2}} \longrightarrow \infty \quad \text{as} \quad t \longrightarrow T^-$.

3 Global existence results

In this section, we study the global existence and the decay of the solution for problem (1.1). We define

$$J(t) = \frac{1}{2}\int_\Omega \alpha(x).\nabla u \nabla u dx - \frac{1}{p+1}||u||_{p+1}^{p+1}, \tag{3.1}$$

and

$$I(t) = \int_\Omega \alpha(x)\nabla u \nabla u dx - ||u||_{p+1}^{p+1}. \tag{3.2}$$

We also define the energy function

$$E(t) = \frac{1}{2}||u_t||^2 + \frac{1}{2}\int_\Omega \alpha(x)\nabla u \nabla u dx - \frac{1}{p+1}||u||_{p+1}^{p+1}, \tag{3.3}$$

and we denote the Nihari space

$$W = \{u : u \in \mathcal{D}_0^{1,2}(\Omega), \quad I(u) > 0\} \cup \{0\}. \tag{3.4}$$

Next, we show the energy function (3.3) which is a nonincreasing function along the solution of (1.1) by the next lemma:

Lemma 3.1. For $t \geq 0$, $E(t)$ is a nonincreasing function and

$$E^{'}(t) = -(||u_t||_{q+1}^{q+1} + ||\nabla u_t||^2) \leq 0. \tag{3.5}$$

Proof . Multiplying the first equation of (1.1) by u_t and integrating over Ω, by using integrating by parts, we get

$$E(t)-E(0)=-\int_0^t ||u_\tau||_{q+1}^{q+1}d\tau-\int_0^t ||\nabla u_\tau||_2^2 d\tau \quad \text{for} \quad t\geq 0. \tag{3.6}$$

□

Lemma 3.2. Let $u_0 \in W$, $u_1 \in L^2(\Omega)$ and we suppose that $(p>1)$ and

$$\beta = C_*\Big(\frac{2(p+1)}{(p-1)}E(0)\Big)^{\frac{p-1}{2}} < 1, \tag{3.7}$$

then, for each $t \geq 0$, $u \in W$.

Proof . For $I(0)>0$ and by the continuity of $u(t)$ so: $I(t)>0$, for some interval near $t=0$. Let $T_m>0$ the maximal time, when (3.2) holds on $[0,T_m]$.
From (3.1) and (3.2), we get

$$J(t)=\frac{1}{p+1}I(t)+\frac{p-1}{2(p+1)}\int_\Omega \alpha(x)\nabla u\nabla u dx, \tag{3.8}$$

by $I(t)>0$, we have

$$J(t)\geq \frac{(p-1)}{2(p+1)}||u||^2_{\mathcal{D}_0^{1,2}}, \tag{3.9}$$

then from $E(t)$ and $E^{'}(t)$, we obtain

$$\begin{aligned} ||u||^2_{\mathcal{D}_0^{1,2}} &\leq \frac{2(p+1)}{(p-1)}J(t)\\ &\leq \frac{2(p+1)}{(p-1)}E(t)\\ &\leq \frac{2(p+1)}{(p-1)}E(0). \end{aligned} \tag{3.10}$$

By using Lemma 3.2 and (3.10), we get

$$\begin{aligned} ||u||_{p+1}^{p+1} &\leq C_*||u||^{p+1}_{\mathcal{D}_0^{1,2}}\\ &= C_*||u||^{p-1}_{\mathcal{D}_0^{1,2}}||u||^2_{\mathcal{D}_0^{1,2}}\\ &\leq C_*\big(\frac{2(p+1)}{(p-1)}E(0)\big)^{\frac{p-1}{2}}||u||^2_{\mathcal{D}_0^{1,2}}\\ &= \beta||u||^2_{\mathcal{D}_0^{1,2}}\\ &< ||u||^2_{\mathcal{D}_0^{1,2}} \quad \forall t\in[0,T_m], \end{aligned} \tag{3.11}$$

we conclude by (3.3) that $I(t)>0$, $\forall\ t\in[0,T_m]$.
When we repeat the procedure, T_m is extended to T. So the proof is completed. □

