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On quasi-subordination for bi-univalency involving generalized distribution series associated with remodelled s-sigmoid function

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Abstract

In this paper, the authors investigate the bi-univalency of the generalized distribution series associated with quasi-subordination and remodelled s-sigmoid function. The early few coefficients are obtained to achieve our goal. The results obtained are new to the history of bi-univalency.

Keywords: Analytic function, Univalent function, Starlike function, Convex function, Quasi-subordination, Logistic sigmoid function, $s-$sigmoid function
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1 Introduction

Special functions deal with an information process that is inspired by the way nervous system such as brain processes information. It comprises of large number of highly interconnected processing elements (neurones) working together to solve a specific problem. The functions are outshinning by other fields like real analysis, algebra, topology, functional analysis, differential equations and so on because it mimicks the way human brain works. They can be programmed to solve a specific problem and it can also be trained by examples.
Special functions can be categorized into three namely, threshold function, ramp function and the logistic sigmoid function. The most important one among all is the logistic sigmoid function because of its gradient descendent learning algorithm. It can be evaluated in different ways, most especially by truncated series expansion. The logistic sigmoid function of the form

$$h(z) = \frac{1}{1+e^{-z}} \tag{1.1}$$

is differentiable and has the following properties:

(i) it outputs real numbers between 0 and 1.

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(ii) it maps a very large input domain to a small range of outputs.
(iii) it never loses information because it is a one-to-one function.
(iv) it increases monotonically.

With all the four properties mentioned above, it shown that logistic sigmoid function is very useful in geometric functions theory. For details, see [5, 8, 10, 16, 17, 21, 24, 25, 27, 28, 29].
Let $E = \{z \in C : |z| < 1\}$ represents the open unit disk and $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in E) \tag{1.2}$$

which are analytic in E and normalized by $f(0) = f'(0) - 1 = 0$.

Suppose $f,\ g \in \mathcal{A}$. We say f is subordinate to g ,written as $f \prec g$ if there exists a Schwarz function $\omega(z)$ which is analytic in E with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$. It follows from Schwarz Lemma that $f(z) \prec g(z)$ $(z \in E)$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$ (see [14]).
Ma and Minda [13] studied and introduced the following class:

$$S^*(\phi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}$$

where ϕ is an analytic function with positive real part in E, $\phi(E)$ is symmetric with respect to the real axis and starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in S^*(\phi)$ is called Ma-Minda starlike with respect to ϕ. The class $C(\phi)$ is the class of functions $f \in A$ for which $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$. The class $S^*(\phi)$ and $C(\phi)$ are well-known subclasses of starlike and convex functions.

For the functions f and ϕ, if there exist analytic functions ψ and ω with the conditions $|\psi(z)| \leq 1,\ \omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in E)$ such that

$$f(z) = \psi(z)\phi(\omega(z)), \tag{1.3}$$

then the function f is said to be quasi-subordinate to ϕ which is demonstrated by

$$f(z) \prec_q \phi(z) \qquad (z \in E).$$

Taking $\psi(z) \equiv 1$, the quasi-subordination given in (1.3) turns to usual subordination $f(z) \prec \phi(z)$. Thus, quasi subordination is working simultaneously as the well known subordination and majorization. For more information, see [3, 9, 11, 12, 15, 18, 23, 26, 7, 31, 32].

Represent by T, the sum of the convergent series of the form

$$T = a_0 + a_1 + a_2 + a_3 + ...,$$

where $a_n \geq 0$ for all $n \in N$. The generalized discrete probability distribution whose probability mass function is given as

$$p(n) = \frac{a_n}{T}, \quad n = 0, 1, 2, 3, \cdots.$$

Clearly, $p(n)$ is a probability mass function because $p(n) \geq 0$ and $\sum_n p(n) = 1$. Now, we introduce the series of the form

$$\gamma(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{1.4}$$

The series given by (1.4) is convergent for $|x| \leq 1$. For special values of a_n, various well known discrete probability distributions such as Poisson distribution, Logarithmic distribution, Zeta distribution, Bernoulli distribution and so on can be obtained.

