Int. J. Nonlinear Anal. Appl. 14 (2023) 9, 345–356
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.7239

Fixed point theorems in ordered non-Archimedean fuzzy metric spaces

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(Communicated by Ali Jabbari)

Abstract

In this paper, we extend very recent fixed point theorems in the setting of ordered non-Archimedean fuzzy metric spaces. We present some fixed point theorems for self-mappings satisfying generalized (ϕ, ψ)-contraction conditions in partially ordered complete non-Archimedean fuzzy metric spaces. On the other hand, we consider a more general class of auxiliary functions in the contractivity condition and we extend recently fixed point theorems for complete ordered non-Archimedean fuzzy metric spaces. Also, we present a few examples to illustrate the validity of the results obtained in the paper.

Keywords: Ordered fuzzy metric space, Coincidence and Common fixed point, Compatible and Weakly compatible functions, weakly increasing
2020 MSC: Primary 47H10; Secondary 54H25

1 Introduction

Fixed points of mappings satisfying contractive conditions in generalized metric spaces are highly useful in large number of mathematical problems of pure and applied mathematics. In 1922, Banach created a famous result called Banach contraction principle in the concept of the fixed point theory [3]. Later, most of the authors introduced many works regarding the fixed point theory in various of spaces. Ran and Reuings [14] have extended the result in this direction, discussed the existence of fixed points for certain maps in ordered metric space and also presented some applications to matrix linear equations. The result of [14] has been extended by Nieto et al. [12] involving nondecreasing mappings and used their results in obtaining a unique solution of a first order differential equation.

There are two well-known extensions of the notion of metric space to frameworks in which imprecise models are considered: fuzzy metric spaces (see [15]) and probabilistic metric spaces [17, 18]. These two concepts are very similar, but they are different in nature. The concept of a fuzzy metric space was introduced in different ways by some authors (see [5, 6]). Gregori and Sapena [6] introduced the notion of fuzzy contractive mappings and gave some fixed point theorems for complete fuzzy metric spaces in the sense of George and Veeramani, and also for Kramosil and Michalek's fuzzy metric spaces which are complete in Grabiec's sense. Mihet [11] developed the class of fuzzy contractive mappings of Gregori and Sapena, considered these mappings in non-Archimedean fuzzy metric spaces in the sense of Kramosil and Michalek, and obtained a fixed point theorem for fuzzy contractive mappings. Lots of different types of fixed point theorems has been presented by many authors by expanding the Banach's result, simultaneously (see [19, 22]).

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Received: March 2022 *Accepted:* September 2022

Recently, Sun and Yang introduced the concept of fuzzy metric spaces and proved two common fixed point theorems for four mappings (see [22]).

Recently, many fixed point theorems have been presented for probabilistic metric space $(X, F, *)$, where F is a distance distribution function. Many of them were inspired by the corresponding results on metric spaces. One of the most attractive and effective ways to introduce contractivity conditions in the probabilistic framework is based on considering some terms like in the following expression:

$$\frac{1}{F(x,y,t)} - 1, \quad \text{where} \quad x, y \in X \text{ and } t > 0$$

see [6, 11, 21], for more details. In this paper, we consider the more general contractivity conditions, replacing the function $t \to \frac{1}{t} - 1$ by an appropriate function h to establish the existence of fixed points for a self-mapping and common fixed points and coincidence points for two self-mappings in ordered complete fuzzy metric space. Our results generalize Theorem 2.1 and 2.2 of [4] and the corollaries of [7, 20].

2 Preliminaries

Before giving our main results, we recall some basic concepts and results in metric space and fuzzy metric spaces.

Definition 2.1. [1] A point $\nu \in X$, is called coincidence (common fixed) point for two self-mappings T and S, if $T\nu = S\nu$ ($\nu = T\nu = S\nu$).

Definition 2.2. [10] A metric space X with a partially ordered relation $\preceq$ is called a partially ordered metric space and is denoted by $(X, \preceq)$.

Definition 2.3. [10] Let $(X, \preceq)$ be a partially ordered metric space.
(i) If any two elements of X are comparable, then it is called a well-ordered set.
(ii) A self-mapping T on X is said to be monotone nondecreasing, if $T(\nu) \preceq T(\mu)$ for all $\nu, \mu \in X$ with $\nu \preceq \mu$.
(iii) Let T and S be two self-mappings on X. Then T is called monotone S-nondecreasing, if $Tx \preceq Ty$ for all $x, y \in X$ with $Sx \preceq Sy$.

Definition 2.4. [9] Let (X, d) be a metric space.
(i) Two self-mappings T and S on X are called compatible, if for all sequence $\{x_n\}$ with $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n$, then $\lim_{n\to\infty} d(TSx_n, STx_n) = 0$.
(ii) A pair of self-mappings (T, S) on X is called weakly compatible, if they commute at their coincidence points, i.e. $T\nu = S\nu$ implies $TS\nu = ST\nu$.

Definition 2.5. [17] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular norm (in short, continuous $t-$norm) if it satisfies the following conditions:

(TN-1) $*$ is commutative and associative,
(TN-2) $*$ is continuous,
(TN-3) $*(a, 1) = a$ for all $a \in [0, 1]$,
(TN-4) $*(a, b) \leq *(c, d)$ whenever $a, b, c, d \in [0, 1]$ with $a \leq c$, $b \leq d$.

