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Comparison of some Bayesian shrinkage estimation for Frechet distribution with simulations

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Abstract

The Frechet distribution is one of the important statistical distributions and it has many applications, especially in the field of distributing the maximum extent of disease precipitation and river drainage, and estimating the parameters of the Frechet distribution is very important, and for that research came in an attempt to compare several methods of estimating the parameter of Frechet distribution based on different Bayesian methods with (square loss, Linux and Unix) functions. In this research, several simulation experiments were conducted according to the difference in (sample size, value of distribution parameters and estimation methods) and the results were compared based on mean square error criteria, it is possible to use other estimation methods such as (moments and percentile), for other distributions such as (Gumbel and Lindley).

Keywords: Frechet distribution, Bayesian Shrinkage Estimators, Square Loss Function, Linux Loss Function, Unix Loss Function. Mean Square Error
2020 MSC: 30D35, 62C10

1 Problem of Research

The estimation of the parameters of the Frechet distribution suffers from problems if the estimation processes are affected by outliers which makes the estimator values move away from the real values with increase in the mean square error values. In addition, all applications of the reliability and survival functions will be unrepresentative of the studied distribution.

2 Objective of Research

The research aims to find estimators for the parameters of the Frechet distribution with minimum mean square error and therefore the applications of the distribution will be representative of their real values and thus increase the ability of the statistical distribution to represent various applications.

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3 Introduction

Frechet distribution is one of the important distributions and has many applications such as (hydrology experiment), many researches has been done such as:

In (2003) each (Nadarajah S, Kotz S.) presented a research that included Studying the skewness and kurtosis of the studied distribution by providing tests of fitting for the Frechet distribution [7].

In (2014) each (Afify AZ, Hamedani G, Ghosh I, Mead M.) presented a research that included the property and applications of the transmuted Marshall-Olkin Fréchet distribution by maximum likelihood method to introduce the estimators for the four parameters [1].

In (2016) each (Yousof HM, Afify AZ, Abd El Hadi NE, Hamedani GG, Butt NS.) presented a research that included Studying the Properties and Applications of the Six-Parameter Fréchet Distribution the maximum likelihood estimation introduce the six estimators for each parameters [10].

In (2020) each (Almetwally EM, Muhammed HZ.) presented a research that included Studying the Properties and Applications of the Six-Parameter Fréchet Distribution the maximum likelihood estimation introduce the six estimators for each parameters [3].

In this research, two methods were presented to estimate the parameters of the Frechet distribution and for a number of simulation experiments according to different values of (sample size and the values of distribution parameters). The results were compared according to the mean square error.

4 Frechete Distribution

The Frechet distribution has been named with the French mathematician Maurice René Fréchet (1878-1973), which he developed in the 1920s as a maximum value distribution (also known as a Type II maximum value distribution). The Frechet distribution is one of the probability distributions of life-time models. Kotz and Nadarajah describe this distribution and discuss its broad applicability in various fields such as natural disasters, Horse racing, precipitation, supermarket queues, wind speed, ocean currents, and life tests are used in failure rate modeling, which is commonly used in biological studies, light signal analysis, and fault modeling [8].

4.1 Probability Density Function

The researcher Drapella [4] and researcher Mudholkar, Srivastava and kollia [6] proposed the name reciprocal of Weibull on the Frechet distribution.

If the random variable (t) has a Weibull distribution, then the variable ($x = 1/t$) it represents the Frechet distribution and its probability density function as follows [8]:

$$f(x.\alpha.\lambda) = \alpha\lambda^{\alpha}x^{-(\alpha+1)}e^{-\left(\frac{\lambda}{x}\right)^{\alpha}}$$

Such that

$x > 0$ and $\alpha, \lambda > 0$

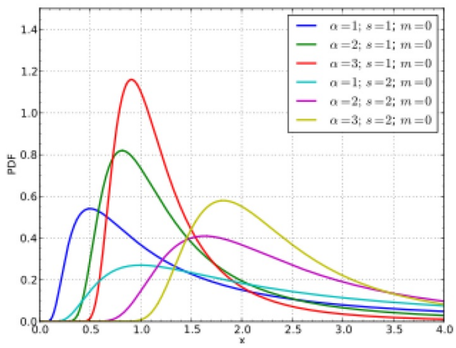

Figure 1: the probability density function for the Frechet distribution with different values of (α, λ)

4.2 Cumulative Density Function

If (x) is a random variable with a Frechet distribution then the cumulative density function is as follows [8]:

$$f(x.\alpha.\lambda) = P(X \leq x) = \int_{-\infty}^{x} f(u)du$$

$$f(x.\alpha.\lambda) = \int_{0}^{x} \alpha\lambda^{\alpha} u^{-(\alpha+1)} e^{-\left(\frac{\lambda}{u}\right)^{\alpha}} du$$

$$f(x.\alpha.\lambda) = e^{-\left(\frac{\lambda}{x}\right)^{\alpha}}$$

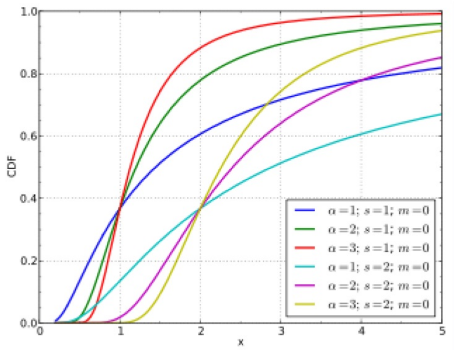

Figure 2: the cumulative density function for the Frechet distribution with different values of (α, λ)