Let the power series whose coefficients are probabilities of the generalized distribution be

$$f_\phi(z) = z + \sum_{n=2}^{\infty} \frac{a_{n-1}}{T} z^n \tag{1.5}$$

which are analytic in E and normalized by $f_\phi(0) = f'_\phi(0) - 1 = 0$. A fair number of publications are made available in literature in this direction. For recent expository works, see [19, 20, 23].

Now we recall the following facts (see [30]):
(1) If X is a discrete random variable that takes the values $x_1,\ x_2,\ x_3, \cdots$ with respective probabilities $p_1,\ p_2,\ p_3, \cdots$, then expectation of X denoted by $E(X)$ is defined as:

$$E(X) = \sum_{n=1}^{\infty} p_n x_n.$$

(2) The moment generating function (m.g.f) of a random variable X denoted by $M_X(t)$ is given by:

$$M_X(t) = E(e^{tX}) = \sum_{n=0}^{\infty} e^{tn} p(n) = \frac{\gamma(e^t)}{T}.$$

Since we are familiar to the fact that univalent functions are one to one, it must possess an inverse. The inverse of univalent functions are invertible therefore it does not need to be defined on the unit disk E. According to Koebe One-Quarter theorem, a disk of radius $\frac{1}{4}$ is in the image of E under every function $f \in S$. Thus, every function $f \in S$ own an inverse function and this inverse function can be defined on a disk of radius $\frac{1}{4}$. The inverse function of f can be expressed by

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + \left(2a_2^2 - a_3\right)\omega^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)\omega^4 + \cdots . \tag{1.6}$$

If f and f^{-1} are univalent in E then the function $f \in A$ is bi-univalent in E. Quite number of results have obtained in this direction which are littered everywhere. Readers can check more details in [1, 6, 12, 25].
Therefore, let the inverse of the power series whose coefficients are probabilities of the generalized distribution be

$$g_\phi(\omega) = f_\phi^{-1}(\omega) = \omega - \frac{a_1}{T}\omega^2 + (2\frac{a_1^2}{T^2} - \frac{a_2}{T})\omega^3 - (5\frac{a_1^3}{T^3} - 5\frac{a_1}{T}\frac{a_2}{T} + \frac{a_3}{T})\omega^4 + ... \tag{1.7}$$

which is of the particular interest for our investigation.
To achieve our aim, we assume the following:
(1) Let $j \in \mathcal{P}$ be the family of all functions j in E for which $\Re\{j(z)\} > 0$ and of the form

$$j(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in E). \tag{1.8}$$

Then $|c_n| \leq 2$, for each n.
(2) Let $\beta_s(z)$ be given by

$$\beta_s(z) = \frac{2}{1 + e_s^{-z}}$$

with the series expansion

$$\beta_s(z) = 1 + \frac{1}{2[1]_s!}z + \left(\frac{1}{4([1]_s!)^2} - \frac{1}{2[2]_s!}\right)z^2 + \left(\frac{1}{8([1]_s!)^3} + \frac{1}{2[3]_s!} - \frac{1}{2[1]_s![2]_s!}\right)z^3 + ... \tag{1.9}$$

which refers to as remodelled s-sigmoid function (see [4, 22]). The history of $s-$analysis are jetitioned in this work because numerous of it have been in the public domain.
(3) Let

$$\psi(z) = B_0 + B_1 z + B_2 z^2 + ... \qquad (B_0 \neq 0;\ z \in E). \tag{1.10}$$

Motivated by above mentioned references, we introduce the following subclasses of class $\mathcal{A}$.
Definition 1: A function $f_\phi \in \mathcal{A}$ given by (1.5) is in the class $\mathcal{S}_\phi^*(\alpha, \beta_s)$ $(\alpha \geq 0)$ if the following quasi-subordinations

$$\frac{zf'_\phi(z)}{f_\phi(z)} + \alpha\frac{z^2 f''_\phi(z)}{f_\phi(z)} - 1 \prec_q (\beta_s(z) - 1) \tag{1.11}$$

and

$$\frac{\omega g'_\phi(\omega)}{g_\phi(\omega)} + \alpha\frac{\omega^2 g''_\phi(\omega)}{g_\phi(\omega)} - 1 \prec_q (\beta_s(\omega) - 1) \tag{1.12}$$

are satisfied for $z, \omega \in E$ and $g_\phi = f_\phi^{-1}$ given by (1.7).