Definition 2.6. [22] A fuzzy metric space is a triple $(X, F, *)$ where X is a nonempty set, $*$ is a continuous $t-$norm and F is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$:

(FM-1) $F(x, y, t) > 0$ for all $t > 0$,
(FM-2) $F(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(FM-3) $F(x, y, t) = F(y, x, t)$ for all $t > 0$,
(FM-4) $F(x, y, t + s) \geq F(x, z, s) * F(z, y, t)$ for all $s, t > 0$,
(FM-5) $F(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

If the triangular inequality (FM-4) is replaced by

$$F(x,y,max\{s,t\}) \geq F(x,z,s) * F(z,y,t)$$

for all $x, y, z \in X$ and all $s, t > 0$ or equivalently,

$$F(x,y,t) \geq F(x,z,t) * F(z,y,t), \tag{2.1}$$

then the triple $(X, F, *)$ is called a non-Archimedean fuzzy metric space [8].

Example 2.7. Let (X, d) be a metric space. Then the triple $(X, F, *)$ is a fuzzy metric space on X where $*(a, b) = ab$ for all $a, b \in [0, 1]$ and $F(x, y, t) = t/(t + d(x, y))$ for all $x, y \in X$ and all $t > 0$. We call this F as the standard fuzzy metric induced by the metric d. Even if we define $a * b = min\{a, b\}$ for all $a, b \in [0, 1]$, the triple $(X, F, *)$ will be a fuzzy metric space.

Definition 2.8. [22] Let $\{x_n\}$ be a sequence in a fuzzy (or a non-Archimedean fuzzy) metric space $(X, F, *)$. We say that:

- $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} F(x_n, x, t) = 1$; i.e., for all $t > 0$ and all $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $F(x_n, x, t) > 1 - \lambda$ for all $n \geq n_0$ (in such a case, we write $\{x_n\} \to x$);
- $\{x_n\}$ is a Cauchy sequence if and only if for all $t > 0$ and all $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $F(x_n, x_m, t) > 1 - \lambda$ for all $n, m \geq n_0$. $\{x_n\}$ is a G-Cauchy sequence if and only if for all $t > 0$ and all $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $F(x_n, x_{n+p}, t) > 1 - \lambda$ for all $n \geq n_0$ and $p > 0$ (i.e., $\lim_{n\to\infty} F(x_n, x_{n+p}, t) = 1$).
- The fuzzy (or the non-Archimedean fuzzy) metric space $(X, F, *)$ is called complete (G-complete) if every Cauchy (G-Cauchy) sequence is convergent.

Lemma 2.9. [22] Let $(X, F, *)$ be a fuzzy metric space. Then $F(x, y, t)$ is nondecreasing with respect to t for all $x, y \in X$.

Lemma 2.10. [22] Let $(X, F, *)$ be a fuzzy metric space. Then F is a continuous function on $X^2 \times (0, \infty)$.

It is easy to prove that a $F(x, y, t)$ in a non-Archimedean fuzzy metric space $(X, F, *)$ is also nondecreasing with respect to t and continuous for all $x, y \in X$.

Definition 2.11. [22] A (complete) fuzzy metric space $(X, F, *)$ with a partially ordered relation $\preceq$ is called a (complete) partially ordered fuzzy metric space and denoted by $(X, F, *, \preceq)$.

The following families of auxiliary functions were considered in [16].

Definition 2.12. Let Φ be the family of all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying:
(1) $\phi(t) = 0$ if and only if $t = 0$;
(2) $\lim_{t\to\infty} \phi(t) = \infty$;
(3) ϕ is continuous at $t = 0$.

Definition 2.13. Let Ψ be the class of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying:
(1) ψ is nondecreasing;
(2) $\psi(0) = 0$
(3) for a sequence $\{a_n\}$ in $[0, \infty)$ whit $\{a_n\} \to 0$, $\{\psi^n(a_n)\} \to 0$ (ψ^n denotes the nth-iterate of ψ)

It worths mentioning that $\psi \in \Psi$ is continuous at $t = 0$. (Proposition 7 of [16])

The following family of auxiliary functions were considered in [16].

Definition 2.14. Let $\mathcal{H}$ be the family of all functions $h : (0,1] \to [0,\infty)$ satisfying the following conditions,
$(\mathcal{H}_1)$ for all sequence $\{a_n\}$ in $(0,1]$, $\{a_n\} \to 1$ if and only if $\{h(a_n)\} \to 0$;
$(\mathcal{H}_2)$ for all sequence $\{a_n\}$ in $(0,1]$, $\{a_n\} \to 0$ if and only if $\{h(a_n)\} \to \infty$.

The previous conditions are guaranteed whenever $h : (0,1] \to [0,\infty)$ is a strictly decreasing bijection such that h and h^{-1} are continuous (in a broad sense, it is sufficient to assume the continuities of h and h^{-1} on the extremes of the respective domains). For instance, this is the case of the function $h(t) = 1/t - 1$ for all $t \in (0,1]$. However, the functions in $\mathcal{H}$ need not to be continuous nor monotone.

Proposition 2.15. [16] If $h \in \mathcal{H}$, then $h(1) = 0$. Furthermore, $h(t) = 0$ if and only if $t = 1$.

3 Main results

In this section, we present an extension of fixed point theorems in several ways: the metric space is more general, the contractivity condition is better and the involved auxiliary functions form a wider class.

