Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 1205–1214
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.27337.3562

Hilfer-Katugampola fractional stochastic differential equations with nonlocal conditions

A.M. Sayed Ahmed[a], Hamdy M. Ahmed[b,*]

[a]Department of Mathematics and Computer Science, Faculty of Sciences, Alexandria University, Alexandria, Egypt

[b]Department of Physics and Engineering Mathematics, Higher Institute of Engineering, El-Shorouk Academy, El-Shorouk City, Cairo, Egypt

(Communicated by Mugur Alexandru Acu)

Abstract

Hilfer-Katugampola-type fractional stochastic differential equations with nonlocal conditions are considered in this paper. By using the fixed point theorem, the existence and uniqueness of solutions for the considered problem are proved. Ulam-Hyers stability for the considered problem is studied. Finally, an example is presented to show our main results.

Keywords: Hilfer-Katugampola fractional derivative, Stochastic differential equation, Ulam-Hyers stability, Fixed point theorem, Nonlocal conditions
2020 MSC: 60G22, 26A33, 34K37, 60H10

1 Introduction

Stochastic differential equations (SDEs) have attracted great interest due to its applications in various fields of science and engineering ([8]-[26]). Fractional differential equations have been widely applied in many fields such as physics, chemical, fluid dynamic and traffic model([19]-[32]). Recently, Hilfer fractional differential equations have attracted the attention of many authors ([12]-[3]). Nowadays, the generalized fractional derivative introduced by Katugampola ([21, 22]) is unified with Hilfer fractional derivative by Oliveira and Capelas de Oliveira is named as Hilfer-Katugampola fractional derivative ([28]). Few authors studied Hilfer-Katugampola fractional differential equations ([28]-[30]).

To the best of our knowledge, there are no results about stochastic implicit Hilfer-Katugampola fractional differential equations with nonlocal conditions. Motivated by the above discussion, the aim of this paper is to study the existence, uniqueness, and Ulam-Hyers stability of the solution of Hilfer-Katugampola-type fractional stochastic implicit differential equations with nonlocal conditions in the form:

$$\begin{aligned} {}^{\rho}D_{0+}^{\alpha,\beta}u(t) &= f(t,u(t),{}^{\rho}D_{0+}^{\alpha,\beta}u(t)) + \int_0^t \sigma(s,u(s),{}^{\rho}D_{0+}^{\alpha,\beta}u(s))dB(s), \qquad t \in \acute{J} := (0,T], \\ \left({}^{\rho}\mathcal{J}_{0+}^{1-\nu}u\right)(0) &= \sum_{i=1}^{m} \zeta_i u(\tau_i), \quad \tau_i \in J := [0,T], \quad \zeta_i \in \mathbb{R}, \quad \nu = \alpha + \beta(1-\alpha). \end{aligned} \tag{1.1}$$

*Corresponding author
Email addresses: ahmed.sibrahim@alexu.edu.eg (A.M. Sayed Ahmed), hamdy_17eg@yahoo.com (Hamdy M. Ahmed)

Received: May 2022 *Accepted:* December 2022

where ${}^{\rho}D_{0+}^{\alpha,\beta}$ is Hilfer-Katugampola fractional derivative of order α and type β $(0 \leq \beta \leq 1)$, and ${}^{\rho}\mathcal{J}_{0+}^{1-\nu}$ is Katugampola fractional integral of order $1-\nu$, $(\nu = \alpha + \beta(1-\alpha))$, $1/2 < \alpha < 1$, $\rho > 0$ and $\tau_i, i = 1, 2, 3, \cdots, m$ are prefixed points satisfying $0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m < T$.

Let $(\Omega, \Im, \mathbb{P})$ be a complete probability space equipped with a filtration $\{\Im_t\}_{t\in J}$ satisfying $\Im_t \subset \Im$. The state $u(\cdot)$ takes values in a real separable Hilbert space $\mathfrak{X}$. Let Y be another separable Hilbert space. Let $\{e_n\}_{n\geq 1}$ is a complete orthonormal basis in Y. Suppose that $\{B(t)\}_{t\geq 0}$ is a cylindrical Y-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $Tr(Q) = \sum_{n=1}^{\infty} \lambda_n = \lambda < \infty$, which satisfies that $Qe_n = \lambda_n e_n$. actually, $B(t) = \sum_{n=0}^{\infty} \sqrt{\lambda_n}\beta_n(t)e_n$, where $\{\beta_n(t)\}_{n=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. Let $\varphi \in \mathfrak{L}_Q(Y, \mathfrak{X})$, where $\mathfrak{L}_Q(Y, \mathfrak{X})$ is the space of all Q-Hilbert Schmidt operators from Y into $\mathfrak{X}$, and define:

$$\|\varphi\|_Q^2 = Tr(\varphi Q \varphi^*) = \sum_{n=1}^{\infty} \left\| \sqrt{\lambda_n}\varphi e_n \right\|^2 .$$

The collection of all strongly measurable, square integrable $\mathfrak{X}$-valued random variables, denoted by $\mathcal{L}_2(\Omega, \mathfrak{X})$, is a Banach space equipped with norm:

$$\|u(\cdot)\|_{\mathcal{L}_2} = \left(\mathbb{E}\, \|u(\cdot;\omega)\|_{\mathfrak{X}}^2\right)^{1/2} .$$

Through this paper, let $f : \acute{J} \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ and $\sigma : J \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{L}_Q(Y, \mathfrak{X})$.

The outline of this paper is structured as follows: Section 2 contains some notations and preliminary facts. In Section 3, we prove the existence and uniqueness solution for equation (1.1). In Section 4, we discuss the Ulam-Hyers stability for equation (1.1). In the end, we consider example to illustrate our main results.