Definition 2: A function $f_\phi \in \mathcal{A}$ as assumed in (1.5), belongs to the class $M_\phi^*(\alpha, \beta_s)$ $(\alpha \geq 0)$ for $\alpha \geq 0$, if the following quasi-subordinations

$$(1-\alpha)\frac{zf_\phi'(z)}{f_\phi(z)} + \alpha\left(1 + \frac{zf_\phi''(z)}{f_\phi'(z)}\right) - 1 \prec_q (\beta_s(z) - 1) \tag{1.13}$$

and

$$(1-\alpha)\frac{\omega g_\phi'(\omega)}{g_\phi(\omega)} + \alpha\left(1 + \frac{\omega g_\phi''(\omega)}{g_\phi'(\omega)}\right) - 1 \prec_q (\beta_s(\omega) - 1) \tag{1.14}$$

are satisfied for $z, \omega \in E$ and $g_\phi = f_\phi^{-1}$ given by (1.7).
For the newly introduced classes, in the next section we conduct coefficient related studies inspired by the numerous recent publications involving similar research such as [2, 10, 16, 28].

2 Main Results

Theorem 2.1. Let the function $f_\phi(z)$ given by (1.5) be in the class $S_\phi^*(\alpha, \beta_s)$ $(\alpha \geq 0)$. Then

$$\left|\frac{a_1}{T}\right| \leq \frac{|B_0|}{2(1+2\alpha)[1]_s!}, \tag{2.1}$$

$$\left|\frac{a_1}{T}\right| \leq \frac{\sqrt{|B_0||[2]_s! - 2([1]_s!)^2|}}{2[1]_s!\sqrt{(1+4\alpha)[2]_s!}}, \tag{2.2}$$

$$\left|\frac{a_1}{T}\right| \leq \frac{\sqrt{[2]_s!}|B_0|}{\sqrt{|2[1]_s![(1+4\alpha)[2]_s!|B_0| - (1+2\alpha)^2([2]_s! - 2([1]_s!)^2 - 2[1]_s![2]_s!)]|}}, \tag{2.3}$$

and

$$\left|\frac{a_2}{T}\right| \leq \frac{|B_0| + |B_1|}{4(1+3\alpha)[1]_s!} + \frac{|B_0||[2]_s! - 2([1]_s!)^2|}{4([1]_s!)^2)[2]_s!(1+4\alpha)}. \tag{2.4}$$

Proof . Let $f_\phi \in \mathcal{A}$ as assumed in (1.5), belongs to the class $S_\phi^*(\alpha, \beta_s)$ for $\alpha \geq 0$. Then by Definition 1 there exist two analytic functions $u(z)$ and $v(\omega) \in P$ so that

$$\frac{zf_\phi'(z)}{f_\phi(z)} + \alpha\frac{z^2 f_\phi''(z)}{f_\phi(z)} - 1 = \psi(z)(\beta_s(u(z)) - 1) \tag{2.5}$$

and

$$\frac{\omega g_\phi'(\omega)}{g_\phi(\omega)} + \alpha\frac{\omega^2 g_\phi''(\omega)}{g_\phi(\omega)} - 1 = \psi(\omega)(\beta_s(v(\omega)) - 1). \tag{2.6}$$