Theorem 3.1. Let $(X, F, *, \preceq)$ be a partially ordered G-complete non-Archimedean fuzzy metric space and let $T : X \to X$ be a continuous and nondecreasing mapping with regards to $\preceq$. Suppose that there exist $c \in (0,1), \phi \in \Phi, \psi \in \Psi$ and $h \in \mathcal{H}$ such that

$$h(F(Tx, Ty, \phi(ct))) \leq \psi(h(M(x,y))) \tag{3.1}$$

for all $x, y \in X$ with $x \preceq y$ and all $t > 0$ and

$$M(x,y) = \max\left\{F(x,y,\phi(t)), \frac{F(x,Tx,\phi(t)) * F(y,Ty,\phi(t))}{1 + F(Tx,Ty,\phi(t))}\right\}. \tag{3.2}$$

If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and also $\lim\limits_{t\to\infty} F(x_0, Tx_0, t) = 1$, then T has at least one fixed point in X.

Proof . If there exists $x_0 \in X$ such that $Tx_0 = x_0$, then the proof is finished. Suppose $x_0 \in X$, such that $x_0 \prec Tx_0$ and $\lim\limits_{t\to\infty} F(x_0, Tx_0, t) = 1$, then construct the sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n$ for $n \geq 0$. Since T is nondecreasing, by using mathematical induction, we get the following

$$\begin{aligned} x_0 \prec Tx_0 = x_1 \preceq Tx_1 = x_2 \preceq .... \preceq Tx_{n-1} = x_n \\ \preceq Tx_n = x_{n+1} \preceq .... \end{aligned} \tag{3.3}$$

If for some $n_0 \in \mathbb{N}$, $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ then x_{n_0} is a fixed point of T and we have nothing to prove. Suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Since $x_n \succ x_{n-1}$ for all $n \geq 1$, by (3.1) we have

$$\begin{aligned} h(F(x_n, x_{n+1}, \phi(ct))) &= h(F(Tx_{n-1}, Tx_n, \phi(ct))) \\ &\leq \psi(h(M(x_{n-1}, x_n))), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{F(x_{n-1}, x_n, \phi(t)), \frac{F(x_{n-1}, Tx_{n-1}, \phi(t)) * F(x_n, Tx_n, \phi(t))}{1 + F(Tx_{n-1}, Tx_n, \phi(t))}\right\} \\ &= \max\left\{F(x_{n-1}, x_n, \phi(t)), \frac{F(x_{n-1}, x_n, \phi(t)) * F(x_n, x_{n+1}, \phi(t))}{1 + F(x_n, x_{n+1}, \phi(t))}\right\}. \end{aligned} \tag{3.5}$$

Since

$$\frac{F(x_{n-1}, x_n, \phi(t)) * F(x_n, x_{n+1}, \phi(t))}{1 + F(x_n, x_{n+1}, \phi(t))} \leq F(x_{n-1}, x_n, \phi(t)),$$

by (3.5) we have

$$M(x_{n-1}, x_n) = F(x_{n-1}, x_n, \phi(t)),$$

and hence from (3.4) again we have

$$h(F(x_n, x_{n+1}, \phi(ct))) \le \psi(h(F(x_{n-1}, x_n, \phi(t)))) \tag{3.6}$$

for all $t > 0$ and all $n \ge 1$. We claim that $\lim\limits_{n\to\infty} F(x_n, x_{n+1}, s) = 1$ for all $s > 0$. In order to prove it, let $s > 0$ be arbitrary. As $\lim\limits_{r\to\infty} c^r s = 0$ and ϕ is continuous at $t = 0$, then $\lim\limits_{r\to\infty} \phi(c^r s) = \phi(0) = 0$. Since $s > 0$, there exists $r \in \mathbb{N}$ such that

$$\phi(c^r s) \le s.$$

Let $n \in \mathbb{N}$ be such that $n > r$. Applying the contractivity (3.6), it follows that

$$h(F(x_n, x_{n+1}, \phi(c^r s))) \le \psi(h(F(x_{n-1}, x_n, \phi(c^{r-1} s)))). \tag{3.7}$$

Repeating this argument, we find that

$$h(F(x_{n-1}, x_n, \phi(c^{r-1} s))) \le \psi(h(F(x_{n-2}, x_{n-1}, \phi(c^{r-2} s)))).$$

As ψ is nondecreasing, then

$$\psi(h(F(x_{n-1}, x_n, \phi(c^{r-1} s)))) \le \psi^2(h(F(x_{n-2}, x_{n-1}, \phi(c^{r-2} s)))). \tag{3.8}$$

Combining inequalities (3.7) and (3.8), we deduce that

$$\begin{aligned} h(F(x_n, x_{n+1}, \phi(c^r s))) &\le \psi(h(F(x_{n-1}, x_n, \phi(c^{r-1} s)))) \\ &\le \psi^2(h(F(x_{n-2}, x_{n-1}, \phi(c^{r-2} s)))). \end{aligned}$$

By repeating this argument n times, we have

$$\begin{aligned} h(F(x_n, x_{n+1}, \phi(c^r s))) &\le \psi^n(h(F(x_0, x_1, \phi(c^{r-n} s)))) \\ &\le \psi^n(h(F(x_0, x_1, \phi(\frac{s}{c^{n-r}})))), \end{aligned} \tag{3.9}$$

for all $n > r$. As a consequence,

$$\begin{aligned} \lim_{n\to\infty} \frac{s}{c^{n-r}} = \infty &\Rightarrow \lim_{n\to\infty} \phi(\frac{s}{c^{n-r}}) = \infty \\ &\Rightarrow \lim_{n\to\infty} F(x_0, x_1, \phi(\frac{s}{c^{n-r}})) = 1 \\ &\Rightarrow \lim_{n\to\infty} h(F(x_0, x_1, \phi(\frac{s}{c^{n-r}}))) = 0. \end{aligned}$$