2 Preliminaries

Throughout this paper, let $\mathcal{C}(J, \mathfrak{X})$ be a Banach space of all continuous functions u from J into $\mathfrak{X}$ and $\mathcal{C}(J, \mathcal{L}_2(\Omega, \mathfrak{X}))$ be a Banach space of all continuous ($\Im_t$-adapted stochastic process) maps from J into $\mathcal{L}_2(\Omega, \mathfrak{X})$ satisfying the condition $\sup_{t\in J} \|u(t)\|^2 < \infty$. Let also $\mathcal{C}_{1-\nu,\rho}(J, \mathcal{L}_2(\Omega, \mathfrak{X}))$ is defined by

$$\mathcal{C}_{1-\nu,\rho}(J, \mathcal{L}_2(\Omega, \mathfrak{X})) := \{u : \acute{J} \to \mathcal{L}_2(\Omega, \mathfrak{X}) : \left(\frac{t^{\rho}}{\rho}\right)^{1-\nu} u(t) \in \mathcal{C}_{1-\nu,\rho}(J, \mathcal{L}_2(\Omega, \mathfrak{X}))\},$$

with the norm

$$\|u\|_{\mathcal{C}_{1-\nu,\rho}} := \left(\sup_{t\in J} \mathbb{E} \left\| \left(\frac{t^{\rho}}{\rho}\right)^{1-\nu} u(t) \right\|^2\right)^{1/2} .$$

Definition 2.1 ([21]). *Let $\alpha, c \in \mathbb{R}$ with $\alpha > 0$ and $\varphi \in X_c^p(a, b)$, where $\varphi \in X_c^p(a, b)$ is the space of Lebesgue measurable functions. The generalized left-sided fractional integral is defined by*

$$\left({}^{\rho}\mathcal{J}_{0+}^{\alpha}\varphi\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1}\varphi(s)}{(t^{\rho} - s^{\rho})^{1-\alpha}} ds, \quad t > 0.$$

where $\Gamma(\cdot)$ is the gamma function which is defined as:

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt, \quad Re(\alpha) > 0.$$

Definition 2.2 ([28],[17]). *The Hilfer-Katugampola fractional derivative with respect to t, with $\rho > 0$, is defined by*

$$\left({}^{\rho}D_{0\pm}^{\alpha,\beta}u\right)(t) = \left(\pm{}^{\rho}\mathcal{J}_{0\pm}^{\beta(1-\alpha)}\delta_{\rho}{}^{\rho}\mathcal{J}_{0\pm}^{(1-\beta)(1-\alpha)}\right)(t) \tag{2.1}$$

where $\delta_{\rho} = \left(t^{\rho-1}\frac{d}{dt}\right)$.

Theorem 2.1 ([15]). *(Krasnoselskii's fixed point theory). Let E be a Banach space, let G be a bounded closed convex subset of E and let T_1, T_2 be mapping from G into E such that $T_1u + T_2v \in G$ for every pair $u, v \in G$. If T_1 is contraction and T_2 is completely continuous, then the equation $T_1u + T_2u = u$ has a solution on G.*

Lemma 2.1 ([24]). *For arbitrary $\mathfrak{L}_Q$-valued predictable process $\sigma(t)$, $t \in [t_1, t_2]$ which satisfies*

$$\mathbb{E}\left(\int_{t_1}^{t_2} \|\sigma(s)\|^2_{\mathfrak{L}_Q} ds\right) < \infty, \qquad 0 \leq t_1 < t_2 \leq T.$$

We have

$$\mathbb{E}\left\|\int_{t_1}^{t_2} \sigma(s)dB(s)\right\|^2 \leq \left(\int_{t_1}^{t_2} \mathbb{E}\,\|\sigma(s)\|^2_{\mathfrak{L}_Q} ds\right).$$

Lemma 2.2 ([14]). *A stochastic process $u \in \mathcal{C}^{\nu}_{1-\nu,\rho}(J, \mathcal{L}_2(\Omega, \mathfrak{X}))$ is a solution of the problem (1.1) if and only if u satisfies the mixed type integral equation*

$$\begin{aligned} u(t) \quad = \quad & \frac{\Upsilon}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s, u(s), \mathbb{K}_u(s))ds \\ & + \frac{\Upsilon}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left(\int_0^s \sigma(\eta, u(\eta), \mathbb{K}_u(\eta))dB(\eta)\right)ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s, u(s), \mathbb{K}_u(s))ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left(\int_0^s \sigma(\eta, u(\eta), \mathbb{K}_u(\eta))dB(\eta)\right)ds. \end{aligned} \tag{2.2}$$

where

$$\Upsilon := \frac{\rho^{\nu-1}}{\rho^{\nu-1}\Gamma(\nu) - \sum_{i=1}^m \zeta_i\left(\tau_i^\rho\right)^{\nu-1}}, \quad \rho^{\nu-1}\Gamma(\nu) \neq \sum_{i=1}^m \zeta_i\left(\tau_i^\rho\right)^{\nu-1},$$

and

$$\mathbb{K}_u(t) := {}^\rho D^{\alpha,\beta}_{0+}u(t) = f(t, u(t), {}^\rho D^{\alpha,\beta}_{0+}u(t)) + \int_0^t \sigma(s, u(s), {}^\rho D^{\alpha,\beta}_{0+}u(s))dB(s).$$

3 Existence and Uniqueness

Firstly, We introduce the following assumptions:

H1. Let $f : \acute{J} \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ be a function such that $f \in \mathcal{C}^{\beta(1-\alpha)}_{1-\nu,\rho}(J, \mathcal{L}_2(\Omega, \mathfrak{X}))$
(i) for any $u \in \mathcal{C}_{1-\nu,\rho}(J, \mathcal{L}_2(\Omega, \mathfrak{X}))$, and for $t \in J$, $u, v \in \mathfrak{X}$, there exist $p, q, \chi \in \mathcal{C}_{1-\nu,\rho}(J, \mathbb{R}_+)$ such that

$$\|f(t, u, v)\|^2 \leq p(t) + q(t)\|u\|^2 + \chi(t)\|v\|^2.$$

with

$$p^* = \left(\sup_{t\in J}\|p(t)\|^2\right)^{1/2}, \quad q^* = \left(\sup_{t\in J}\|q(t)\|^2\right)^{1/2} \quad and \quad \chi^* = \left(\sup_{t\in J}\|\chi(t)\|^2\right)^{1/2} < 1.$$

(ii) There exist constants $K_1, K_1^* > 0$ such that

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\|^2 \leq K_1\|u_1 - u_2\|^2 + K_1^*\|v_1 - v_2\|^2$$

for any $u_1, v_1, u_2, v_2 \in \mathfrak{X}$.