Expanding the left hand sides of (2.5) and (2.6), we get

$$\frac{zf_\phi'(z)}{f_\phi(z)} + \alpha\frac{z^2 f_\phi''(z)}{f_\phi(z)} - 1 = (1+2\alpha)\frac{a_1}{T}z + \left(2(1+3\alpha)\frac{a_2}{T} - (1+2\alpha)\frac{a_1^2}{T^2}\right)z^2 \tag{2.7}$$

and

$$\frac{\omega g_\phi'(\omega)}{g_\phi(\omega)} + \alpha\frac{\omega^2 g_\phi''(\omega)}{g_\phi(\omega)} - 1 = -(1+2\alpha)\frac{a_1}{T}\omega + \left(-2(1+3\alpha)\frac{a_2}{T} + (3+10\alpha)\frac{a_1^2}{T^2}\right)\omega^2. \tag{2.8}$$

Let the functions $p,\ q \in P$ given by

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + \ldots \quad and \quad q(\omega) = \frac{1+v(\omega)}{1-v(\omega)} = 1 + d_1\omega + d_2\omega^2 + \ldots \quad (z,\ \omega \in E). \tag{2.9}$$

Equivalently, from (2.9), we obtain

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{c_1}{2}z+\left(\frac{c_2}{2}-\frac{c_1^2}{4}\right)z^2+... \tag{2.10}$$

and

$$v(\omega)=\frac{q(\omega)-1}{q(\omega)+1}=\frac{d_1}{2}\omega+\left(\frac{d_2}{2}-\frac{d_1^2}{4}\right)\omega^2+.... \tag{2.11}$$

Then, by (2.10) to (2.11) and (1.9), we get

$$\beta_s(u(z))=1+\frac{c_1}{4[1]_s!}z+\left(\frac{c_2}{4[1]_s!}+\frac{[2]_s!-2([1]_s!)^2-2[1]_s![2]_s!}{16([1]_s!)^2[2]_s!}c_1^2\right)z^2+... \tag{2.12}$$

and

$$\beta_s(v(\omega))=1+\frac{d_1}{4[1]_s!}\omega+\left(\frac{d_2}{4[1]_s!}+\frac{[2]_s!-2([1]_s!)^2-2[1]_s![2]_s!}{16([1]_s!)^2[2]_s!}d_1^2\right)\omega^2+.... \tag{2.13}$$

Furthermore, we see that

$$\psi(z)[\beta_s(u(z))-1]=\frac{B_0c_1}{4[1]_s!}z+\left(\left(\frac{c_2}{4[1]_s!}+\frac{[2]_s!-2([1]_s!)^2-2[1]_s![2]_s!}{16([1]_s!)^2[2]_s!}c_1^2\right)B_0+\frac{B_1c_1}{4[1]_s!}\right)z^2+... \tag{2.14}$$

and

$$\psi(\omega)[\beta_s(v(\omega))-1]=\frac{B_0d_1}{4[1]_s!}\omega+\left(\left(\frac{d_2}{4[1]_s!}+\frac{[2]_s!-2([1]_s!)^2-2[1]_s![2]_s!}{16([1]_s!)^2[2]_s!}d_1^2\right)B_0+\frac{B_1d_1}{4[1]_s!}\right)\omega^2+.... \tag{2.15}$$

Therefore, from (2.7) to (2.8) and (2.14) to (2.15), we get

$$(1+2\alpha)\frac{a_1}{T}=\frac{B_0c_1}{4[1]_s!} \tag{2.16}$$

$$2(1+3\alpha)\frac{a_2}{T}-(1+2\alpha)\frac{a_1^2}{T^2}=\left(\frac{c_2}{4[1]_s!}+\frac{[2]_s!-2([1]_s!)^2-2[1]_s![2]_s!}{16([1]_s!)^2[2]_s!}c_1^2\right)B_0+\frac{B_1c_1}{4[1]_s!} \tag{2.17}$$

$$-(1+2\alpha)\frac{a_1}{T}=\frac{B_0d_1}{4[1]_s!} \tag{2.18}$$

and

$$-2(1+3\alpha)\frac{a_2}{T}+(3+10\alpha)\frac{a_1^2}{T^2}=\left(\frac{d_2}{4[1]_s!}+\frac{[2]_s!-2([1]_s!)^2-2[1]_s![2]_s!}{16([1]_s!)^2[2]_s!}d_1^2\right)B_0+\frac{B_1d_1}{4[1]_s!}. \tag{2.19}$$