As the sequence $\{a_n = h(F(x_0, x_1, \phi(\frac{s}{c^{n-r}})))\} \to 0$ we have $\{\psi^n(a_n)\} \to 0$. Since $h \in \mathcal{H}$, by (3.9) we deduce that

$$\lim_{n\to\infty} h(F(x_n, x_{n+1}, \phi(c^r s))) = 0.$$

In particular, as $h \in \mathcal{H}$, condition $(\mathcal{H}_1)$ implies that

$$\lim_{n\to\infty} F(x_n, x_{n+1}, \phi(c^r s)) = 1.$$

Taking into account $\phi(c^r s) < s$, we observe that

$$F(x_n, x_{n+1}, \phi(c^r s)) \le F(x_n, x_{n+1}, s) \le 1.$$

Therefore,

$$\lim_{n\to\infty} F(x_n, x_{n+1}, s) = 1$$

which means that $\{x_n\}$ is a G-Cauchy sequence in X, [16, Lemma 15]. Since X is G-complete, there exists $x \in X$ such that $\{x_n\} \to x$. Also, the continuity of T implies that

$$Tx = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x.$$

Therefore, x is a fixed point of T in X. □

Example 3.2. Let $X = [0,1)$, $*(a,b) = \min\{a,b\}$, and

$$F(x,y,t) = \begin{cases} 1, & \text{if } x = y \\ \frac{1}{1+\max\{x,y\}}, & \text{otherwise,} \end{cases}$$

where $x, y \in X$ and $t > 0$. It is easy to prove that $(X, F, *, \preceq)$ is a complete partially ordered non-Archimedean fuzzy metric space with usual ordering. Define $T : X \to X$ by $T(x) = x/2$ for all $x \in X$. Assume that $\psi(t) = \phi(t) = t$ for all $t \in [0,\infty)$ and let $h : (0,1] \to [0,\infty)$ be a strictly decreasing bijection between $(0,1]$ and $[0,\infty)$ such that h and h^{-1} are continuous (for instance, $h(t) = 1/t - 1$, $t \in (0,1]$, but any other function verifying these properties yields the same result). In this context, the contractivity conditions (3.1) and (3.2) are equivalent to $h(F(Tx,Ty,\phi(ct))) \leq \psi(h(M(x,y)))$ if and only if $h(F(Tx,Ty,ct)) \leq h(M(x,y))$ if and only if $F(Tx,Ty,ct) \geq M(x,y) \geq F(x,y,t)$, for all $c \in (0,1)$ and $x, y \in X$, whit $x \neq y$ and for all $t > 0$,

$$\begin{aligned} F(Tx,Ty,ct) = F(\frac{x}{2}, \frac{y}{2}, ct) &= \frac{1}{1+\max\{\frac{x}{2},\frac{y}{2}\}} \geq \frac{1}{1+\max\{x,y\}} \\ &= F(x,y,t). \end{aligned}$$

In the case $x = y$ it is trivial. As a result, the contractivity condition is verified. Also, all the assumptions made in Theorem 3.1 are satisfied and hence, it guarantees that T has a unique fixed point (which is $x = 0$).

By weakening the continuity property of a map T in Theorem 3.1, we have the following result.

Theorem 3.3. In Theorem 3.1 let X has the property that, for every nondecreasing sequence $\{x_n\}$ with $\{x_n\} \to x$, we have $x_n \preceq x$ for all $n \in \mathbb{N}$, i.e., $x = \sup\ x_n$. Then a non-continuous map T has a fixed point in X.

Proof . From Theorem 3.1 we take the same sequence $\{x_n\}$ in X such that $x_0 \preceq x_1 \preceq x_2 \preceq ... \preceq x_n \preceq x_{n+1} \preceq ...$, i.e., the sequence $\{x_n\}$ is nondecreasing and converges to some x in X. Thus, from hypotheses we have $x = \sup\ x_n$. Next, we prove that x is a fixed point of T, that is $Tx = x$. From (3.1) we have

$$h(F(x_{n+1}, Tx, \phi(ct))) = h(F(Tx_n, Tx, \phi(ct))) \leq \psi(h(M(x_n, x))), \tag{3.10}$$

for all $t > 0$ and $n \in \mathbb{N}$, where

$$\begin{aligned} M(x_n, x) &= \max\left\{F(x_n, x, \phi(t)), \frac{F(x_n, Tx_n, \phi(t)) * F(x, Tx, \phi(t))}{1 + F(Tx_n, Tx, \phi(t))}\right\} \\ &= \max\left\{F(x_n, x, \phi(t)), \frac{F(x_n, x_{n+1}, \phi(t)) * F(x, Tx, \phi(t))}{1 + F(x_{n+1}, Tx, \phi(t))}\right\}. \end{aligned}$$

By Lemma 2.10, F is a continuous function on $X^2 \times (0,\infty)$. Letting $n \to \infty$ and since $\lim_{n\to\infty} x_n = x$ we get

$$\lim_{n\to\infty} M(x_n, x) = \max\left\{F(x, x, \phi(t)), \frac{F(x, x, \phi(t)) * F(x, Tx, \phi(t))}{1 + F(x, Tx, \phi(t))}\right\} = 1.$$

From $(\mathcal{H}_1)$, $\lim_{n\to\infty} h(M(x_n, x)) = 0$ and since ψ is continuous at $t = 0$, we have

$$\lim_{n\to\infty} \psi(h(M(x_n, x))) = 0.$$

Then by (3.10) we deduce

$$\lim_{n\to\infty} h(F(x_{n+1}, Tx, \phi(ct))) = 0,$$

which by $(\mathcal{H}_1)$ yields $\lim_{n\to\infty} F(x_{n+1}, Tx, \phi(ct)) = 1$ for all $t > 0$. Easily, we conclude that $\lim_{n\to\infty} F(x_{n+1}, Tx, t) = 1$ for all $t > 0$, which means $x = \lim_{n\to\infty} x_{n+1} = Tx$ and T has a fixed point x in X. □

The uniqueness of an existing fixed point in Theorems 3.1 and 3.3, can be obtained, if the set of fixed pints of T, $Fix(T)$, is well-ordered.