4.3 Distribution Moments

The distribution moments can be [2]

$$EX^k = \int_{0}^{\infty} X^k f(x)dx$$

$$EX^k = \lambda^k \Gamma\left(1 - \frac{k}{\alpha}\right), \quad k = 1, 2, 3, ...$$

When $(k = 1)$ then

$$EX = \lambda\Gamma\left(1 - \frac{1}{\alpha}\right)$$

with $\alpha > 1$. When $(k = 2)$ then

$$EX^2 = \lambda^2\Gamma\left(1 - \frac{2}{\alpha}\right).$$

The Mean will be

$$\lambda\Gamma\left(1 - \frac{1}{\alpha}\right).$$

The Median will be

$$\frac{\lambda}{\sqrt[\alpha]{\log_e(2)}}.$$

The Mode will be

$$\lambda\left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{\alpha}}.$$

The variance will be

$$Var(X) = \lambda^2 \left[\Gamma\left(1-\frac{2}{\alpha}\right) - \left(\Gamma^2\left(1-\frac{1}{\alpha}\right)\right)\right]$$

with $\alpha > 2$. The Skewness will be

$$\frac{\Gamma\left(1-\frac{3}{\alpha}\right) - 3\Gamma\left(1-\frac{1}{\alpha}\right) + 2\Gamma^3\left(1-\frac{1}{\alpha}\right)}{\sqrt{\left(\Gamma\left(1-\frac{2}{\alpha}\right) - \Gamma^2\left(1-\frac{1}{\alpha}\right)\right)^3}}$$

with $\alpha > 3$. The Kurtosis will be

$$-6 + \frac{\Gamma\left(1-\frac{4}{\alpha}\right) - 4\Gamma\left(1-\frac{3}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right) + 3\Gamma^2\left(1-\frac{2}{\alpha}\right)}{\left[\Gamma\left(1-\frac{4}{\alpha}\right) - \Gamma^2\left(1-\frac{1}{\alpha}\right)\right]^2}$$

with $\alpha > 4$

4.4 Maximum Likelihood Estimator

The Maximum Likelihood Estimation method is one of the important and commonly used methods of estimating, this method depends on finding the parameter values that maximize the likelihood function, the likelihood function can be defined with the following [5]:

Let $(x_1.x_2....x_n)$ is a random sample of size (n) drawn from a population with a density function $(f(x,\alpha,\lambda))$ such that each observation follows the Frechet distribution then the likelihood function can be

$$L(x_1.x_2....x_n : \alpha.\lambda) = \prod_{i=1}^{n} \alpha\lambda^\alpha x^{-(\alpha+1)} e^{-\left(\frac{\lambda}{x}\right)^\alpha} = \alpha^n \lambda^{n\alpha} e^{-\sum_{i=1}^{n}\left(\frac{\lambda}{x}\right)^\alpha} \prod_{i=1}^{n} x_i^{-(\alpha+1)}$$

Tacking logarithm for the previous equation we get

$$\ln L = n\ln\alpha + n\alpha\ln\lambda - \sum_{i=1}^{n}\left(\frac{\lambda}{x}\right)^\alpha - (\alpha+1)\sum_{i=1}^{n}\ln x_i$$

By finding the partial derivative of the function relative to the two parameters (α, λ), we get the following:

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n\alpha}{\hat{\lambda}} - \alpha\hat{\lambda}^{\alpha-1}\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^\alpha$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\hat{\alpha}} + n\ln(\lambda) - \sum_{i=1}^{n}\ln(x_i) - \left\{\lambda^\alpha \sum_{i=1}^{n}(x_i^{-\alpha})(-1)\ln(x_i) + \left(\sum_{i=1}^{n}(x_i^{-\alpha})\right)(\lambda^\alpha \ln(\lambda))\right\}$$

Set the derivative equal to zero we get

$$\frac{n\alpha}{\hat{\lambda}} - \alpha\hat{\lambda}^{\alpha-1}\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^\alpha = 0$$

$$\frac{n}{\hat{\alpha}} + n\ln\hat{\lambda} - \sum_{i=1}^{n}\ln x_i + \hat{\lambda}^\alpha\sum_{i=1}^{n}x_i^{-\alpha}(-1)\ln x_i - \lambda^\alpha\ln\hat{\lambda}\sum_{i=1}^{n}x_i^{-\alpha} = 0$$

The maximum likelihood estimator will be

$$\hat{\lambda}_{mle} = \left|\frac{n}{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^\alpha}\right|^{\frac{1}{\alpha}}$$

$$g(\alpha) = \frac{n}{\hat{\alpha}} - \sum_{i=1}^{n}\ln x_i + \frac{n\sum_{i=1}^{n}x_i^{-\alpha}\ln x_i}{\sum_{i=1}^{n}x_i^{-\alpha}}$$

$$\acute{g}(\alpha) = \frac{\partial g(\alpha)}{\partial \alpha}$$

$$\acute{g}(\alpha) = -\frac{n}{\hat{\alpha}} - \frac{n(\sum_{i=1}^{n} \ln x_i^{-\alpha}) \sum_{i=1}^{n} \ln x_i^{-\alpha} \ln x_i^2 - ((\sum_{i=1}^{n} \ln x_i^{-\alpha}) \ln x_i)^2}{(\sum_{i=1}^{n} x_i^{-\alpha})^2}$$