In view of (2.16) and (2.18), we can express

$$\frac{a_1}{T}=\frac{B_0c_1}{4(1+2\alpha)[1]_s!}=-\frac{B_0d_1}{4(1+2\alpha)[1]_s!} \tag{2.20}$$

such that

$$c_1=-d_1 \tag{2.21}$$

and

$$B_0^2(c_1^2+d_1^2)=32(1+2\alpha)^2([1]_s!)^2\frac{a_1^2}{T^2}. \tag{2.22}$$

By (2.17) and (2.19), we obtain

$$\frac{(c_2+d_2)B_0}{4[1]_s!}+\frac{[2]_s!-2([1]_s!)^2-2[1]_s![2]_s!}{16([1]_s!)^2[2]_s!}(c_1^2+d_1^2)B_0+\frac{(c_1+d_1)B_1}{4[1]_s!}=2(1+4\alpha)\frac{a_1^2}{T^2}. \tag{2.23}$$

From (2.21),

$$c_1+d_1=0.$$

Therefore, (2.23) reduces to

$$\frac{(c_2+d_2)B_0}{4[1]_s!}+\frac{[2]_s!-2([1]_s!)^2-2[1]_s![2]_s!}{16([1]_s!)^2[2]_s!}(c_1^2+d_1^2)B_0=2(1+4\alpha)\frac{a_1^2}{T^2}. \tag{2.24}$$

Then, together with (2.22) to (2.24), we get

$$\frac{a_1^2}{T^2}=\frac{(c_2+d_2)[2]_s!B_0^2}{8[1]_s![(1+4\alpha)[2]_s!B_0-(1+2\alpha)^2([2]_s!-2([1]_s!)^2-2[1]_s![2]_s!)]}. \tag{2.25}$$

Moreover, it implies from (2.20) and (2.22) to (2.25) that

$$\left|\frac{a_1}{T}\right|\leq\frac{|B_0|}{2(1+2\alpha)[1]_s!},$$

$$\left|\frac{a_1}{T}\right|\leq\frac{\sqrt{|B_0||[2]_s!-2([1]_s!)^2|}}{2[1]_s!\sqrt{(1+4\alpha)[2]_s!}},$$

and

$$\left|\frac{a_1}{T}\right|\leq\frac{\sqrt{[2]_s!}|B_0|}{\sqrt{|2[1]_s![(1+4\alpha)[2]_s!|B_0|-(1+2\alpha)^2([2]_s!-2([1]_s!)^2-2[1]_s![2]_s!)]|}}.$$

Similarly from (2.17), (2.19) and (2.21), it implies that

$$\frac{(c_2-d_2)B_0}{4[1]_s!}+\frac{(c_1-d_1)B_1}{4[1]_s!}=4(1+3\alpha)\left(\frac{a_2}{T}-\frac{a_1^2}{T^2}\right). \tag{2.26}$$

Hence, from (2.22) and (2.26), we have

$$\frac{a_2}{T}=\frac{(c_2-d_2)B_0+(c_1-d_1)B_1}{16(1+3\alpha)[1]_s!}+\frac{(c_1^2+d_1^2)B_0^2}{32(1+2\alpha)^2([1]_s!)^2}. \tag{2.27}$$

Further, from (2.27), we remark that

$$\left|\frac{a_2}{T}\right|\leq\frac{|B_0|+|B_1|}{4(1+3\alpha)[1]_s!}+\frac{B_0^2}{4(1+2\alpha)^2([1]_s!)^2}. \tag{2.28}$$