Lemma 3.3. If the assumptions of the Lemma 3.2 hold, so there exists $\eta_1=1-\beta$ such that

$$||u||_{p+1}^{p+1}\leq(1-\eta_1)||u||^2_{\mathcal{D}_0^{1,2}}.$$

Proof . We have

$$||u||_{p+1}^{p+1}\leq \beta||u||^2_{\mathcal{D}_0^{1,2}},$$

when, we put $\eta_1=1-\beta$, we obtain the following results. □

Remark 3.4. We have

$$||u||_{p+1}^{p+1}\leq(1-\eta_1)||u||^2_{\mathcal{D}_0^{1,2}},$$

then, we can deduce that

$$||u||^2_{\mathcal{D}_0^{1,2}}\leq\frac{1}{\eta_1}I(t). \tag{3.12}$$

Theorem 3.5. Let $u_0 \in W$ satisfying Lemma 3.2, and we suppose that $2 < p+1 < \dfrac{2n}{n-2}$, $n > 2$ holds, then the solution of problem (1.1) is global.

Proof . We have

$$\begin{aligned} E(0) \geq E(t) &= \frac{1}{2}||u_t||_2^2 + \frac{1}{2}\int_\Omega \alpha(x)\nabla u \nabla u dx - \frac{1}{p+1}||u||_{p+1}^{p+1} \\ &= \frac{1}{2}||u_t||_2^2 + \frac{p-1}{2(p+1)}\int_\Omega \alpha(x)\nabla u\nabla u dx + \frac{1}{p+1}I(t) \\ &\geq \frac{1}{2}||u_t||_2^2 + \frac{(p-1)}{2(p+1)}||u||^2_{\mathcal{D}_0^{1,2}}, \end{aligned}$$

since $I(t) \geq 0$, so

$$||u_t||_2^2 + ||u||^2_{\mathcal{D}_0^{1,2}} \leq CE(0),$$

where $C = \max\{2; \frac{2(p+1)}{(p-1)}\}$. By using Theorem 2.4, we obtain the global existence result. □

Theorem 3.6. Let $u_0 \in W$, we suppose that $2 < p+1 < \frac{2n}{n-2}$, $n > 2$ and (3.7) holds.
So, we have the following decay estimates:

$$E(t) \leq \begin{cases} E(0)e^{-w_0[t-1]^+}, & \text{if } q = 1, \\ \left(E(0)^{-\lambda} + C_7^{-1}\lambda[t-1]^+\right)^{-\frac{1}{\lambda}}, & \text{if } q > 1, \end{cases}$$

where w_0, λ, and C_7 are positive constants which will be defined later.

Proof . By integrating $E'(t)$ over $[t, t+1]$, $t > 0$, we get

$$E(t) - E(t+1) = D^{q+1}(t), \tag{3.13}$$

where

$$D^{q+1}(t) = \int_t^{t+1} \left(\|u_\tau\|_{q+1}^{q+1} + \|\nabla u_\tau\|_2^2\right) d\tau. \tag{3.14}$$

By using $D^{q+1}(t)$ and Hölder's inequality, we observe that

$$\int_t^{t+1}\int_\Omega |u_t|^2 \, dxdt \leq C_0 D^2(t), \tag{3.15}$$

where $C_0 > 0$.

Then, there exists $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$||u_t(t_i)||_2 \leq CD(t), \quad i = 1, 2. \tag{3.16}$$

We multiply the first equation of (1.1) by u and integrate over $\Omega \times [t_1, t_2]$, and by integration by parts, we obtain

$$\begin{aligned} \int_{t_1}^{t_2} I(t)dt = -\Bigg[&\int_{t_1}^{t_2}\int_\Omega u.u_{tt}dxdt + \int_{t_1}^{t_2}\int_\Omega \nabla u_t \nabla u dxdt \\ &+ \int_{t_1}^{t_2}\int_\Omega |u_t|^{q-1}u_t u dxdt\Bigg]. \end{aligned} \tag{3.17}$$