Theorem 3.4. If in Theorems 3.1 and 3.3, $Fix(T)$, is well-ordered and $\lim_{t\to\infty} F(x,y,t) = 1$ for all $x, y \in Fix(T)$ and also $h \in \mathcal{H}$ is decreasing, then T has a unique fixed point in X.

Proof . Assume that T has two different fixed points x^* and y^*, such that $x^* \preceq y^*$. It follows from (3.1) that

$$h(F(x^*, y^*, \phi(ct))) = h(F(Tx^*, Ty^*, \phi(ct))) \leq \psi(h(M(x^*, y^*))) \tag{3.11}$$

for all $t > 0$ and

$$\begin{aligned} M(x^*, y^*) &= \max\left\{F(x^*, y^*, \phi(t)), \frac{F(x^*, Tx^*, \phi(t)) * F(y^*, Ty^*, \phi(t))}{1 + F(Tx^*, Ty^*, \phi(t))}\right\} \\ &= \max\left\{F(x^*, y^*, \phi(t)), \frac{1}{1 + F(x^*, y^*, \phi(t))}\right\}. \end{aligned} \tag{3.12}$$

Since h is decreasing, from (3.11) and (3.12) we get

$$h(F(x^*, y^*, \phi(ct))) \leq \psi(h(F(x^*, y^*, \phi(t)))),$$

for all $t > 0$. Therefore

$$h(F(x^*, y^*, \phi(t))) \leq \psi(h(F(x^*, y^*, \phi(\frac{t}{c})))).$$

Repeating this argument, since ψ is nondecreasing, we deduce that

$$\begin{aligned} h(F(x^*, y^*, \phi(t))) &\leq \psi(h(F(x^*, y^*, \phi(\frac{t}{c})))) \\ &\leq \psi^2(h(F(x^*, y^*, \phi(\frac{t}{c^2})))) \\ &\vdots \\ &\leq \psi^n(h(F(x^*, y^*, \phi(\frac{t}{c^n})))), \end{aligned} \tag{3.13}$$

for all $t > 0$. On the other hand, from the hypotheses, we have $\lim_{n\to\infty} \frac{t}{c^n} = \infty$. Hence, $\lim_{n\to\infty} \phi(\frac{t}{c^n}) = \infty$. Then $\lim_{n\to\infty} F(x^*, y^*, \phi(\frac{t}{c^n})) = 1$. This implies that

$$\lim_{n\to\infty} h(F(x^*, y^*, \phi(\frac{t}{c^n}))) = 0.$$

Thus, $\lim_{n\to\infty} \psi(h(F(x^*, y^*, \phi(\frac{t}{c^n})))) = 0$, for all $t > 0$. Then by (3.13) we get $\lim\limits_{n\to\infty} h(F(x^*, y^*, \phi(t))) = 0$ and hence $\lim\limits_{n\to\infty} F(x^*, y^*, \phi(t)) = 1$ for all $t > 0$. Therefore, we conclude that $\lim\limits_{n\to\infty} F(x^*, y^*, t) = 1$, for all $t > 0$, which means that $x^* = y^*$ by virtue of (FM-2). Thus, T can only have one fixed point in X. □

We have the following results, which are the generalizations of Theorems 3.1 and 3.3 in the partially ordered non-Archimedean fuzzy metric spaces.

Theorem 3.5. Let $(X, F, *, \preceq)$ be a partially ordered non-Archimedean fuzzy metric space and suppose $T, S : X \to X$ are continuous mappings such that
(i) for some $c \in (0, 1), \phi \in \Phi, \psi \in \Psi$, and $h \in \mathcal{H}$ with

$$h(F(Tx, Ty, \phi(ct))) \leq \psi(h(M_S(x, y))) \tag{3.14}$$

for all $x, y \in X$ with $Sx \preceq Sy$ and all $t > 0$ and

$$M_S(x, y) = \max\left\{F(Sx, Sy, \phi(t)), \frac{F(Sx, Tx, \phi(t)) * F(Sy, Ty, \phi(t))}{1 + F(Tx, Ty, \phi(t))}\right\} \tag{3.15}$$

(ii) $TX \subseteq SX$ and SX is a G-complete subspace of X,
(iii) T is a monotone S-nondecreasing mapping,
(iv) T and S are compatible.

If there exists $x_0 \in X$ such that $Sx_0 \preceq Tx_0$ and $\lim\limits_{t\to\infty} F(Sx_0, Tx_0, t) = 1$, then T and S have a coincidence point in X.