On the other hand, by (2.24) and (2.27) , we obtain

$$\frac{a_2}{T}=\frac{(c_2-d_2)B_0+(c_1-d_1)B_1}{16(1+3\alpha)[1]_s!}+\frac{4[1]_s![2]_s!(c_2+d_2)B_0+([2]_s!-2([1]_s!)^2-2[1]_s![2]_s!)(c_1^2+d_1^2)B_0}{32([1]_s!)^2[2]_s!(1+4\alpha)}. \tag{2.29}$$

Therefore, from (2.29), we see that

$$\left|\frac{a_2}{T}\right|\leq\frac{|B_0|+|B_1|}{4(1+3\alpha)[1]_s!}+\frac{|B_0|([2]_s!-2([1]_s!)^2)}{4([1]_s!)^2)[2]_s!(1+4\alpha)}.$$

This completes the proof. □

Theorem 2.2. Let the function $f_\phi(z)\in\mathcal{A}$ given by (1.5) belongs to the class $M_\phi^*(\alpha,\beta_s)$ $(\alpha\geq 0)$. Then

$$\left|\frac{a_1}{T}\right|\leq\frac{|B_0|}{2(1+\alpha)[1]_s!}, \tag{2.30}$$

$$\left|\frac{a_1}{T}\right|\leq\frac{\sqrt{|B_0||[2]_s!-2([1]_s!)^2|}}{2[1]_s!\sqrt{2(1+2\alpha)[2]_s!}}, \tag{2.31}$$

$$\left|\frac{a_1}{T}\right|\leq\frac{\sqrt{[2]_s!}|B_0|}{\sqrt{|2[1]_s![(1+2\alpha)[2]_s!|B_0|-(1+\alpha)^2([2]_s!-2([1]_s!)^2-2[1]_s![2]_s!)]|}}, \tag{2.32}$$

$$\left|\frac{a_2}{T}\right| \leq \frac{|B_0|+|B_1|}{4(1+2\alpha)[1]_s!} + \frac{B_0^2}{4(1+\alpha)^2([1]_s!)^2}. \tag{2.33}$$

and

$$\left|\frac{a_2}{T}\right| \leq \frac{|B_0|+|B_1|}{4(1+2\alpha)[1]_s!} + \frac{|B_0||[2]_s! - 2([1]_s!)^2|}{8([1]_s!)^2)[2]_s!(1+2\alpha)}. \tag{2.34}$$

Proof . If $f_\phi \in \mathcal{A}$ as assumed in (1.5), belongs to the class $M_\phi(\alpha, \beta_s)$ for $\alpha \geq 0$, then by Definition 2 there exist two analytic functions $u(z)$ and $v(\omega) \in P$ so that

$$(1-\alpha)\frac{zf'_\phi(z)}{f_\phi(z)} + \alpha\left(1 + \frac{zf''_\phi(z)}{f'_\phi(z)}\right) - 1 = \psi(z)(\beta_s(u(z)) - 1) \tag{2.35}$$

and

$$(1-\alpha)\frac{\omega g'_\phi(\omega)}{g_\phi(\omega)} + \alpha\left(1 + \frac{\omega g''_\phi(\omega)}{g'_\phi(\omega)}\right) - 1 = \psi(\omega)(\beta_s(v(\omega)) - 1). \tag{2.36}$$

Expanding the left hand sides of (2.35) and (2.36), we get

$$\frac{zf'_\phi(z)}{f_\phi(z)} + \alpha\frac{z^2 f''_\phi(z)}{f_\phi(z)} - 1 = (1+\alpha)\frac{a_1}{T}z + \left(2(1+2\alpha)\frac{a_2}{T} - (1+3\alpha)\frac{a_1^2}{T^2}\right)z^2 \tag{2.37}$$

and

$$\frac{\omega g'_\phi(\omega)}{g_\phi(\omega)} + \alpha\frac{\omega^2 g''_\phi(\omega)}{g_\phi(\omega)} - 1 = -(1+\alpha)\frac{a_1}{T}\omega + \left(-2(1+2\alpha)\frac{a_2}{T} + (3+5\alpha)\frac{a_1^2}{T^2}\right)\omega^2. \tag{2.38}$$