Now, we use (1.1) and integrate by parts then apply the Cauchy-Schwarz inequality in the first term and we use Hölder inequality in the second term, we get

$$\begin{aligned} \int_{t_1}^{t_2} I(t)dt \leq\ & ||u_t(t_1)||_2||u(t_1)||_2 + ||u_t(t_2)||_2||u(t_2)||_2 \\ &+ \int_{t_1}^{t_2} ||u_t(t)||_2^2 dt + \int_{t_1}^{t_2} ||\nabla u_t||_2||\nabla u||_2 dt \\ &- \int_{t_1}^{t_2}\int_\Omega |u_t|^{q-1}u_t u dxdt. \end{aligned} \tag{3.18}$$

Our goal, now is to estimate the last term in the inequality. By using Hölder inequality, we obtain

$$\int_{t_1}^{t_2}\int_{\Omega}|u_t|^{q-1}u_t u dxdt \leq \int_{t_1}^{t_2}||u_t||_{q+1}^{q}||u(t)||_{q+1}dt. \tag{3.19}$$

After that, by using (3.10), we obtain

$$\begin{aligned}\int_{t_1}^{t_2}||u_t(t)||_{q+1}^{q}||u(t)||_{q+1}dt &\leq C_*\int_{t_1}^{t_2}||u_t(t)||_{q+1}^{q}||u||_{\mathcal{D}_0^{1,2}}dt\\ &\leq C_*\int_{t_1}^{t_2}||u_t(t)||_{q+1}^{q}||u||_{\mathcal{D}_0^{1,2}}^{2}dt\\ &\leq C_*\big(\frac{2(p+1)}{(p-1)}E(0)\big)^{\frac{1}{2}}\int_{t_1}^{t_2}||u_t(t)||_{q+1}^{q}E^{\frac{1}{2}}(s)dt\\ &\leq C_*\big(\frac{2(p+1)}{(p-1)}E(0)\big)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{2}}(s)D^{q}(t). \end{aligned}\tag{3.20}$$

Next, we estimate the fourth term of the right hand side of (3.18), we find

$$\begin{aligned}\int_{t_1}^{t_2}||\nabla u_t||_2||\nabla u||_2dt &\leq C_*\int_{t_1}^{t_2}||\nabla u_t||_2||u||_{\mathcal{D}_0^{1,2}}^{2}dt\\ &\leq C_*\big(\frac{2(p+1)}{(p-1)}E(0)\big)^{\frac{1}{2}}\int_{t_1}^{t_2}||\nabla u_t||_2E^{\frac{1}{2}}(s)dt\\ &\leq C_*\big(\frac{2(p+1)}{(p-1)}E(0)\big)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{2}}(s)\int_{t_1}^{t_2}||\nabla u_t||_2dt,\end{aligned}$$

then

$$\begin{aligned}\int_{t_1}^{t2}\|\nabla u_t\|_2\,dt &\leq \left(\int_{t_1}^{t_2}1dt\right)^{\frac{1}{2}}\left(\int_{t_1}^{t_2}\|\nabla u_t\|_2^2\,dt\right)^{\frac{1}{2}}\\ &\leq CD(t).\end{aligned}$$

After that

$$\int_{t_1}^{t_2}||\nabla u_t||_2||\nabla u||_2dt \leq CC_*\left(\frac{2(p+1)}{(p-1)}E(0)\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{2}}(s)D(t), \tag{3.21}$$

so, we have

$$||u_t\,(t_i)\,||_2||u\,(t_i)\,||_2 \leq C_1D(t)\sup_{t_1\leq s\leq t_2}E^{\frac{1}{2}}(s), \tag{3.22}$$

with $C_1 = 2C_*\left(\frac{2(p+1)}{(p-1)}E(0)\right)^{\frac{1}{2}}$.

Then

$$\begin{aligned}\int_{t_1}^{t_2}I(t)dt \leq &C_1D(t)\sup_{t_1\leq s\leq t_2}E^{\frac{1}{2}}(s)+D^2(t)\\ &+CC_*\left(\frac{2(p+1)}{(p-1)}E(0)\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{2}}(s)D(t)\\ &+C_*\left(\frac{2(p+1)}{(p-1)}E(0)\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E^{\frac{1}{2}}(s)D^{q}(t).\end{aligned}\tag{3.23}$$

On the other hand, we have

$$E(t) \leq \frac{1}{2}\|u_t\|_2^2 + C_3I(t), \tag{3.24}$$

with $C_3 = \frac{1}{\eta_1}\frac{(p-1)}{2(p+1)} + \frac{1}{p+1}$.