H2. The function $\sigma : J \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{L}_Q(Y, \mathfrak{X})$ satisfies
(i) There exist $l, m, n \in \mathcal{C}_{1-\nu,\rho}(J, \mathbb{R}_+)$ such that

$$\|\sigma(t, u, v)\|^2_{\mathfrak{L}_Q} \leq l(t) + m(t)\|u\|^2 + n(t)\|v\|^2.$$

with

$$l^* = \left(\sup_{t\in J}\|l(t)\|^2\right)^{1/2}, \quad m^* = \left(\sup_{t\in J}\|m(t)\|^2\right)^{1/2} \quad and \quad n^* = \left(\sup_{t\in J}\|n(t)\|^2\right)^{1/2} < 1.$$

(ii) There exist constants $K_2, K_2^* > 0$ such that

$$\|\sigma(t,u_1,v_1)-\sigma(t,u_2,v_2)\|^2_{\mathfrak{L}_Q} \leq K_2\|u_1-u_2\|^2 + K_2^*\|v_1-v_2\|^2$$

for any $u_1, v_1, u_2, v_2 \in \mathfrak{X}$.

Theorem 3.1. *Assume that (H1) and (H2) are satisfied. Then problem (1.1) has at least one mild solution in* $\mathcal{C}^{\nu}_{1-\nu,\rho}(J,\mathcal{L}_2(\Omega,\mathfrak{X})) \subset \mathcal{C}^{\alpha,\beta}_{1-\nu,\rho}(J,\mathcal{L}_2(\Omega,\mathfrak{X}))$.

Proof . Consider the operator $N : \mathcal{C}_{1-\nu,\rho}(J,\mathcal{L}_2(\Omega,\mathfrak{X})) \to \mathcal{C}_{1-\nu,\rho}(J,\mathcal{L}_2(\Omega,\mathfrak{X}))$ defined by

$$\begin{aligned} N(u)(t) &= \frac{\Upsilon}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^m \zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}f(s,u(s),\mathbb{K}_u(s))ds \\ &+\frac{\Upsilon}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^m \zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\left(\int_0^s\sigma(\eta,u(\eta),\mathbb{K}_u(\eta))dB(\eta)\right)ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}f(s,u(s),\mathbb{K}_u(s))ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\left(\int_0^s\sigma(\eta,u(\eta),\mathbb{K}_u(\eta))dB(\eta)\right)ds. \end{aligned} \tag{3.1}$$

Consider the ball

$$B_k = \{u \in \mathcal{C}_{1-\nu,\rho}(J,\mathcal{L}_2(\Omega,\mathfrak{X})) : \|u(t)\|_{\mathcal{C}_{1-\nu,\rho}} \leq k\}.$$

Now we subdivide the operator N into two operators $\mathcal{A}$ and $\mathcal{B}$ on B_k as follows:

$$\begin{aligned} \mathcal{A}(u)(t) &= \frac{\Upsilon}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^m \zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}f(s,u(s),\mathbb{K}_u(s))ds \\ &+\frac{\Upsilon}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^m \zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\left(\int_0^s\sigma(\eta,u(\eta),\mathbb{K}_u(\eta))dB(\eta)\right)ds. \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}(u)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}f(s,u(s),\mathbb{K}_u(s))ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\left(\int_0^s\sigma(\eta,u(\eta),\mathbb{K}_u(\eta))dB(\eta)\right)ds. \end{aligned}$$

The proof is divided into several steps.

Step 1. $\mathcal{A}u + \mathcal{B}v \in B_k$, for every $u, v \in B_k$.

$$\mathbb{E}\left\|\mathcal{A}(u)(t)\left(\frac{t^\rho}{\rho}\right)^{1-\nu}\right\|^2 \leq I_1 + I_2.$$

where

$$\begin{aligned} I_1 &= \frac{2m\mid\Upsilon\mid^2}{\Gamma^2(\alpha)}\sum_{i=1}^m\zeta_i^2\mathbb{E}\left\|\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}f(s,u(s),\mathbb{K}_u(s))ds\right\|^2 \\ &\leq \frac{2m\mid\Upsilon\mid^2}{\Gamma^2(\alpha)}\sum_{i=1}^m\zeta_i^2\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{2\alpha-2}s^{2\rho-2}\mathbb{E}\,\|f(s,u(s),\mathbb{K}_u(s))\|^2_{\mathfrak{L}_Q}ds \end{aligned}$$

and

$$\begin{aligned} I_2 &= \frac{2m\mid\Upsilon\mid^2}{\Gamma^2(\alpha)}\sum_{i=1}^m\zeta_i^2\mathbb{E}\left\|\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\left(\int_0^s\sigma(\eta,u(\eta),\mathbb{K}_u(\eta))dB(\eta)\right)ds\right\|^2 \\ &\leq \frac{2m\mid\Upsilon\mid^2 TrQ}{\Gamma^2(\alpha)}\sum_{i=1}^m\zeta_i^2\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{2\alpha-2}s^{2\rho-2}\mathbb{E}\,\|\sigma(s,u(s),\mathbb{K}_u(s))\|^2_{\mathfrak{L}_Q}ds. \end{aligned}$$

We first estimate

$$\begin{aligned}
\mathbb{E}\left\|\mathbb{K}_u(t)\right\|^2 &= \mathbb{E}\left\|f(t,u(t),\mathbb{K}_u(s))+\int_0^t\sigma(s,u(s),\mathbb{K}_u(s))dB(s)\right\|^2\\
&\le 2\left(\mathbb{E}\left\|f(t,u(t),\mathbb{K}_u(s))\right\|^2+TTrQ\mathbb{E}\left\|\sigma(s,u(s),\mathbb{K}_u(s))\right\|^2_{\mathfrak{L}_Q}\right)\\
&\le 2\left(p(t)+q(t)\mathbb{E}\left\|u\right\|^2+\chi(t)\mathbb{E}\left\|\mathbb{K}_u(t)\right\|^2+TTrQ\left(l(t)+m(t)\mathbb{E}\left\|u\right\|^2+n(t)\mathbb{E}\left\|\mathbb{K}_u(t)\right\|^2\right)\right).
\end{aligned}$$