Let the functions $p,\ q \in P$ by

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + ... \quad and \quad q(\omega) = \frac{1+v(\omega)}{1-v(\omega)} = 1 + d_1\omega + d_2\omega^2 + ... \quad (z,\ \omega \in E).$$

Equivalently, from (2.9), we obtain

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{c_1}{2}z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)z^2 + ...$$

and

$$v(\omega) = \frac{q(\omega)-1}{q(\omega)+1} = \frac{d_1}{2}\omega + \left(\frac{d_2}{2} - \frac{d_1^2}{4}\right)\omega^2 + ...$$

Then, by (2.10)-(2.11) and (1.9), we get

$$\beta_s(u(z)) = 1 + \frac{c_1}{4[1]_s!}z + \left(\frac{c_2}{4[1]_s!} + \frac{[2]_s! - 2([1]_s!)^2 - 2[1]_s![2]_s!}{16([1]_s!)^2[2]_s!}c_1^2\right)z^2 + ...$$

and

$$\beta_s(v(\omega)) = 1 + \frac{d_1}{4[1]_s!}\omega + \left(\frac{d_2}{4[1]_s!} + \frac{[2]_s! - 2([1]_s!)^2 - 2[1]_s![2]_s!}{16([1]_s!)^2[2]_s!}d_1^2\right)\omega^2 + ...$$

Furthermore, we see that

$$\psi(z)[\beta_s(u(z)) - 1] = \frac{B_0 c_1}{4[1]_s!}z + \left(\left(\frac{c_2}{4[1]_s!} + \frac{[2]_s! - 2([1]_s!)^2 - 2[1]_s![2]_s!}{16([1]_s!)^2[2]_s!}c_1^2\right)B_0 + \frac{B_1 c_1}{4[1]_s!}\right)z^2 + ...$$

and

$$\psi(\omega)[\beta_s(v(\omega)) - 1] = \frac{B_0 d_1}{4[1]_s!}\omega + \left(\left(\frac{d_2}{4[1]_s!} + \frac{[2]_s! - 2([1]_s!)^2 - 2[1]_s![2]_s!}{16([1]_s!)^2[2]_s!}d_1^2\right)B_0 + \frac{B_1 d_1}{4[1]_s!}\right)\omega^2 + ...$$

Therefore, from (2.7)-(2.8) and (2.14)-(2.15), we get

$$(1+\alpha)\frac{a_1}{T} = \frac{B_0 c_1}{4[1]_s!} \tag{2.39}$$

$$2(1+2\alpha)\frac{a_2}{T}-(1+3\alpha)\frac{a_1^2}{T^2}=\left(\frac{c_2}{4[1]_s!}+\frac{[2]_s!-2([1]_s!)^2-2[1]_s![2]_s!}{16([1]_s!)^2[2]_s!}c_1^2\right)B_0+\frac{B_1c_1}{4[1]_s!} \tag{2.40}$$

$$-(1+\alpha)\frac{a_1}{T}=\frac{B_0d_1}{4[1]_s!} \tag{2.41}$$

and

$$-2(1+2\alpha)\frac{a_2}{T}+(3+5\alpha)\frac{a_1^2}{T^2}=\left(\frac{d_2}{4[1]_s!}+\frac{[2]_s!-2([1]_s!)^2-2[1]_s![2]_s!}{16([1]_s!)^2[2]_s!}d_1^2\right)B_0+\frac{B_1d_1}{4[1]_s!}. \tag{2.42}$$