Proof . Suppose that there exists $x_0 \in X$ such that $Sx_0 \preceq Tx_0$. Define $Sx_0 = y_0$ and $Tx_0 = y_1$. Using the assumption $TX \subseteq SX$, there exists some $x_1 \in X$ such that $Sx_1 = y_1$. Construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Tx_n = y_{n+1} = Sx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is a monotone S-nondecreasing mapping, from $Sx_0 \preceq Tx_0 = Sx_1$, we have $Tx_0 \preceq Tx_1$ and so, $Sx_1 \preceq Sx_2$. Using mathematical induction, we get the following

$$y_0 \preceq y_1 \preceq y_2 \preceq \dots .$$

Since y_n are comparable for all $n \in \mathbb{N} \cup \{0\}$, we can use inequality (3.14) to write

$$h(F(y_n, y_{n+1}, \phi(ct))) = h(F(Tx_{n-1}, Tx_n, \phi(ct))) \leq \psi(h(M_S(x_{n-1}, x_n)))$$

for all $n \in \mathbb{N}$ and all $t > 0$ and by (3.15)

$$\begin{aligned} M_S(x_{n-1}, x_n) &= \max\left\{F(Sx_{n-1}, Sx_n, \phi(t)), \frac{F(Sx_{n-1}, Tx_{n-1}, \phi(t)) * F(Sx_n, Tx_n, \phi(t))}{1 + F(Tx_{n-1}, Tx_n, \phi(t))}\right\} \\ &= \max\left\{F(y_{n-1}, y_n, \phi(t)), \frac{F(y_{n-1}, y_n, \phi(t)) * F(y_n, y_{n+1}, \phi(t))}{1 + F(y_n, y_{n+1}, \phi(t))}\right\} \\ &= F(y_{n-1}, y_n, \phi(t)). \end{aligned}$$

Therefore,

$$h(F(y_n, y_{n+1}, \phi(ct))) \leq \psi(h(F(y_{n-1}, y_n, \phi(t)))),$$

for all $n \in \mathbb{N}$ and all $t > 0$. Using the same argument given in the proof of Theorem 3.1, we can prove that $\lim_{n\to\infty} F(y_n, y_{n+1}, t) = 1$, for all $t > 0$. Lemma 15 in [16] guarantees that $\{y_n = Sx_n\}$ is a G-Cauchy sequence in SX. As SX is G-complete, there exists $z \in SX$, such that $\{Sx_n\} \to z$ as $n \to \infty$. We will prove that $Tz = Sz$. Since $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_{n+1} = z$, by compatibility of T and S we have

$$\lim_{n\to\infty} F(TSx_n, STx_n, t) = 1$$

for all $t > 0$. Furthermore, by use of continuity of T and S and by (FM-4),

$$F(Tz, Sz, t) \geq F(Tz, TSx_n, t) * F(TSx_n, STx_n, t) * F(STx_n, Sz, t),$$

for all $n \in \mathbb{N}$ and all $t > 0$. Finally, letting $n \to \infty$, we get $F(Tz, Sz, t) = 1$ for all $t > 0$, which means that z is a coincidence point for T and S in X. □

Replacing the condition of being weakly compatible instead of compatibility in Theorem 3.5, we obtain the following result.

Corollary 3.6. Assume in Theorem 3.5, $\lim_{t\to\infty} F(Tx, Ty, t) = 1$ for all coincidence points of T and S and $h \in \mathcal{H}$ is decreasing. If X has the property that for every nondecreasing sequence $\{Sx_n\}$ in X such that $\lim_{n\to\infty} Sx_n = Sx$ implies that $Sx_n \preceq Sx$ for all $n \in \mathbb{N}$, that is $Sx = sup\ Sx_n$. If T and S are weakly compatible for every coincidence point ν of T and S with $S\nu \preceq S(S\nu)$, then T and S have common fixed point in X. Furthermore, the set of common fixed point of T and S is well-ordered if and only if T and S have one common fixed point in X.

Proof . From Theorem 3.5 we conclude that there exists $\nu \in X$ such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_{n+1} = S\nu$. By hypotheses and (3.14) we have $Sx_n \preceq S\nu$ and

$$h(F(Tx_n, T\nu, \phi(ct))) \leq \psi(h(M_S(x_n, \nu))),$$

for all $n \in \mathbb{N}$ and all $t > 0$ and by (3.15)

$$M_S(x_n, \nu) = \max\left\{F(Sx_n, S\nu, \phi(t)), \frac{F(S\nu, T\nu, \phi(t)) * F(Sx_n, Tx_n, \phi(t))}{1 + F(Tx_n, T\nu, \phi(t))}\right\}.$$

Since $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_{n+1} = S\nu$, we get

$$\lim_{n\to\infty} M_S(x_n, \nu) = 1,$$

which yields $\lim_{n\to\infty} h(F(Tx_n, T\nu, \phi(ct))) = 0$, for all $t > 0$. Therefore, $\lim_{n\to\infty} F(Tx_n, T\nu, \phi(ct)) = 1$, for all $t > 0$ and we conclude that $T\nu = S\nu$. Hence, ν is coincidence point of T and S. Next, assume that $S\nu \preceq S(S\nu)$ and T and S are weakly compatible. Let $T\nu = S\nu = \mu$, then $S\mu = ST\nu = TS\nu = T\mu$. By using the assumption $S\nu \preceq S(S\nu)$, and (3.14) we have

$$h(F(T\nu, T\mu, \phi(ct))) \le \psi(h(M_S(\nu, \mu))),$$

for all $t > 0$ and by (3.15)

$$\begin{aligned} M_S(\nu,\mu) &= \max\left\{F(S\nu, S\mu, \phi(t)), \frac{F(S\nu, T\nu, \phi(t)) * F(S\mu, T\mu, \phi(t))}{1 + F(T\nu, T\mu, \phi(t))}\right\} \\ &= \max\left\{F(T\nu, T\mu, \phi(t)), \frac{1}{1 + F(T\nu, T\mu, \phi(t))}\right\}. \end{aligned}$$