Thus, we have

$$\mathbb{E}\left\|\mathbb{K}_u(t)\right\|^2\le\frac{2\left((p^*+TTrQl^*)+(q^*+TTrQm^*)\mathbb{E}\left\|u\right\|^2\right)}{1-2(\chi^*+TTrQn^*)}. \tag{3.2}$$

Using $(H2)$ and equation (3.2), $\mathbb{B}(\cdot,\cdot)$ is the Beta function and let $a=p^*+TTrQl^*$, $b=q^*+TTrQm^*$ and $c=1-2(\chi^*+TTrQn^*)$, we get

$$I_1\le\frac{2m\mid\Upsilon\mid^2}{c}\sum_{i=1}^m\zeta_i^2S,$$

where

$$S=\frac{(p^*c+2a\chi^*)\mathbb{B}(\frac{2\rho-1}{\rho},2\alpha-1)\left((\tau_i^\rho/\rho)^{2\alpha-1}\tau_i^{\rho-1}\right)}{\Gamma^2(\alpha)}+\mathbb{B}(\frac{2\nu\rho-1}{\rho},2\alpha-1)\frac{(q^*c+2b\chi^*)\left((\tau_i^\rho/\rho)^{2\alpha+2\nu-3}\tau_i^{\rho-1}\right)}{\Gamma^2(\alpha)}\left\|u\right\|^2_{\mathcal{C}_{1-\nu,\rho}}.$$

Similarly for I_2, by using $(H2)$, we can obtain

$$I_2\le\frac{2m\mid\Upsilon\mid^2 TTrQ}{c}\sum_{i=1}^m\zeta_i^2\varrho,$$

where

$$\varrho=\frac{(l^*c+2an^*)\mathbb{B}(2,2\alpha-1)\left((\tau_i^\rho/\rho)^{2\alpha}\rho\right)}{\Gamma^2(\alpha)}+\mathbb{B}(2\nu,2\alpha-1)\frac{(m^*c+2bn^*)\left((\tau_i^\rho/\rho)^{2\alpha+2\nu-2}\rho\right)}{\Gamma^2(\alpha)}\left\|u\right\|^2_{\mathcal{C}_{1-\nu,\rho}}.$$

For operator $\mathcal{B}$,

$$\mathbb{E}\left\|\mathcal{B}(u)(t)\left(\frac{t^\rho}{\rho}\right)^{1-\nu}\right\|^2\le I_3+I_4, \tag{3.3}$$

where

$$\begin{aligned}
I_3 &= \frac{\left(\frac{t^\rho}{\rho}\right)^{2-2\nu}}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}f(s,u(s),\mathbb{K}_u(s))ds\right\|^2,\\
I_4 &= \frac{\left(\frac{t^\rho}{\rho}\right)^{2-2\nu}}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\left(\int_0^s\sigma(\eta,u(\eta),\mathbb{K}_u(\eta))dB(\eta)\right)ds\right\|^2.
\end{aligned}$$

By using $(H1)$ and $(H2)$, we can obtain

$$I_3\le\frac{(p^*c+2a\chi^*)\left((T^\rho/\rho)^{2\alpha-2\nu+1}T^{\rho-1}\right)}{c\Gamma^2(\alpha)}\mathbb{B}(\frac{2\rho-1}{\rho},2\alpha-1)+\mathbb{B}(\frac{2\rho\nu-1}{\rho},2\alpha-1)\frac{(q^*c+2b\chi^*)\left((T^\rho/\rho)^{2\alpha-1}T^{\rho-1}\right)}{c\Gamma^2(\alpha)}\left\|u\right\|^2_{\mathcal{C}_{1-\nu,\rho}} \tag{3.4}$$

and

$$\begin{aligned}
I_4 \le{}& \frac{(l^*c+2an^*)\left((T^\rho/\rho)^{2\alpha-2\nu+1}T^{\rho+1}TrQ\right)}{c\Gamma^2(\alpha)}\mathbb{B}(\frac{2\rho-1}{\rho},2\alpha-1)\\
&+\mathbb{B}(2\nu,2\alpha-1)\frac{(m^*c+2bn^*)\left((T^\rho/\rho)^{2\alpha-1}T^{\rho+1}TrQ\right)}{c\Gamma^2(\alpha)}\left\|u\right\|^2_{\mathcal{C}_{1-\nu,\rho}}.
\end{aligned} \tag{3.5}$$

Then, by substituting (3.4) and (3.5) into (3.3), we can get the result.

Step 2. $\mathcal{A}$ is a contraction mapping. For any $u,v\in B_k$,

$$\mathbb{E}\left\|\left((\mathcal{A}u)(t)-(\mathcal{A}v)(t)\right)(t^\rho/\rho)^{1-\nu}\right\|^2\le\mathbb{V}_1+\mathbb{V}_2,$$

where

$$\mathbb{V}_1=\frac{2m\mid\Upsilon\mid^2}{\Gamma^2(\alpha)}\sum_{i=1}^m\zeta_i^2\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{2\alpha-2}s^{2\rho-2}\mathbb{E}\left\|f(s,u(s),\mathbb{K}_u(s))-f(s,v(s),\mathbb{K}_v(s))\right\|^2ds$$

and

$$\mathbb{V}_2=\frac{2m\mid\Upsilon\mid^2}{\Gamma^2(\alpha)}\sum_{i=1}^m\zeta_i^2\mathbb{E}\left\|\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\left(\int_0^s\left(\sigma(\eta,u(\eta),\mathbb{K}_u(\eta))-\sigma(\eta,v(\eta),\mathbb{K}_v(\eta))\right)dB(\eta)\right)ds\right\|^2.$$