Maximum likelihood estimator can be obtained by using Newton Raphson's method

4.5 Bayesian Shrinkage Estimators under Squared Loss Function (BSE1)

The squared loss function is an analogue loss function, which is also called the squared error loss function whose formula is as follows [9]:

$$L(b.\hat{b}_{s1}) = [\hat{b}_{s1} - b]^2$$

The general Bayesian estimator will be as follows

$$E[L(b.\hat{b}_{s1})] = E[\hat{b}_{s1} - b]^2$$

$$\rho(b.\hat{b}_{s1}) = E[\hat{b}_{s1} - b]^2 = \hat{b}_{s1}^2 - 2\hat{b}_{s1} Eb + Eb^2$$

$$f(b/x) = \frac{t^n b^{n-1} e^{-tb}}{\Gamma(n)}$$

$$Eb = \frac{n}{t} . Eb^2 = \frac{n(n+1)}{t^2}$$

$$\rho(b.\hat{b}_{s1}^2) = \hat{b}_{s1}^2 - 2\hat{b}_{s1} \frac{n}{t} + \frac{n(n+1)}{t^2}$$

$$\frac{\partial \rho(b.\hat{b}_{s1}^2)}{\partial \hat{b}_{s1}^2} = 2\hat{b}_{s1} - 2\frac{n}{t} + zero$$

$$\frac{\partial \rho(b.\hat{b}_{s1}^2)}{\partial \hat{b}_{s1}^2} = 0$$

$$2\hat{b}_{s1} - 2\frac{n}{t} = 0$$

The general Bayesian estimator is in the following form

$$\hat{b}_{s1} = \frac{n}{t}$$

such that

$$t = \sum_{i=1}^{n} \left(\frac{1}{x_i}\right)^{\alpha}.$$

The loss function of Bayesian shrinkage estimators under squared loss function will be:

$$\hat{b}_{sh1} = k(\hat{b}_{s1} - b_0) + b_0$$

$$\begin{aligned}\rho_1(b.\hat{b}_{sh}) &= E[\hat{b}_{sh1} - b_0]^2\\ &= E[k(\hat{b}_{s1} - b_0) + b_0]^2\\ &= E[k\hat{b}_{s1} - kb_0 + b_0]^2\\ &= E[k\hat{b}_{s1} + (1-k)b_0]^2\\ &= k^2\hat{b}_{s1}^2 + (2k+2k^2)b_0\hat{b}_{s1} + (1-k)^2b_0^2 - 2k\hat{b}_{s1}\frac{n}{t} + (2k-2)\frac{n}{t}b_0 + \frac{n(n+1)}{t^2}\end{aligned}$$

$$\frac{\partial\rho_1(b.\hat{b}_{sh})}{\partial k} = 2k\hat{b}_{s1}^2 + (2+4k)b_0\hat{b}_{s1} - 2(1-k)b_0^2 - \frac{2n}{t}[\hat{b}_{s1} - b_0]$$

$$\frac{\partial\rho_1(b.\hat{b}_{sh})}{\partial k} = 0$$

$$2k\hat{b}_{s1}^2 + (2+4k)b_0\hat{b}_{s1} - 2(1-k)b_0^2 - \frac{2n}{t}[\hat{b}_{s1} - b_0] = 0$$

$$2k\hat{b}_{s1}^2 + 2b_0\hat{b}_{s1} + 4kb_0\hat{b}_{s1} - 2b_0^2 + 2kb_0^2 - \frac{2n}{t}[\hat{b}_{s1} - b_0] = 0$$

$$k[2\hat{b}_{s1}^2 + 4b_0\hat{b}_{s1} + 2b_0^2] + 2b_0\hat{b}_{s1} - 2b_0^2 - \frac{2n}{t}[\hat{b}_{s1} - b_0] = 0$$

$$k[2\hat{b}_{s1}^2 + 4b_0\hat{b}_{s1} + 2b_0^2] = 2b_0^2 + \frac{2n}{t}[\hat{b}_{s1} - b_0] - 2b_0\hat{b}_{s1}$$

$$\begin{aligned}k &= \frac{2b_0^2 + \frac{2n}{t}[\hat{b}_{s1} - b_0] - 2b_0\hat{b}_{s1}}{2\hat{b}_{s1}^2 + 4b_0\hat{b}_{s1} + 2b_0^2}\\ k &= \frac{2[b_0^2 + \frac{n}{t}[\hat{b}_{s1} - b_0] - b_0\hat{b}_{s1}]}{2[\hat{b}_{s1}^2 + 2b_0\hat{b}_{s1} + b_0^2]}\\ k &= \frac{b_0^2 + \frac{n}{t}[\hat{b}_{s1} - b_0] - b_0\hat{b}_{s1}}{\hat{b}_{s1}^2 + 2b_0\hat{b}_{s1} + b_0^2}\\ k &= \frac{b_0^2 + \frac{n}{t}\hat{b}_{s1} - \frac{n}{t}b_0 - b_0\hat{b}_{s1}}{(\hat{b}_{s1}^2 + b_0)^2} * \frac{t}{t}\\ k &= \frac{tb_0^2 + n\hat{b}_{s1} - nb_0 - tb_0\hat{b}_{s1}}{t(\hat{b}_{s1}^2 + b_0)^2}\end{aligned}$$