By integration over $[t_1, t_2]$, we have

$$\int_{t_1}^{t_2}E(t)dt \leq \frac{1}{2}\int_{t_1}^{t_2}\|u_t\|_2^2\,dt + C_3\int_{t_1}^{t_2}I(t)dt. \tag{3.25}$$

Then, we obtain

$$\begin{aligned}\int_{t_1}^{t_2} E(t)dt \leq & \frac{1}{2}CD^2(t) + C_3\left[C_1D(t)\sup_{t_1\leq s\leq t_2} E^{\frac{1}{2}}(s) + D^2(t)\right.\\ & + CC_*\left(\frac{2(p+1)}{(p-1)}E(0)\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2} E^{\frac{1}{2}}(s)D(t)\\ & \left.+C_*\left(\frac{2(p+1)}{(p-1)}E(0)\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2} E^{\frac{1}{2}}(s)D^q(t)\right]. \end{aligned} \tag{3.26}$$

Next, we integrate over $[t, t_2]$, we get

$$E(t) = E(t_2) + \int_t^{t_2}\left(\|u_\tau\|_{q+1}^{q+1} + \|\nabla u_\tau\|_2^2\right)d\tau, \tag{3.27}$$

since $t_2 - t_1 \geq \frac{1}{2}$, we conclude

$$\int_{t_1}^{t_2} E(t)dt \geq (t_2 - t_1)E(t_2) \geq \frac{1}{2}E(t_2),$$

so

$$E(t_2) \leq 2\int_{t_1}^{t_2} E(t)dt. \tag{3.28}$$

Consequently,

$$\begin{aligned}E(t) &\leq 2\int_{t_1}^{t_2} E(t)dt + \int_{t_1}^{t_2}\left(||u_\tau||_{q+1}^{q+1} + ||\nabla u_\tau||^2\right)d\tau\\ &= 2\int_{t_1}^{t_2} E(t)dt + D^{q+1}(t). \end{aligned}\tag{3.29}$$

Hence, we have

$$E(t) \leq \left(\frac{1}{2}C + C_3\right)D^2(t) + D^{q+1}(t) + C_4\left[D(t) + D^q(t)\right]E^{\frac{1}{2}}(t). \tag{3.30}$$

Then, we get

$$E(t) \leq C_5\left[2D^2(t) + D^{q+1}(t) + D^{2q}(t)\right], \tag{3.31}$$

since $E(t)$ is nonincreasing function and $E(t) \geq 0$ on $[0, \infty)$,

$$D^{q+1}(t) = E(t) - E(t+1) \leq E(0). \tag{3.32}$$

Then, we have

$$D(t) \leq E^{\frac{1}{q+1}}(0). \tag{3.33}$$

So

$$\begin{aligned}E(t) &\leq C_5\left[2D^2(t) + D^{q+1}(t) + D^{2q}(t)\right]\\ &\leq C_5D^2(t)\left[2 + D^{q-1}(t) + D^{2(q-1)}(t)\right]\\ &\leq C_5D^2(t)\left[2 + E^{(q-1)\times\frac{1}{q+1}}(0) + E^{2(q-1)\left(\frac{1}{q+1}\right)}(0)\right]\\ &= C_6D^2(t). \end{aligned}$$

Then, we get

$$E^{1+\frac{(q-1)}{2}}(t) \leq C_7D^{q+1}(t). \tag{3.34}$$

case 1 : When $q = 1$

$$E(t) \leq C_7D^2(t) = C_7[E(t) - E(t+1)],$$

we get

$$E(t) \leq E(0)e^{-w_0[t-1]^+},$$

where $w_0 = \ln(\frac{C_7}{C_7-1})$.
case 2 : When $q > 1$, we apply Lemma 2.1 to (3.27), we obtain

$$E(t) \leq \left(E(0)^{-\lambda} + C_7^{-1}\lambda[t-1]^+\right)^{-\frac{1}{\lambda}},$$

where $\lambda = \frac{(q-1)}{2}$.