By assumptions $(H2)$ and $(H3)$, we have

$$\mathbb{V}_1\le\frac{2m\mid\Upsilon\mid^2\mathbb{B}(\frac{2\nu\rho-1}{\rho},2\alpha-1)(K_1b^*+K_1^*a^*)}{\Gamma^2(\alpha)b^*}\sum_{i=1}^m\zeta_i^2G_1\left\|u-v\right\|^2_{\mathcal{C}_{1-\nu,\rho}},$$

where

$$G_1 = \left((\tau_i^\rho/\rho)^{2\alpha+2\nu-3}\tau_i^{\rho-1}\right), \quad a^* = 2(K_1 + TTrQK_2), \quad b^* = 1 - 2(K_1^* + TTrQK_2^*)$$

and

$$\mathbb{V}_2 = \frac{2m \mid \Upsilon \mid^2 \mathbb{B}(2\nu, 2\alpha-1)TrQ(K_2b^* + K_2^*a^*)}{b^*\Gamma^2(\alpha)} \sum_{i=1}^{m} \zeta_i^2 G_2 \|u - v\|^2_{\mathcal{C}_{1-\nu,\rho}},$$

where

$$G_2 = \left((\tau_i^\rho/\rho)^{2\alpha+2\nu-2}\rho\right).$$

Step 3. The operator $\mathcal{B}$ is compact and continuous.
According to Step 1 and inequality (3.3) we can obtain that operator $\mathcal{B}$ is uniformly bounded and equicontinuous and then by using Arzela-Ascoli theorem, we get, the operator $\mathcal{B}$ is compact on B_k.

It is follows that from Theorem 2.1 that the problem (1.1) has at least one solution.

Theorem 3.2. *Assume that $(H1)$ and $(H2)$ hold. Then the problem (1.1) has a unique solution provided that*

$$\Xi < 1, \tag{3.6}$$

where

$$\begin{aligned}
\Xi \; = \; & \frac{4m\Upsilon^2(K_1b^* + K_1^*a^*)\mathbb{B}(\frac{2\nu\rho-1}{\rho}, 2\alpha-1)}{b^*\Gamma^2(\alpha)} \sum_{i=1}^{m} \zeta_i^2 \left(\tau_i^\rho/\rho\right)^{2\alpha+2\nu-3} \tau_i^{\rho-1} \\
& + \frac{4m\Upsilon^2 TrQ(K_2b^* + K_2^*a^*)\mathbb{B}(2\nu, 2\alpha-1)}{b^*\Gamma^2(\alpha)} \sum_{i=1}^{m} \zeta_i^2 \left(\tau_i^\rho/\rho\right)^{2\alpha+2\nu-2} \rho \\
& + \frac{4(K_1b^* + K_1^*a^*)\mathbb{B}(\frac{2\nu\rho-1}{\rho}, 2\alpha-1)}{b^*\Gamma^2(\alpha)} \left(T^\rho/\rho\right)^{2\alpha-1} T^{\rho-1} \\
& + \frac{4TrQ(K_2b^* + K_2^*a^*)\mathbb{B}(2\nu, 2\alpha-1)}{b^*\Gamma^2(\alpha)} \left(T^\rho/\rho\right)^{2\alpha} \rho.
\end{aligned}$$

Proof . Consider the well-defined operator $N : \mathcal{C}_{1-\nu,\rho}(J, \mathcal{L}_2(\Omega, \mathfrak{X})) \to \mathcal{C}_{1-\nu,\rho}(J, \mathcal{L}_2(\Omega, \mathfrak{X}))$, which is given as follows:

$$\begin{aligned}
N(u)(t) \; = \; & \frac{\Upsilon}{\Gamma(\alpha)} \left(\frac{t^\rho}{\rho}\right)^{\nu-1} \sum_{i=1}^{m} \zeta_i \int_0^{\tau_i} \left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s, u(s), \mathbb{K}_u(s))ds \\
& + \frac{\Upsilon}{\Gamma(\alpha)} \left(\frac{t^\rho}{\rho}\right)^{\nu-1} \sum_{i=1}^{m} \zeta_i \int_0^{\tau_i} \left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \left(\int_0^s \sigma(\eta, u(\eta), \mathbb{K}_u(\eta))dB(\eta)\right) ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s, u(s), \mathbb{K}_u(s))ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \left(\int_0^s \sigma(\eta, u(\eta), \mathbb{K}_u(\eta))dB(\eta)\right) ds.
\end{aligned}$$

Clearly, the fixed points of the operator N are solutions of the problem (1.1), according to Lemma 2.2 and Theorem 3.1. Now it remains to show that the solution is unique. Let $u, v \in \mathcal{C}_{1-\nu,\rho}(J, \mathcal{L}_2(\Omega, \mathfrak{X}))$, then we get

$$\mathbb{E}\left\|\left((Nu)(t) - (Nv)(t)\right)(t^\rho/\rho)^{1-\nu}\right\|^2 \leq 4\vartheta,$$

where

$$\begin{aligned}
\vartheta \; = \; & \frac{m \mid \Upsilon \mid^2}{\Gamma^2(\alpha)} \sum_{i=1}^{m} \zeta_i^2 \int_0^{\tau_i} \left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{2\alpha-2} s^{2\rho-2}\mathbb{E}\left\|(f(s, u(s), \mathbb{K}_u(s)) - f(s, v(s), \mathbb{K}_v(s)))\right\|^2 ds \\
& + \frac{(t^\rho/\rho)^{2-2\nu}}{\Gamma^2(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{2\alpha-2} s^{2\rho-2}\mathbb{E}\left\|(f(s, u(s), \mathbb{K}_u(s)) - f(s, v(s), \mathbb{K}_v(s)))\right\|^2 ds \\
& + \frac{m \mid \Upsilon \mid^2 TrQ}{\Gamma^2(\alpha)} \sum_{i=1}^{m} \zeta_i^2 \int_0^{\tau_i} \left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{2\alpha-2} s^{2\rho-2}\mathbb{E}\left\|(\sigma(s, u(s), \mathbb{K}_u(s)) - \sigma(s, v(s), \mathbb{K}_v(s)))\right\|^2 ds \\
& + \frac{(t^\rho/\rho)^{2-2\nu} TrQ}{\Gamma^2(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{2\alpha-2} s^{2\rho-2}\mathbb{E}\left\|(\sigma(s, u(s), \mathbb{K}_u(s)) - \sigma(s, v(s), \mathbb{K}_v(s)))\right\|^2 ds.
\end{aligned} \tag{3.7}$$