such that

$$t = \sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\alpha}$$

$$\hat{b}_{s1} = \frac{n}{t}$$

$$\hat{b}_{sh1} = \frac{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\alpha} b_0^2 + \frac{n^2}{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\alpha}} - nb_0 - \sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\alpha} b_0 \frac{n}{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\alpha}}}{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\alpha}\left(\frac{n}{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\alpha}} + b_0\right)^2}$$

$$\hat{b}_{sh1} = \frac{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\alpha} b_0^2 + \frac{n^2}{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\alpha}} - 2nb_0}{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\alpha}\left(\frac{n}{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\alpha}} + b_0\right)^2}.$$

4.6 Bayesian Shrinkage Estimators under LINEX Loss Function (BSE2)

It is one of the important estimators in which the loss function of the pessimist estimator for this distribution is under the shadow of the functions Exponential, which are nonlinear functions that can be taken into account and thus affect the values of each of the parameters The obtained initial and final results, and the LINEX loss function, states the following [9]:

$$L_2 = e^{a\Delta} - \alpha\Delta - 1$$

(Δ) represent the change amount with

$$\Delta = \frac{\hat{b}_{s2}}{b} - 1$$

($a\Delta$) represent the first change in the (LINEX) loss function

($\alpha\Delta$) represent the second change in the (LINEX) loss function

$$\rho_2(\hat{b}.\hat{b}_{s2}) = E[e^{a\Delta} - \alpha\Delta - 1]$$

$$\rho_2(\hat{b}.\hat{b}_{s2}) = E\left[e^{a\left(\frac{\hat{b}_{s2}}{b}-1\right)} - \alpha\left(\frac{\hat{b}_{s2}}{b} - 1\right) - 1\right] = e^{-a}E\left[e^{a\left(\frac{\hat{b}_{s2}}{b}\right)}\right] - \alpha\hat{b}_{s2}E\left[\frac{1}{b}\right] + \alpha - 1$$

such that

$$b \sim \Gamma(n,t)$$

and

$$\frac{1}{b} \sim inv\Gamma(n,t).$$

The probability density function

$$f\left(\frac{1}{b}\right) = \frac{t^n}{\Gamma(n)} b^{-n-1} e^{t/b}.$$

The Expected value will be

$$E\left(e^{a\frac{\hat{b}_{s2}}{b}}\right) = \frac{t^n}{\Gamma(n)} \int_0^\infty e^{\frac{t-a\hat{b}_{s2}}{b}} b^{-n-1} db = \left[\frac{t}{t - a\hat{b}_{s2}}\right]^n$$

$$E\left(\frac{1}{b}\right) = \frac{t}{n-1}$$

$$\rho(b.\hat{b}_{s2}) = e^{-a}\left[\frac{t}{t - a\hat{b}_{s2}}\right]^n - \alpha\hat{b}_{s2}\left[\frac{t}{n-1}\right] + \alpha - 1$$

The partial derivative will be

$$\frac{\partial\rho(b.\hat{b}_{s2})}{\partial\hat{b}_{s2}} = -nt^n e^{-a}(t - a\hat{b}_{s2})^{-n-1}(-a) - \frac{\alpha t}{n-1} = ant^n e^{-a}(t - a\hat{b}_{s2})^{-n-1} - \frac{\alpha t}{n-1}$$

Tacking $\left(\frac{\partial\rho(b.\hat{b}_{s2})}{\partial\hat{b}_{s2}} = 0\right)$ then

$$ant^n e^{-a}(t - a\hat{b}_{s2})^{-n-1} - \frac{\alpha t}{n-1} = 0$$

$$\hat{b}_{s2} = \frac{1}{\alpha}\left[t - (nt^{n-1}e^{-a})^{\frac{1}{n+1}}\right]$$

The shrinkage estimator under (LINEX) loss function will be

$$\rho_2(\hat{b}.\hat{b}_{sh2}) = E\left[e^{a\left(\frac{\hat{b}_{sh2}}{b}-1\right)} - \alpha\left(\frac{\hat{b}_{sh2}}{b}-1\right)-1\right]$$

$$\rho_2(\hat{b}.\hat{b}_{sh2}) = e^{-a}E\left[e^{a\left(\frac{\hat{b}_{sh2}}{b}\right)}\right] - \alpha\hat{b}_{sh2}E\left[\frac{1}{b}\right]+\alpha-1$$

$$E\left[e^{a\left(\frac{\hat{b}_{sh2}}{b}\right)}\right] = \left[\frac{t}{t-a\hat{b}_{sh2}}\right]^n .E\left[\frac{1}{b}\right] = \frac{t}{n-1}$$

$$\rho_2(\hat{b}.\hat{b}_{sh2}) = e^{-a}\left[\frac{t}{t-a\hat{b}_{sh2}}\right]^n - \alpha\hat{b}_{sh2}\left[\frac{t}{n-1}\right]+\alpha-1$$