By the hypotheses, as in the proof of Theorem 3.4, we obtain $F(T\nu, T\mu, t) = 1$ for all $t > 0$ and so, $\mu = S\nu = T\nu = T\mu = S\mu$. Therefore, μ is a common fixed point of T and S. Eventually, by following Theorem 3.4 we deduce that T and S have one and only one common fixed point if and only if the set of common fixed points of T and S is well-ordered. □

In 2010, Altunet al. [2] contributed in this field by defining notion of weakly increasing mappings. In addition, by using implicit relations, they derived some results (both for weakly increasing and nondecreasing operators) in a partially ordered metric space. Their results are of considerable interest for others related to these areas.

Definition 3.7. [2] Suppose E is a non-empty set and $\preceq$ is a partially ordered relation on set E. Then maps $T, S : E \to E$ are weakly increasing if for all $x \in E$, $Tx \preceq STx$ and $Sx \preceq TSx$.

Theorem 3.8. Let $(X, F, *, \preceq)$ be a partially ordered G-complete non-Archimedean fuzzy metric space and suppose $T, S : X \to X$ are weakly increasing mappings such that
(i) for some $c \in (0, 1), \phi \in \Phi, \psi \in \Psi$, and $h \in \mathcal{H}$ with

$$h(F(Tx, Sy, \phi(ct))) \le \psi(h(M(x, y)))$$

and

$$M(x,y) = \max\left\{F(x, y, \phi(t)), \frac{F(x, Tx, \phi(t)) * F(y, Sy, \phi(t))}{1 + F(Tx, Sy, \phi(t))}\right\}.$$

for every comparable pair $x, y \in X$ and all $t > 0$,
(ii) there exists $x_0 \in X$, with $Tx_0 \preceq STx_0$ and $\lim_{t\to\infty} F(Tx_0, STx_0, t) = 1$,
(iii) for every nondecreasing sequence $\{\nu_n\} \subset X$ whit $\nu_n \to \nu$, we have $\nu_n \preceq \nu$ for all $n \in \mathbb{N}$, i.e., $\nu = sup\ \nu_n$.

Then T and S have at least one common fixed point in X.

Proof . Let $x_0 \in X$ be the point such that $Tx_0 \preceq STx_0$ and $\lim_{t\to\infty} F(Tx_0, STx_0, t) = 1$. Define $Tx_0 = x_1$ and $Sx_1 = x_2$. Construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$x_{2n+1} = Tx_{2n} = y_{2n} \tag{3.16}$$

and

$$x_{2n+2} = Sx_{2n+1} = y_{2n+1} \tag{3.17}$$

for all $n \in \mathbb{N}$. Since T and S are weakly increasing functions,

$$y_0 = x_1 = Tx_0 \preceq STx_0 = Sx_1 = x_2 = y_1,$$

and

$$y_1 = Sx_1 \preceq TSx_1 = Tx_2 = x_3 = y_2.$$

Continuing this process, we obtain

$$y_0 \preceq y_1 \preceq y_2 \preceq \dots \preceq y_{2n} \preceq y_{2n+1} \preceq \dots .$$

If there exists some $n_0 \in \mathbb{N}$, such that $y_{2n_0} = y_{2n_0+1}$ then it implies by (3.16) and (3.17) that $Sx_{2n_0+1} = x_{2n_0+1}$. Thus, if $z = x_{2n_0+1}$ we have $Sz = z$. Furthermore, we assert that $z \in X$ whit $Sz = z$ implies that $Tz = z$. To get this, setting $x = y = z$ in condition (i), we obtain

$$h(F(Tz, Sz, \phi(ct))) \leq \psi(h(M(z, z)))$$

for all $t > 0$ and

$$M(z, z) = \max\left\{F(z, z, \phi(t)), \frac{F(z, Tz, \phi(t)) * F(z, Sz, \phi(t))}{1 + F(Tz, Sz, \phi(t))}\right\} = 1.$$

Hence, $F(Tz, Sz, \phi(ct)) = 1$ for all $t > 0$ and by the same argument given in the proof of Theorem 3.1, $F(Tz, Sz, t) = 1$ for all $t > 0$ and then $Tz = Sz$. From (FM-4) we have

$$F(Tz, z, t) \geq F(Tz, Sz, t) * F(Sz, z, t),$$

for all $t > 0$ which yields $F(Tz, z, t) = 1$. Hence we get $Tz = Sz = z$ and the existence part of the proof is finished. On the contrary case, assume that $y_{2n} \neq y_{2n+1}$, for all $n \in \mathbb{N}$. Applying the contractivity condition (i), $x = y_{2n}$ and $y = y_{2n+1}$, it follows that

$$\begin{aligned} h(F(y_{2n}, y_{2n+1}, \phi(ct))) &= h(F(Tx_{2n}, Sx_{2n+1}, \phi(ct))) \\ &\leq \psi(h(M(x_{2n}, x_{2n+1}))), \end{aligned}$$

for all $t > 0$ and

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= max\left\{F(x_{2n}, x_{2n+1}, \phi(t)), \frac{F(x_{2n}, Tx_{2n}, \phi(t)) * F(x_{2n+1}, Sx_{2n+1}, \phi(t))}{1 + F(Tx_{2n}, Sx_{2n+1}, \phi(t))}\right\} \\ &= F(x_{2n}, x_{2n+1}, \phi(t)) \\ &= F(y_{2n-1}, y_{2n}, \phi(t)). \end{aligned}$$