We can also estimate using the assumptions $(H2)$ and $(H3)$,

$$\mathbb{E}\left\|\mathbb{K}_u(s) - \mathbb{K}_v(s)\right\|^2 \leq \frac{2(K_1 + TTrQK_2)}{1 - 2(K_1^* + TTrQK_2^*)}\mathbb{E}\left\|u - v\right\|^2. \tag{3.8}$$

By substituting (3.8) into inequalities (3.7) and using $(H2, H3)$, we get

$$\mathbb{E}\left\|(Nu)(t) - (Nv)(t)\,(t^\rho/\rho)^{1-\nu}\right\|^2 \leq \Xi \left\|u - v\right\|^2_{\mathcal{C}_{1-\nu,\rho}}.$$

It follows from inequality (3.6), that the operator N is contraction. By well-known Banach contraction principle, we can deduce that N has a unique fixed point, which is the solution of the problem (1.1).

4 Ulam's Stability Results

In this section, we prove the Ulam stability result for the fractional stochastic differential equations (1.1). Now, we give definition of Ulam-Hyers stable (U.H.S.) for the Hilfer-Katugampola fractional stochastic differential equations (1.1). For $\epsilon > 0$, we consider the following inequality:

$$\mathbb{E}\left\|{}^\rho D^{\alpha,\beta}_{0+} Z(t) - f(t, Z(t), {}^\rho D^{\alpha,\beta}_{0+} Z(t)) - \int_0^t \sigma(s, Z(s), \mathbb{K}_Z(s))dB(s)\right\|^2 \leq \epsilon. \tag{4.1}$$

Definition 4.1. *The problem (1.1) is said to be U.H.S. if there exists the real number $C_u > 0$ such that for all $\epsilon > 0$ and for each solution $Z \in \mathcal{C}^\nu_{1-\nu,\rho}(J, \mathcal{L}_2(\Omega, \mathfrak{X}))$ of the inequality (4.1) there exists the solution $u \in \mathcal{C}^\nu_{1-\nu,\rho}(J, \mathcal{L}_2(\Omega, \mathfrak{X}))$ of the problem (1.1) with*

$$\mathbb{E}\left\|Z(t) - u(t)\right\|^2 \leq C_u \epsilon, \qquad t \in J.$$

Remark 4.1. *The function $Z \in \mathcal{C}^\nu_{1-\nu,\rho}(J, \mathcal{L}_2(\Omega, \mathfrak{X}))$ is a solution of the inequality (4.1), if and only if, there exists the function $g \in \mathcal{C}^\nu_{1-\nu,\rho}(J, \mathcal{L}_2(\Omega, \mathfrak{X}))$ such that*

(a) $\mathbb{E}\left\|g(t)\right\|^2 \leq \epsilon, \qquad t \in J.$

(b) ${}^\rho D^{\alpha,\beta}_{0+} Z(t) = f(t, Z(t), {}^\rho D^{\alpha,\beta}_{0+} Z(t)) + \int_0^t \sigma(s, Z(s), \mathbb{K}_Z(s))dB(s) + g(t), \qquad t \in J.$

Lemma 4.1. *Let $\rho > 0$, $1/2 < \alpha < 1$, and $0 \leq \beta \leq 1$. If the function $Z \in \mathcal{C}^\nu_{1-\nu,\rho}(J, \mathcal{L}_2(\Omega, \mathfrak{X}))$ is the solution of the inequality (4.1), then Z is the solution of the following integral inequality*

$$\mathbb{E}\left\|Z(t) - \mathfrak{Z}\right\|^2 \leq \kappa(\nu, \rho, T)\frac{\epsilon}{\Gamma^2(\alpha)},$$

where

$$\begin{aligned}
\mathfrak{Z} &= \frac{\Upsilon}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^m \zeta_i \int_0^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s, Z(s), \mathbb{K}_Z(s))ds \\
&\quad + \frac{\Upsilon}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^m \zeta_i \int_0^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left(\int_0^s \sigma(\eta, Z(\eta), \mathbb{K}_Z(\eta))dB(\eta)\right)ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s, Z(s), \mathbb{K}_Z(s))ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left(\int_0^s \sigma(\eta, Z(\eta), \mathbb{K}_Z(\eta))dB(\eta)\right)ds.
\end{aligned}$$

Proof . In view of Remark 4.1, we have

$${}^\rho D^{\alpha,\beta}_{0+} Z(t) = f(t, Z(t), {}^\rho D^{\alpha,\beta}_{0+} Z(t)) + \int_0^t \sigma(s, Z(s), \mathbb{K}_Z(s))dB(s) + g(t) = \mathbb{K}_Z(t) + g(t). \qquad t \in \acute{J}.$$

Then,

$$Z(t) = \mathfrak{Z} + \frac{\Upsilon(t^\rho/\rho)^{\nu-1}}{\Gamma(\alpha)}\sum_{i=1}^m \zeta_i \int_0^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s)ds.$$

From this, we get the following:

$$\mathbb{E}\left\|Z(t) - \mathfrak{Z}\right\|^2 = \mathbb{E}\left\|\mathfrak{B}\right\|^2,$$

where

$$\mathfrak{B} = \frac{\Upsilon(t^\rho/\rho)^{\nu-1}}{\Gamma(\alpha)}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}g(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}g(s)ds$$

and

$$\mathbb{E}\,\|\mathfrak{B}\|^2 \leq\ 2\left(m\mid\Upsilon\mid^2\mid\zeta\mid(T^\rho/\rho)^{2\alpha+2\nu-3}T^{\rho-1}\mathbb{B}(\frac{2\rho-1}{\rho},2\alpha-1)\right),+2\left((T^\rho/\rho)^{2\alpha-1}T^{\rho-1}\mathbb{B}(\frac{2\rho-1}{\rho},2\alpha-1)\right) = \mathfrak{K}.$$

Then,

$$\mathbb{E}\,\|Z(t)-\mathfrak{Z}\|^2 \leq \mathfrak{K}\frac{\epsilon}{\Gamma^2(\alpha)}.$$