$$\hat{b}_{sh2} = k(\hat{b}_{s2}-b_0)+b_0$$

$$\hat{b}_{s2} = \frac{1}{\alpha}\left[t-(nt^{n-1}e^{-a})^{\frac{1}{n+1}}\right]$$

$$\rho_2(b.\hat{b}_{sh2}) = e^{-a}t^n[t-ak(\hat{b}_{s2}-b_0)-ab_0]^{-n} - \alpha\left(\frac{t}{n-1}\right)k(\hat{b}_{s2}-b_0)-\alpha b_0\left(\frac{t}{n-1}\right)$$

$$\frac{\partial\rho_2(b.\hat{b}_{sh2})}{\partial k} = an(\hat{b}_{s2}-b_0)e^{-a}t^n[t-ak(\hat{b}_{s2}-b_0)-ab_0]^{-n-1}\alpha t\frac{(\hat{b}_{s2}-b_0)}{n-1}$$

$$k = \frac{(t-ab_0)-[n(n-1)t^{n-1}e^{-a}]^{\left(\frac{1}{n+1}\right)}}{a\left(\left[\left(\frac{1}{\alpha}\right)\left[t-(nt^{n-1}e^{-a})^{\frac{1}{n+1}}\right]\right]-b_0\right)}$$

$$\hat{b}_{sh2} = k(\hat{b}_{s2}-b_0)+b_0$$

$$\hat{b}_{sh2} = \left[\frac{(t-ab_0)-[n(n-1)t^{n-1}e^{-a}]^{\left(\frac{1}{n+1}\right)}}{a\left(\left[\left(\frac{1}{\alpha}\right)\left[t-(nt^{n-1}e^{-a})^{\frac{1}{n+1}}\right]\right]-b_0\right)}\right]$$

Tacking ($t=\sum_{i=1}^n\left(\frac{1}{x_i}\right)^\alpha$)

$$\hat{b}_{sh2} = \left[\frac{\left(\sum_{i=1}^n\left(\frac{1}{x_i}\right)^\alpha - ab_0\right) - \left[n(n-1)\left(\sum_{i=1}^n\left(\frac{1}{x_i}\right)^\alpha\right)^{n-1}e^{-a}\right]^{\left(\frac{1}{n+1}\right)}}{a\left(\left[\left(\frac{1}{\alpha}\right)\left[\sum_{i=1}^n\left(\frac{1}{x_i}\right)^\alpha - \left(n\left[\sum_{i=1}^n\left(\frac{1}{x_i}\right)^\alpha\right]^{n-1}e^{-a}\right)^{\frac{1}{n+1}}\right]\right]-b_0\right)}\right]$$
$$*\left[\left(\left(\frac{1}{\alpha}\right)\left[\sum_{i=1}^n\left(\frac{1}{x_i}\right)^\alpha - \left(n\left[\sum_{i=1}^n\left(\frac{1}{x_i}\right)^\alpha\right]^{n-1}e^{-a}\right)^{\frac{1}{n+1}}\right]\right)-b_0\right]+b_0$$

5 Simulation Experiments

For the purpose of generating a sample of size (n) that follows the Frechete distribution with certain parameters, using the Cumulative distribution function of the distribution such that [2]

$$R = e^{-\left(\frac{\lambda}{x}\right)^{\alpha}}$$

with

$(\lambda.\alpha)$ represent the parameter of the distribution,

(R) represent the randomized function with uniform distribution with interval (0-1),

and

$$x = -\left[\frac{\lambda}{Ln(R)}\right]^{\frac{1}{\alpha}}$$

such that (x) follow Frechet distribution with $(\lambda.\alpha)$. Simulation experiments with $(n_1 = 50.n_2 = 100.n_3 = 150)$, $(\alpha_1 = 0.25.\alpha_2 = 0.5.\alpha_3 = 0.75)$, $(\beta_1 = 0.5.\beta_2 = 1.\beta_3 = 1.5)$. The best estimation method with $(\min(\varphi_{ik}))$ and $(\min(MSE))$ such that

$$\varphi_{ik} = \min_j |\theta_k - m_j|$$

$i = 1.2.....27$ No. of simulation experiments

$j = 1.2$ No. of Estimation methods

$k = 1$ or 2 No. of Parameters

$\theta_1 = \alpha$

$\theta_2 = \beta$

The mean square error is [5]

$$MSE = \frac{\sum_{it=1}^{rep}[\hat{\theta}_k - \theta_k]^2}{rep}$$

such that

$rep =$ No. of Iterations

$\hat{\theta}_k =$ he Estimator value

$\theta_k =$ the true value

6 Experimental Results

Simulation results it can be placed in the following tables and figures

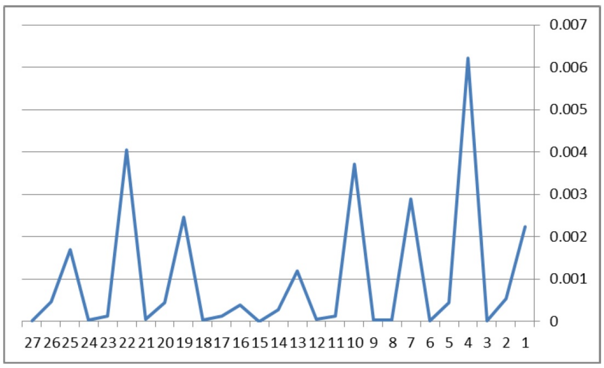

Figure 3: the best first estimator for each simulation experiment and for each method