The proof of Theorem 3.6 is completed.

□

Example 3.1: We consider the problem (1.1) in $\mathbb{R}^3$ with $\alpha(x) = 1$, $p = 1$ and $q = 1$ with $u_0 = cosh(x) * cosh(y) * cosh(z)$, so the Theorem 3.6 is applicable.

4 Blow up results

Definition 4.1. A solution u of the problem (1.1) is called blow up with $(q = 1)$ if there exists T^* a finite time such that

$$\lim_{t\to T^{*-}}\left[\int_\Omega u^2dx + \int_0^t (||u||_2^2 + ||\nabla u||_2^2)d\tau\right] = \infty. \tag{4.1}$$

We put

$$A(t) = \int_\Omega u^2dx + \int_0^t (||u||_2^2 + ||\nabla u||_2^2)d\tau \quad \text{for} \quad t \geq 0. \tag{4.2}$$

Lemma 4.2. We assume that: $2 < p+1 < \dfrac{2n}{n-2}$, $n > 2$, $q = 1$ and $0 \leq 4\delta \leq p-1$, so, we have

$$A^{''}(t) \geq 4(\delta+1)\int_\Omega u_t^2dx - 4(2\delta+1)E(0) + 4(2\delta+1)\int_0^t (||\nabla u_t||_2^2 + ||u_t||_2^2)d\tau. \tag{4.3}$$

Proof . By taking the first and second derivative of $A(t)$, we obtain

$$A^{'}(t) = 2\int_\Omega uu_tdx + ||\nabla u||_2^2 + ||u||_2^2, \tag{4.4}$$

$$\begin{aligned} A^{''}(t) =& 2\int_\Omega u_t^2dx + 2\int_\Omega uu_{tt}dx + 2\int_\Omega (\nabla u\nabla u_t)dx + 2\int_\Omega (|u|u_t)dx. \\ =& 2||u_t||_2^2 - 2\int_\Omega \alpha(x)\nabla u\nabla udx + 2||u||_{p+1}^{p+1}. \end{aligned} \tag{4.5}$$

From (4.5) and (3.3), we get

$$\begin{aligned} A^{''}(t) =& 4(\delta+1)\int_\Omega u_t^2dx - 4(2\delta+1)E(0) + 4(2\delta+1)\int_0^t (||\nabla u_t||_2^2 + ||u_t||_2^2)d\tau \\ &+ 4\delta\int_\Omega \alpha(x)\nabla u\nabla udx + \big(2 - \frac{8\delta+4}{p+1}\big)||u||_{p+1}^{p+1}. \end{aligned}$$

Where $0 < 4\delta \leq p-1$, we get (4.3). □

Lemma 4.3. We assume that $2 < p+1 < \frac{2n}{n-2}$ and $n > 2$, so we have

1. If $E(0) < 0$, then $A^{'}(t) > ||u_0||_2^2$ for $t > t^*$, where $t_0 = t^*$ is given by

$$t^* = \max\{0, \frac{A^{'}(t) - ||u_0||_2^2}{(8\delta+4)E(0)}\}.$$

2. If $E(0) > 0$ and
$$A^{'}(0) > r_2[A(0) + \frac{G}{4\delta + 4}] + ||u_0||_2^2, \tag{4.6}$$
holds, where G and t^* will be defined later.
Then, $A^{'}(t) > ||u_0||_2^2$ where $t_0 = 0$.

Proof .