Theorem 4.1. *Assume that the hypotheses $(H1)$ and $(H2)$ and equation (3.6) are satisfied. Then, the problem (1.1) is U.H.S.*

Proof . Let $\epsilon > 0$ and for any $t \in J$, the function $Z \in \mathcal{C}^\nu_{1-\nu,\rho}(J,\mathcal{L}_2(\Omega,\mathfrak{X}))$ satisfies the inequality (4.1). From Theorem 3.2, $u \in \mathcal{C}^\nu_{1-\nu,\rho}(J,\mathcal{L}_2(\Omega,\mathfrak{X}))$ is the unique solution of the problem (1.1). By using Lemma 2.2, we have

$$\begin{aligned} u(t) &= A_u + \frac{\Upsilon}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left(\int_0^s\sigma(\eta,u(\eta),\mathbb{K}_u(\eta))dB(\eta)\right)ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left(\int_0^s\sigma(\eta,u(\eta),\mathbb{K}_u(\eta))dB(\eta)\right)ds, \end{aligned}$$

where

$$\begin{aligned} A_u &= \frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}f(s,u(s),\mathbb{K}_u(s))ds \\ &\quad + \frac{\Upsilon}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}f(s,u(s),\mathbb{K}_u(s))ds. \end{aligned}$$

Now, if $u(\tau_i) = Z(\tau_i)$ and ${}^\rho\mathcal{J}^{1-\nu}_{0+}u(0) = {}^\rho\mathcal{J}^{1-\nu}_{0+}Z(0)$, then $A_u = A_Z$. Clearly,

$$\begin{aligned} \mathbb{E}\,\|A_u - A_Z\|^2 &\leq \frac{2m\Upsilon^2\left(\frac{t^\rho}{\rho}\right)^{2\nu-2}}{\Gamma^2(\alpha)}\sum_{i=1}^{m}\zeta_i^2\int_0^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{2\alpha-2} s^{2\rho-2}\mathbb{E}\,\|(f(s,u(s),\mathbb{K}_u(s)) - f(s,Z(s),\mathbb{K}_Z(s)))\|^2\,ds \\ &\quad + \frac{2}{\Gamma^2(\alpha)}\int_0^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{2\alpha-2} s^{2\rho-2}\mathbb{E}\,\|(f(s,u(s),\mathbb{K}_u(s)) - f(s,Z(s),\mathbb{K}_Z(s)))\|^2\,ds \end{aligned}$$

and by using $(H1, H2)$ and inequality (3.8), we get

$$\mathbb{E}\,\|A_u - A_Z\|^2 = 0.$$

Thus, $A_u = A_Z$. Then, we have

$$\begin{aligned} u(t) &= A_Z + \frac{\Upsilon}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left(\int_0^s\sigma(\eta,u(\eta),\mathbb{K}_u(\eta))dB(\eta)\right)ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left(\int_0^s\sigma(\eta,u(\eta),\mathbb{K}_u(\eta))dB(\eta)\right)ds. \end{aligned}$$

By applying Lemma 4.1 and integration of inequality (4.1) for any $t \in J$ we can obtain

$$\mathbb{E}\,\|Z(t) - u(t)\|^2 \leq C_u\epsilon,$$

which completes the proof.

5 Example

In this section, we give an example to illustrate our main results. We consider the following implicit stochastic Hilfer-Katugampola-type fractional differential equation

$$
{}^{2}D_{0+}^{\frac{2}{3},\frac{1}{2}}u(t) = \frac{1}{\sqrt{10}}\left(t\cos(t)\frac{\|u(t)\|}{1+\|u(t)\|} + \frac{\left\|{}^{2}D_{1+}^{\frac{2}{3},\frac{1}{2}}u(t)\right\|}{1+\left\|{}^{2}D_{1+}^{\frac{2}{3},\frac{1}{2}}u(t)\right\|}\right) + \int_{0}^{t}\frac{(s^{1/3}\sin(u(s)) + {}^{2}D_{1+}^{\frac{2}{3},\frac{1}{2}}u(t))}{20}dB(s), \quad t \in (0,1],
$$

$$
\left({}^{2}\mathcal{J}_{0+}^{\frac{1}{6}}u\right)(0) = 2u(3/2) \quad ,\nu = \alpha + \beta(1-\alpha). \tag{5.1}
$$

Here, ${}^{\rho}D_{0+}^{\alpha,\beta}$ is the Hilfer-Katugampola fractional derivative, $\rho = 2, \alpha = 2/3, \beta = 1/2$ and $\nu = 5/6$. Let $B(t)$ denote the standard one-dimensional Brownian motion process in $\mathcal{L}_2([0,\pi])$ defined on $(\Omega, \Im, \mathbb{P})$. Set

$$
f(t,u,v) = \frac{1}{\sqrt{10}}\left(t\cos(t)\frac{u(t)}{1+u(t)} + \frac{v(t)}{1+v(t)}\right), \qquad \sigma(t,u,v) = \frac{t^{1/3}\sin(u(t)) + v}{20}.
$$

Therefore, (5.1) can be reformulated as the system (1.1). Clearly, for $u, v \in \mathbb{R}_+$ and $t \in (0,1]$, the functions f, σ satisfy all of the assumptions of Theorem 3.2 and Theorem 4.1. For $u, v, \bar{u}, \bar{v} \in \mathbb{R}_+$ and $t \in (0,1]$, we have

$$
\|f(t,u,v) - f(t,\bar{u},\bar{v})\|^2 \leq \frac{1}{10}\left(\|u-\bar{u}\|^2 + \|v-\bar{v}\|^2\right).
$$

Hence, the condition $(H1(ii))$ is satisfied with $K_1 = K_1^* = 1/10$. In addition, for $t \in (0,1]$,

$$
\|f(t,u,v)\|^2 \leq \frac{1}{10}\left(\|u\|^2 + \|v\|^2\right),
$$

so condition $(H1(i))$ is satisfied with $p(t) = 0, q(t) = \chi(t) = 1/10$, and $q^* = \chi^* = 1/10 < 1$. Similarly, the conditions $(H2(i,ii))$ are satisfied for the function σ defined above. As a result, the assumptions $(H1)$ and $(H2)$ hold, and we can see that $\mid \Xi \mid < 1$. Hence, from Theorem 3.2, it follows that the implicit stochastic Hilfer-Katugampola-type fractional differential equation (5.1) has a unique solution for $t \in (0,1]$. Furthermore, it implies from Theorem 4.1, that the problem (5.1) is U.H.S.