By observing the previous table, for the best first estimator it becomes clear to us that the (27) different experiments, according to each of (the value of the first parameter, the second parameter and sample size) and for the estimation

Table 1: Estimators of the first parameter and the best estimator for each simulation experiment and for each method

S_i	α	β	n	m_1	m_2	φ_{i1}	i
1	0.25	0.5	50	0.245522	0.252243	0.002243	2
2	0.25	0.5	100	0.305928	0.249457	0.000543	2
3	0.25	0.5	150	0.249986	0.250076	1.4E-05	1
4	0.25	1	50	0.281919	0.256223	0.006223	2
5	0.25	1	100	0.249411	0.249549	0.000451	2
6	0.25	1	150	0.249935	0.249988	1.2E-05	2
7	0.25	1.5	50	0.282566	0.252884	0.002884	2
8	0.25	1.5	100	0.248687	0.24997	3E-05	2
9	0.25	1.5	150	0.249935	0.250031	3.1E-05	2
10	0.5	0.5	50	0.507021	0.503713	0.003713	2
11	0.5	0.5	100	0.499877	0.500147	0.000123	1
12	0.5	0.5	150	0.499698	0.50006	6E-05	2
13	0.5	1	50	0.493872	0.501188	0.001188	2
14	0.5	1	100	0.637824	0.500272	0.000272	2
15	0.5	1	150	0.511337	0.499997	3E-06	2
16	0.5	1.5	50	0.499607	0.506443	0.000393	1
17	0.5	1.5	100	0.507722	0.500136	0.000136	2
18	0.5	1.5	150	0.506058	0.500029	2.9E-05	2
19	0.75	0.5	50	0.747537	0.74675	0.002463	1
20	0.75	0.5	100	1.094763	0.750449	0.000449	2
21	0.75	0.5	150	0.749951	0.750079	4.9E-05	1
22	0.75	1	50	0.754042	0.755719	0.004042	1
23	0.75	1	100	1.030459	0.749875	0.000125	2
24	0.75	1	150	0.74744	0.749965	3.5E-05	2
25	0.75	1.5	50	0.730049	0.751691	0.001691	2
26	0.75	1.5	100	0.75695	0.749544	0.000456	2
27	0.75	1.5	150	0.739076	0.750017	1.7E-05	2

methods, it becomes clear to us that the best estimator and according to the value of (φ_{i1}) is the best in comparison between two methods and for each simulated experiment of the executed experiments.

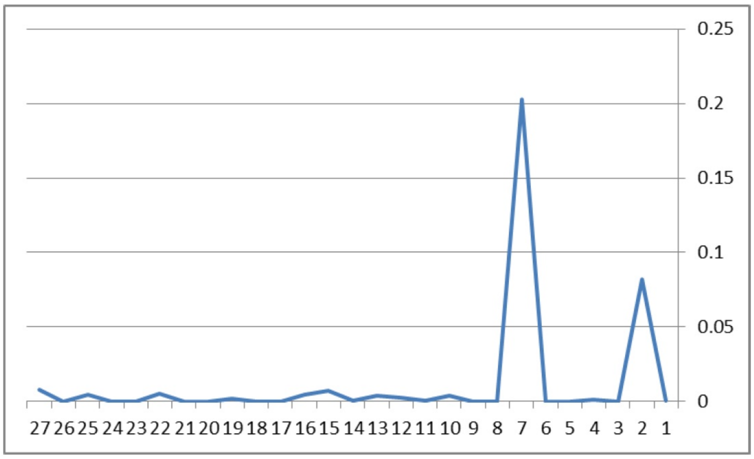

Figure 4: the best second estimator for each simulation experiment and for each method

By observing the previous table, the best second estimator according to (27) different experiments, with (the value of the first parameter, the second parameter and sample size) and for the estimation methods, it becomes clear to us that the best estimator and according to the value of (φ_{i2}) is the best in comparison between two methods and for each simulated experiment of the executed experiments.

Table 2: Estimators of the second parameter and the best estimator for each simulation experiment and for each method