1. When $E(0) < 0$, then for $t \geq 0$, $A^{''}(t) \geq -4(1+2\delta)E(0)$, by integration over $[0,t]$, we obtain
$$A^{'}(t) \geq A'(0) - 4(2\delta + 1)E(0)t, \quad t \geq 0.$$
Then, we get $A^{'}(t) > ||u_0||_2^2$ for $t > t^*$, with:
$$t^* = \max\{0, \frac{A^{'}(t) - ||u_0||_2^2}{4(2\delta + 1)E(0)}\}. \tag{4.7}$$

2. When $E(0) > 0$, Firstly, we find
$$2\int_0^t \int_\Omega uu_t dx d\tau = ||u||_2^2 - ||u_0||_2^2. \tag{4.8}$$
By using the Hölder inequality and the Young inequality, we obtain
$$||u||_2^2 \leq ||u_0||_2^2 + \int_0^t (||u||_2^2 + ||u_t||_2^2) d\tau. \tag{4.9}$$
Now, we use (4.9), Hölder's inequality and Young's inequality, we get
$$A^{'}(t) \leq A(t) + \int_\Omega u_t^2 dx + \int_0^t ||u_t||_2^2 d\tau + ||u_0||_2^2. \tag{4.10}$$
By (4.9) and (4.3), we have
$$A''(t) - 4(\delta+1)A'(t) + ||u_0||_2^2 A(t) + (8\delta+4)E(0) + (4\delta+4)||u_0||_2^2 \geq 0, \tag{4.11}$$
with
$$G = (8\delta + 4)E(0) + (4\delta + 4)||u_0||_2^2. \tag{4.12}$$
Let
$$b(t) = A(t) + \frac{G}{4(\delta+1)}, \text{for} \quad t > 0.$$
Where $b(t)$ satisfies Lemma 2.2.
Finaly, from (4.6) we obtain $A^{'}(t) > ||u_0||_2^2$ for $t > 0$ and r_2 is given in Lemma 2.2.

□

Theorem 4.4. We assume that $2 < p+1 < \frac{2n}{n-2}$ and $2 < n$, so we have two cases

1. Case 1: If $E(0) < 0$, then the solution u blows up in finite time T^* in the sense of $\lim_{t \longrightarrow T^{*-}} A(t) = \infty$ and
$$T^* \leq t_0 - \frac{L(t_0)}{L'(t_0)}. \tag{4.13}$$
Moreover, if $L(t_0) < \min \left|1, (-\frac{a}{b})^{\frac{1}{2}}\right|$, we have:
$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\kappa}{\kappa - L(t_0)}, \tag{4.14}$$
where $\kappa = (-\frac{a}{b})^{\frac{1}{2}}$.

2. Case 2: If $0 < E(0) < \dfrac{\left(F^{'}(t_0)\right)^2}{8F(t_0)}$ and (4.6) holds, then the solution u blows up in finite time T^* in the sense of $\lim_{t\longrightarrow T^{*-}} A(t) = \infty$, and

$$T^* \leq \frac{L\left(t_0\right)}{\sqrt{a}} \text{ or } T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\mu\delta}{\sqrt{a}}\{1 - \left[1 + \mu L\left(t_0\right)\right]^{-\frac{1}{2\delta}}\}. \tag{4.15}$$

With:

$$a = \delta^2 L^{2+\frac{2}{\delta}}\left(t_0\right)\left[\left(F'(t_0)\right)^2 - 8E(0)L^{-\frac{1}{\delta}}\left(t_0\right)\right] > 0, \tag{4.16}$$

$$b = 8\delta^2 E(0), \tag{4.17}$$

$$\mu = (\frac{a}{b})^{2+\frac{1}{\delta}}. \tag{4.18}$$

Proof . Let

$$L(t) = \left[A(t) + (T_1 - t)\,||u_0||_2^2\right]^{-\delta}, t \in \left[0, T_1\right], \tag{4.19}$$

where

$$F(t) = A(t) + (T_1 - t)||u_0||_2^2,$$

with T_1 is a stricty positive constant which will be defined later.
Then, by taking the second derivative of $L(t)$, we obtain

$$\begin{aligned} L'(t) &= -\delta\left[A(t) + (T_1 - t)\left\|u_0\right\|_2^2\right]^{-\delta-1}\left[A'(t) - \left\|u_0\right\|_2^2\right] \\ &= -\delta L^{1+\frac{1}{\delta}}(t)\left[A'(t) - \left\|u_0\right\|_2^2\right]. \end{aligned} \tag{4.20}$$