In view of (2.39) and (2.41), we can express

$$\frac{a_1}{T}=\frac{B_0c_1}{4(1+\alpha)[1]_s!}=-\frac{B_0d_1}{4(1+\alpha)[1]_s!} \tag{2.43}$$

such that

$$c_1=-d_1 \tag{2.44}$$

and

$$B_0^2(c_1^2+d_1^2)=32(1+\alpha)^2([1]_s!)^2\frac{a_1^2}{T^2}. \tag{2.45}$$

By (2.40) and (2.42), we obtain

$$\frac{(c_2+d_2)B_0}{4[1]_s!}+\frac{[2]_s!-2([1]_s!)^2-2[1]_s![2]_s!}{16([1]_s!)^2[2]_s!}(c_1^2+d_1^2)B_0+\frac{(c_1+d_1)B_1}{4[1]_s!}=4(1+2\alpha)\frac{a_1^2}{T^2}. \tag{2.46}$$

Then, together with (2.45)-(2.46), we get

$$\frac{a_1^2}{T^2}=\frac{(c_2+d_2)[2]_s!B_0^2}{8[1]_s![(1+2\alpha)[2]_s!B_0-(1+\alpha)^2([2]_s!-2([1]_s!)^2-2[1]_s![2]_s!)]}. \tag{2.47}$$

Moreover, it implies from (2.43) and (2.45) to (2.47) that

$$\left|\frac{a_1}{T}\right|\leq\frac{|B_0|}{2(1+\alpha)[1]_s!},$$

$$\left|\frac{a_1}{T}\right|\leq\frac{\sqrt{|B_0|([2]_s!-2([1]_s!)^2)}}{2[1]_s!\sqrt{2(1+2\alpha)[2]_s!}},$$

and

$$\left|\frac{a_1}{T}\right|\leq\frac{\sqrt{[2]_s!}|B_0|}{\sqrt{2[1]_s![(1+2\alpha)[2]_s!|B_0|-(1+\alpha)^2([2]_s!-2([1]_s!)^2-2[1]_s![2]_s!)]}},$$

then (2.30)-(2.32) holds. Similarly from (2.40), (2.42) and (2.44), it implies that

$$\frac{(c_2-d_2)B_0}{4[1]_s!}+\frac{(c_1-d_1)B_1}{4[1]_s!}=4(1+2\alpha)\left(\frac{a_2}{T}-\frac{a_1^2}{T^2}\right). \tag{2.48}$$

Hence, from (2.45) and (2.48), we have

$$\frac{a_2}{T}=\frac{(c_2-d_2)B_0+(c_1-d_1)B_1}{16(1+2\alpha)[1]_s!}+\frac{(c_1^2+d_1^2)B_0^2}{32(1+\alpha)^2([1]_s!)^2}. \tag{2.49}$$

Further, from (2.49), we remark that

$$\left|\frac{a_2}{T}\right|\leq\frac{|B_0|+|B_1|}{4(1+2\alpha)[1]_s!}+\frac{B_0^2}{4(1+\alpha)^2([1]_s!)^2}. \tag{2.50}$$

On the other hand, by (2.46) and (2.49) , we obtain

$$\frac{a_2}{T}=\frac{(c_2-d_2)B_0+(c_1-d_1)B_1}{16(1+2\alpha)[1]_s!}+\frac{4[1]_s![2]_s!(c_2+d_2)B_0+([2]_s!-2([1]_s!)^2-2[1]_s![2]_s!)(c_1^2+d_1^2)B_0}{64([1]_s!)^2[2]_s!(1+2\alpha)}. \tag{2.51}$$

Therefore, from (2.51), we see that

$$\left|\frac{a_2}{T}\right| \leq \frac{|B_0|+|B_1|}{4(1+2\alpha)[1]_s!} + \frac{|B_0|([2]_s! - 2([1]_s!)^2)}{8([1]_s!)^2)[2]_s!(1+2\alpha)}. \tag{2.52}$$

This proves the assertion of Theorem 2. That completes the proof. □

3 Concluding Remark

In this paper, we introduced two subclasses of bi-univalent function by making use of generalized distribution series associated with remodelled s-sigmoid function. The first two initial coefficient bounds are investigated. Efforts can be made in the same line to obtain the upper bound of Fekete-Szegö inequality and second Hankel determinant.

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