S_i	α	β	n	m_1	m_2	φ_{i2}	i
1	0.25	0.5	50	0.496287	0.500552	0.000552	2
2	0.25	0.5	100	0.582271	0.590681	0.082271	1
3	0.25	0.5	150	0.451545	0.500015	1.5E-05	2
4	0.25	1	50	0.892469	1.001512	0.001512	2
5	0.25	1	100	0.99511	0.999896	0.000104	2
6	0.25	1	150	0.930385	1.000001	1E-06	2
7	0.25	1.5	50	1.297575	1.2900663	0.202425	1
8	0.25	1.5	100	1.367761	1.500199	0.000199	2
9	0.25	1.5	150	1.511172	1.500061	6.1E-05	2
10	0.5	0.5	50	0.599307	0.496384	0.003616	2
11	0.5	0.5	100	0.565042	0.500336	0.000336	2
12	0.5	0.5	150	0.502863	0.489948	0.002863	1
13	0.5	1	50	1.358993	0.99598	0.00402	2
14	0.5	1	100	1.015104	1.000656	0.000656	2
15	0.5	1	150	0.993064	0.989985	0.006936	1
16	0.5	1.5	50	1.2785	1.504665	0.004665	2
17	0.5	1.5	100	1.345097	1.500041	4.1E-05	2
18	0.5	1.5	150	1.479525	1.499978	2.2E-05	2
19	0.75	0.5	50	0.6177	0.501621	0.001621	2
20	0.75	0.5	100	0.489329	0.500121	0.000121	2
21	0.75	0.5	150	0.485618	0.499988	1.2E-05	2
22	0.75	1	50	0.994869	0.946855	0.005131	1
23	0.75	1	100	0.857445	0.999997	3E-06	2
24	0.75	1	150	1.000246	0.999911	8.9E-05	2
25	0.75	1.5	50	1.594612	1.495426	0.004574	2
26	0.75	1.5	100	1.635403	1.499847	0.000153	2
27	0.75	1.5	150	1.507953	1.600055	0.007953	1

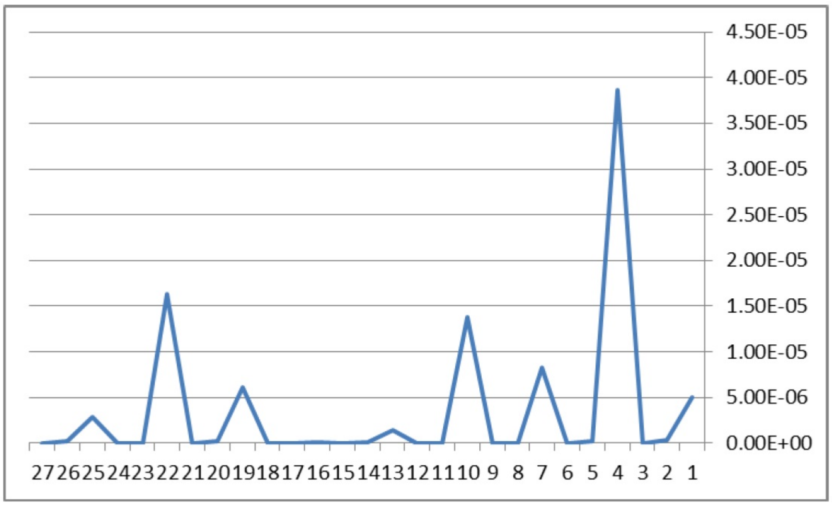

Figure 5: the best estimation method in second parameter for each simulation experiment and for each method

By observing the previous table, the best estimation method for the first estimator according to (27) different experiments, with (the value of the first parameter, the second parameter and sample size) and for the estimation methods, it becomes clear to us that the best estimator and according to the minimum value of the mean square error comparison between two methods and for each simulated experiment of the executed experiments.

By observing the previous table, the best estimation method for the second estimator according to (27) different

Table 3: Mean square error of the first parameter and the minimum mean square error for each simulation experiment and for each method

S_i	α	β	n	m_1	m_2	min	i
1	0.25	0.5	50	2.00E-05	5.03E-06	5.03E-06	2
2	0.25	0.5	100	3.13E-03	2.95E-07	2.95E-07	2
3	0.25	0.5	150	1.84E-10	5.83E-09	1.84E-10	1
4	0.25	1	50	1.02E-03	3.87E-05	3.87E-05	2
5	0.25	1	100	3.47E-07	2.03E-07	2.03E-07	2
6	0.25	1	150	4.23E-09	1.44E-10	1.44E-10	2
7	0.25	1.5	50	1.06E-03	8.32E-06	8.32E-06	2
8	0.25	1.5	100	1.72E-06	8.71E-10	8.71E-10	2
9	0.25	1.5	150	4.27E-09	9.69E-10	9.69E-10	2
10	0.5	0.5	50	4.93E-05	1.38E-05	1.38E-05	2
11	0.5	0.5	100	1.52E-08	2.17E-08	1.52E-08	1
12	0.5	0.5	150	9.09E-08	3.64E-09	3.64E-09	2
13	0.5	1	50	3.76E-05	1.41E-06	1.41E-06	2
14	0.5	1	100	1.90E-02	7.38E-08	7.38E-08	2
15	0.5	1	150	1.29E-04	6.89E-12	6.89E-12	2
16	0.5	1.5	50	1.54E-07	4.15E-05	1.54E-07	1
17	0.5	1.5	100	5.96E-05	1.85E-08	1.85E-08	2
18	0.5	1.5	150	3.67E-05	8.59E-10	8.59E-10	2
19	0.75	0.5	50	6.07E-06	1.06E-05	6.07E-06	1
20	0.75	0.5	100	0.118861	2.01E-07	2.01E-07	2
21	0.75	0.5	150	2.35E-09	6.29E-09	2.35E-09	1
22	0.75	1	50	1.63E-05	3.27E-05	1.63E-05	1
23	0.75	1	100	7.87E-02	1.57E-08	1.57E-08	2
24	0.75	1	150	6.55E-06	1.20E-09	1.20E-09	2
25	0.75	1.5	50	3.98E-04	2.86E-06	2.86E-06	2
26	0.75	1.5	100	4.83E-05	2.08E-07	2.08E-07	2
27	0.75	1.5	150	1.19E-04	2.93E-10	2.93E-10	2