$$\begin{aligned} L^{''}(t) &= -\delta L^{1+\frac{2}{\delta}}(t)A''(t)\left[A(t) + (T_1 - t)\left\|u_0\right\|_2^2\right] \\ &\quad + \delta L^{1+\frac{2}{\delta}}(t)(1+\delta)[A'(t) - ||u_0||_2^2]^2 \\ L''(t) &= -\delta L^{1+\frac{2}{\delta}}(t)W(t), \end{aligned} \tag{4.21}$$

with

$$W(t) = A^{''}(t)F(t) - (1+\delta)(F^{'}(t))^2. \tag{4.22}$$

We define

$$M_u = \int_\Omega u^2 dx, \quad N_u = \int_\Omega u_t^2 dx,$$
$$P_u = \int_0^t \|u\|^2 dt, \quad Q_u = \int_0^t \left\|u_t\right\|^2 dt.$$

From (4.4), (4.7) and by Hölder's inequality, we obtain

$$\begin{aligned} A'(t) &= 2\int_\Omega u u_t dx + ||u_0|\,|_2^2 + 2\int_0^t \int_\Omega u u_t dx dt \\ &\leq 2\left((N_u M_u)^{\frac{1}{2}} + (P_u Q_u)^{\frac{1}{2}}\right) + ||u_0||_2^2. \end{aligned} \tag{4.23}$$

When case 1 holds and by Lemma 4.2 we have

$$A''(t) \geq -4(1+2\delta)E(0) + 4(1+\delta)\left(N_u + Q_u\right). \tag{4.24}$$

Then, from (4.19), (4.22) and (4.24), we get

$$\begin{aligned} W(t) \geq &\left[-4(1+2\delta)E(0) + 4(1+\delta)\left(N_u + Q_u\right)\right]L^{-\frac{1}{\delta}}(t) \\ &-4(1+\delta)\left((N_u M_u)^{\frac{1}{2}} + (P_u Q_u)^{\frac{1}{2}}\right)^2. \end{aligned}$$

From $A(t)$, we have

$$A(t) = \int_\Omega u^2 dx + \int_0^t \int_\Omega u^2 dx ds = M_u,$$

and by $L(t)$, we obtain

$$W(t) \geq -4(1+2\delta)E(0)L^{-\frac{1}{\delta}}(t) + 4(1+\delta)[(N_u+Q_u)(T_1-t)||u_0||_2^2 + \Lambda(t)],$$

with

$$\Lambda(t) = (N_u+Q_u)(M_u+P_u) - \left((N_uM_u)^{\frac{1}{2}} + (P_uQ_u)^{\frac{1}{2}}\right)^2.$$

Where $\Lambda(t)$ being nonnegative function and by the Schwarz inequality, we obtain

$$W(t) \geq -4(1+2\delta)E(0)L^{-\frac{1}{\delta}}(t), \quad \text{for} \quad t \geq t_0.$$

Thus, by $L^{''}(t)$, we obtain

$$L^{''}(t) \leq 4\delta(1+2\delta)E(0)L^{1+\frac{1}{\delta}}(t), \quad \text{for} \quad t \geq t_0. \tag{4.25}$$

We have

$$L'(t) < 0, \quad t \geq t_0.$$

Multiplying (4.25) by $L'(t)$ and integrating over $[t_0; t]$, we obtain

$$L^{'2}(t) \geq a + bL^{2+\frac{1}{\delta}}(t), \quad \text{for} \quad t \geq t_0,$$

with a, b are defined.
By using the steps of case 1, when case 2 holds, then if and only if

$$E(0) < \frac{(F'(t_0))^2}{8F(t_0)},$$

we get $a > 0$.
After that, we use Lemma 2.3, so there exists T^* such that $\lim_{t\longrightarrow T^{*-}} L(t) = 0$ and according to the sign of $E(0)$, the upper bound of T^* is estimated. So (4.1) holds. □
Example 4.1: We consider the problem (1.1) in $\mathbb{R}^2$ with $\alpha(x) = 1$, $p = 1$ and satisfied $0 < \delta \leq \frac{1}{2}$, so the Theorem 4.4 is applicable.

5 Conclusion

In this paper, we obtained the local and global existence, decay of solutions and blow up time for a quasilinear hyperbolic equation with source terms in a bounded domain. This improves and extends many results in the literature.

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