References

[1] S. Abbas, M. Benchohra, J. Lazreg and Y. Zhou, *A survey on Hadamard and Hilfer fractional differential equations: analysis and stability*, Chaos Solitons Fractals **102** (2017), 47–71.

[2] M.I. Abbas and M.A. Ragusa, *Nonlinear fractional differential inclusions with nonsingular Mittag-Leffler kernel*, AIMS Math. 7 (2022), no. 11, 20328–20340.

[3] H.M. Ahmed and M.M. El-Borai, *Hilfer fractional stochastic integro-differential equations*, Appl. Math. Comput. **331** (2018), 182–189.

[4] H.M. Ahmed, M.M. El-Borai, H.M. El-Owaidy and A.S. Ghanem, *Impulsive Hilfer fractional differential equations*, Adv. Differ. Equ. **2018** (2018), no. 1, 1–20.

[5] S.P. Bhairat, *New approach to existence of a solution to generalize Cauchy type problem*, J. Math. Model. **8** (2020), no. 4.

[6] S.P. Bhairat and D.-B. Dhaigude, *Existence of solutions of generalized fractional differential equation with a nonlocal initial condition*, Math. Bohemica **144** (2019), no. 2, 203–220.

[7] S.P. Bhairat and D.-B. Dhaigude, *Existence and stability results for fractional differential equations involving generalized Katugampola derivative*, Stud. Univ. Babeş-Bolyai Math. **65** (2020), no. 1, 29—46.

[8] J. Cao, Q. Yang, Z. Huang and Q. Liu, *Asymptotically almost periodic solutions of stochastic functional differential equations*, Appl. Math. Comput. **218** (2011), 1499–1511.

[9] J. Cao, Q. Yang and Z. Huang, *On almost periodic mild solutions for stochastic functional differential equations*, Nonlinear Anal. TMA. **13** (2012), 275–286.

[10] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[11] D. Delbosco and L. Rodino, *Existence and uniqueness for a nonlinear fractional differential equation*, J. Math. Anal. Appl. **204** (1996), 609–625.

[12] K.M. Furati, M.D. Kassim and N. Tatar, *Existence and uniqueness for a problem involving Hilfer factional derivative*, Comput. Math. Appl. **64** (2012), 1616–1626.

[13] Z.Y. Gao and X.L. Yu, *Existence results for BVP of a class of Hilfer fractional differential equations*, J. Appl. Math. Comput. **56** (2018), 217–233.

[14] H. Gou and Y. Li., *Study on Hilfer-Katugampola fractional implicit differential equations with nonlocal conditions*, Bull. Sci. Math. **167** (2021), 102944.

[15] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, New York, 2003.

[16] H. Gu and J.J. Trujillo, *Existence of mild solution for evolution equation with Hilfer fractional derivative*, Appl. Math. Comput. **257** (2015), 344–354.

[17] S. Harikrishnan, K. Kanagarajan and E.M. Elsayed, *Existence of solutions of nonlocal initial value problems for differential equations with Hilfer-Katugampola fractional derivative*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. **113** (2019).

[18] J.H. He, *Some applications of nonlinear fractional differential equations and their approximations*, Bull. Sci. Technol. **15** (1999), no. 2, 86–90.

[19] R. Hilfer, *Application of fractional calculus in physics*, World Scientific, New Jersey, 2001.

[20] R. Hilfer, Y. Luchko and Z. Tomovski, *Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives*, Fract. Calc. Appl. Anal. **12** (2009), no. 3, 299–318.

[21] U.N. Katugampola, *New approach to a generalized fractional integral*, Appl. Math. Comput. *218* (2011), 860–865.

[22] U.N. Katugampola, *A new approach to generalized fractional derivatives*, Bull. Math. Anal. Appl. **6** (2014), 1–15.

[23] N.A. Khan, A. Ara and A. Mahmood, *Approximate solution of time-fractional chemical engineering equations: a comparative study*, Int. J. Chem. Reactor Eng. **8** (2010), Article A19.

[24] R. Kruse, *Strong and Weak Approximation of Semilinear Stochastic Evolution Equations*, Lecture Notes in Mathematics, Springer, 2014.

[25] Y.J. Liu , C.G. Yan anf W.H. Jiang , *Existence of the Unique nontrivial solution for mixed fractional differential equations*, J. Function Spaces **2021** (2021).

[26] X. Mao, *Stochastic Differential Equations and their Applications*, Horwood Publishing, Chichester, 1997.

[27] K.B. Oldham , *Fractional differential equations in electrochemistry*, Adv. Eng. Softw. **41** (2010), no. 1, 9–12.

[28] D.S. Oliveira, E. Capelas de Oliveira, *Hilfer-Katugampola fractional derivatives*, Comput. Appl. Math. **37** (2018), 3672–3690.

[29] J. Sabatier, O.P. Agrawal and J.A.T. Machado, *Advances in fractional calculus*, Springer, Dordrecht, The Netherlands, 2007.

[30] A.Y.A. Salamooni and D. D. Pawar, *Existence and stability results for Hilfer-Katugampola-type fractional implicit differential equations with nonlocal conditions*, J. Nonlinear Sci. Appl. **14** (2021), 124–138.

[31] D. Vivek, K. Kanagarajan and E.M. Elsayed, *Some existence and stability results for Hilfer-fractional implicit differential equations with nonlocal conditions*, Mediterr. J. Math. **15** (2018), no. 1, 1–21.

[32] O.K. Wanassi and F. Toumi, *Existence of positive solutions for a singular nonlinear semipositone fractional differential equations with parametric dependence*, Filomat **35** (2021), no. 4, 1141–1154.

[33] J. Wang and Y. Zhang, *Nonlocal initial value problems for differential equations with Hilfer fractional derivative*, Appl. Math. Comput. **266** (2015), 850–859.