Hence,

$$h(F(y_{2n}, y_{2n+1}, \phi(ct))) \leq \psi(h(F(y_{2n-1}, y_{2n}, \phi(t)))), \tag{3.18}$$

for all $t > 0$ and all $n \in \mathbb{N}$. Therefore, by the same argument given in the proof of Theorem 3.1 we observe that $\lim_{n\to\infty} F(y_{2n}, y_{2n+1}, t) = 1$ for all $t > 0$, which means that $\{y_n\}$ is a G-Cauchy sequence in X by lemma [16]. As X is G-complete, there exists $z \in X$ such that $\{y_n\} \to z$. We claim that z is a common fixed point of T and S. To prove it, observe that for all $t > 0$ and $n \in \mathbb{N}$,

$$F(Sz, z, t) \geq F(Sz, y_{2n}, t) * F(y_{2n}, z, t). \tag{3.19}$$

It is clear that $\lim_{n\to\infty} F(y_{2n}, z, t) = 1$ for all $t > 0$. Let us to show that the first factor in (3.19) converges to 1, whenever $n \to \infty$. By condition (iii) and applying the contractivity (i) for $x = x_{2n}$ and $y = z$ we have

$$h(F(y_{2n}, Sz, \phi(ct))) = h(F(Tx_{2n}, Sz, \phi(ct))) \leq \psi(h(M(x_{2n}, z))), \tag{3.20}$$

for all $t > 0$ and

$$M(x_{2n}, z) = max\left\{F(x_{2n}, z, \phi(t)), \frac{F(x_{2n}, Tx_{2n}, \phi(t)) * F(z, Sz, \phi(t))}{1 + F(Tx_{2n}, Sz, \phi(t))}\right\}. \tag{3.21}$$

Letting $n \to \infty$ in (3.21) and from $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n_1} = z$ we get

$$\lim_{n\to\infty} M(x_{2n}, z) = 1,$$

and by (3.20) we deduce that $\lim_{n\to\infty} F(y_{2n}, Sz, \phi(ct)) = 1$, for all $t > 0$. Thus, $\lim_{n\to\infty} F(y_{2n}, Sz, t) = 1$ for all $t > 0$ and then, by letting $n \to \infty$ in (3.19) we have $F(Sz, z, t) = 1$ for all $t > 0$. By applying the condition (i) it is easy to prove that $Tz = Sz = z$ and hence z is a common fixed point of T and S in X and the proof is completed. □

Theorem 3.9. Assume in Theorem 3.8, h is a decreasing function, and also assume that the set of common fixed points of T and S is well-ordered and for all pair (ν, μ) from the common fixed points of T and S, $\lim_{t\to\infty} F(\nu, \mu, t) = 1$. Then T and S have a unique common fixed point.

Proof . Suppose ν and μ are common fixed points of T and S. Applying the contractivity condition (i) in Theorem 3.8, we get

$$h(F(T\nu, S\mu, \phi(ct))) \leq \psi(h(M(\nu, \mu))),$$

for all $t > 0$ and since h is decreasing function,

$$\begin{aligned} M(\nu, \mu) &= \max\left\{F(\nu, \mu, \phi(t)), \frac{F(\nu, T\nu, \phi(t)) * F(\mu, S\mu, \phi(t))}{1 + F(T\nu, S\mu, \phi(t))}\right\} \\ &\leq F(\nu, \mu, \phi(t)). \end{aligned}$$

Hence,

$$h(F(\nu, \mu, \phi(ct))) \leq \psi(h(F(\nu, \mu, \phi(t)))),$$

for all $t > 0$. Thus, by the same argument given in the proof of Theorem 3.4 we deduce that $F(\nu, \mu, t) = 1$ for all $t > 0$. This proves that T and S have a unique common fixed point in X. $\square$

Example 3.10. Let $(X, F, *)$ be the complete partially ordered non-Archimedean fuzzy metric space introduced in Example 3.2. Let $T, S : X \to X$ defined by $T(x) = x$ and $S(x) = \sqrt{x}$. Then clearly T and S are weakly increasing mappings, but not nondecreasing (see [2]). Assume that $\psi(t) = \phi(t) = t$ for all $t \in [0, \infty)$ and let $h : (0, 1] \to [0, \infty)$ be whatever strictly decreasing bijection between $(0, 1]$ and $[0, \infty)$ such that h and h^{-1} are continuous. In this context, the contractivity condition (i) in Theorem 3.8 is equivalent to $h(F(Tx, Sy, \phi(ct))) \leq \psi(h(M(x, y)))$ if and only if $h(F(Tx, Sy, ct)) \leq h(M(x, y))$ if and only if $F(Tx, Sy, ct) \geq M(x, y) \geq F(x, y, t)$, for all $c \in (0, 1)$ and $x, y \in X$ whit $x \neq y$ and for all $t > 0$,

$$\begin{aligned} F(Tx, Sy, ct) &= F(x, \sqrt{y}, ct) \\ &= \frac{1}{1 + \max\{x, \sqrt{y}\}} \\ &\geq \frac{1}{1 + \max\{x, y\}} \\ &= F(x, y, t). \end{aligned}$$

In the case $x = y$ it is trivial. As a result, the contractivity condition is verified. Also, all the assumptions made in Theorem 3.8 or 3.9 are satisfied and hence, it guarantees that T and S have a unique common fixed point (which is $x = 0$).

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