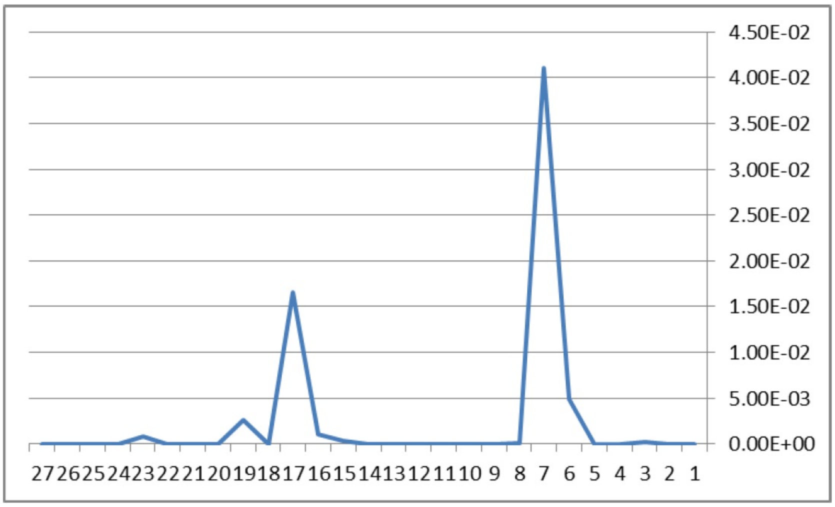

Figure 6: the best estimation method in second parameter for each simulation experiment and for each method

experiments, with (the value of the first parameter, the second parameter and sample size) and for the estimation methods, it becomes clear to us that the best estimator and according to the minimum value of the mean square error comparison between two methods and for each simulated experiment of the executed experiments.

From the previous tables and figures with (54) simulation experiments the best estimation method for each (first and second) parameter with minimum mean square error we get for the first estimation method the best was (14) of

Table 4: Mean square error of the second parameter and the minimum mean square error for each simulation experiment and for each method

S_i	α	β	n	m_1	m_2	min	i
1	0.25	0.5	50	1.38E-05	3.04E-07	3.04E-07	2
2	0.25	0.5	100	6.77E-03	6.55E-09	6.55E-09	2
3	0.25	0.5	150	2.35E-03	2.30E-04	2.30E-04	2
4	0.25	1	50	2.29E-06	1.04E-02	2.29E-06	1
5	0.25	1	100	1.08E-08	2.39E-05	1.08E-08	1
6	0.25	1	150	4.85E-03	1.23E-02	4.85E-03	1
7	0.25	1.5	50	4.10E-02	4.39E-02	4.10E-02	1
8	0.25	1.5	100	8.71E-05	3.96E-02	8.71E-05	1
9	0.25	1.5	150	1.25E-04	3.71E-09	3.71E-09	2
10	0.5	0.5	50	9.86E-03	1.31E-05	1.31E-05	2
11	0.5	0.5	100	4.23E-03	1.13E-07	1.13E-07	2
12	0.5	0.5	150	1.06E-02	2.66E-09	2.66E-09	2
13	0.5	1	50	0.128876	1.62E-05	1.62E-05	2
14	0.5	1	100	2.28E-04	4.31E-07	4.31E-07	2
15	0.5	1	150	2.87E-04	2.30E-03	2.87E-04	1
16	0.5	1.5	50	4.91E-02	1.02E-03	1.02E-03	2
17	0.5	1.5	100	2.40E-02	1.65E-02	1.65E-02	2
18	0.5	1.5	150	9.22E-06	4.73E-03	9.22E-06	1
19	0.75	0.5	50	1.39E-02	2.63E-03	2.63E-03	2
20	0.75	0.5	100	1.14E-04	1.46E-08	1.46E-08	2
21	0.75	0.5	150	2.07E-04	1.52E-10	1.52E-10	2
22	0.75	1	50	9.45E-06	9.89E-03	9.45E-06	1
23	0.75	1	100	2.03E-02	7.88E-04	7.88E-04	2
24	0.75	1	150	6.04E-08	7.85E-09	7.85E-09	2
25	0.75	1.5	50	8.95E-03	2.09E-05	2.09E-05	2
26	0.75	1.5	100	0.018334	2.33E-08	2.33E-08	2
27	0.75	1.5	150	0.028208	3.07E-09	3.07E-09	2

(54) with (0.259259259%) and the second estimation method the best was (40) of (54) with (0.740740741%).

7 Conclusions and Suggestions

After applying simulation experiments many conclusions and suggestions are obtained, which are:

1. Bayesian Shrinkage Estimators under LINEX Loss Function (BSE2) method was the best because it was the best according to (0.740740741%)
2. Each of Bayesian Shrinkage Estimation methods depend on (sample size, the parameter values)
3. The two estimation methods gives best estimators according to ($\varphi_{i1}.\varphi_{i2}$) criteria.
4. The two estimation methods give best estimators according to mean square error.
5. Estimation methods can be applied to other distributions such as (Gumbel, weibull) distributions.
6. Other estimation methods such that (moment, percentage) can be applied
7. Applied functions such that (reliability, hazard) can be applied.

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