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Strong convergence result for split inclusion problems in Banach spaces

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Abstract

By using Halpern's type iteration process, an iterative algorithm is proposed to study the split inclusion problem and fixed points of a relatively nonexpansive mapping in Banach spaces. This method uses dynamic stepsize that is generated at each iteration by simple computations, which allows it to be easily implemented without the prior information of the operator norm. Then, the main result is used to study the fixed points of a countable family of relatively nonexpansive mappings and the semigroup of relatively nonexpansive mappings. Finally, a numerical example is provided to illustrate the main result.

Keywords: Split feasibility problem, Uniformly convexity, Uniformly smoothness, Fixed point problem
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1 Introduction

Let H_1 and H_2 be two Hilbert space. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two maximal monotone operators and $A : H_1 \to H_2$ be a bounded linear operator. Consider the following split inclusion problem (SIP) introduced by Moudafi [25] in Hilbert space:

$$\text{To find } x^* \in H_1 \text{ such that } 0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*). \tag{1.1}$$

Let the solution set of (1.1) is denoted by Γ. In fact, we know that the SIP is a generalization of the inclusion problem and the split feasibility problem (SFP). Now, we provide some special cases of SIP (1.1).
Let $f : H_1 \to \mathbb{R} \cup \{\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{\infty\}$ be proper, lower semicontinuous and convex functions. If we take $B_1 = \partial f$ and $B_2 = \partial g$, where ∂f and ∂g are the subdifferential of f and g, then the SIP (1.1) becomes the following proximal split feasibility problem:

$$\text{To find } x^* \in \arg\min f \text{ such that } Ax^* \in \arg\min g, \tag{1.2}$$

where $\arg\min f = \{x \in H_1 : f(x) \le f(y), \forall y \in H_1\}$ and $\arg\min g = \{x \in H_2 : g(x) \le g(y), \forall y \in H_2\}$. In particular, if we take $f(x) = \frac{1}{2}\|M(x) - b\|^2$ and $g(x) = \frac{1}{2}\|N(x) - c\|^2$, where M and N are matrices, and $b, c \in H_1$, then the (1.2) becomes the least square problem.

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Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. If $B_1 = N_C, B_2 = N_Q$, where N_C and N_Q are the normal cones of C and Q, respectively, then we have the following the SFP:

$$\text{To find } x^* \in C \text{ such that } Ax^* \in Q.$$

This problem was first introduced in a finite dimensional Hilbert space by Censor and Elfving [13] for modeling inverse problems in radiation therapy treatment planning, which arise from phase retrieval and in medical image reconstruction, especially intensity modulated therapy [12].

In 2011, to solve the SIP (1.1) Byrne et al. [11] proved some weak convergence results in infinite dimensional Hilbert spaces. For given $x_1 \in H_1$, the sequence $\{x_n\}_{n=1}^{\infty}$ is defined by,

$$x_{n+1} = J_{\lambda}^{B_1}(x_n - \gamma A^*(I - J_{\lambda}^{B_2})Ax_n), \ \forall\ n \geq 1,$$

where $\lambda > 0$ and $\gamma \in (0, \frac{2}{\|A\|^2})$ and $J_{\lambda}^{B_1}$ is resolvent operator of B_1. In order to obtain strong convergence, Kazmi and Rizvi [19] proposed following algorithm for solving SIP (1.1) and fixed points of a nonexpansive mapping, for given $x_1 \in H_1$:

$$\begin{cases} u_n = J_{\lambda}^{B_1}(x_n - \gamma A^*(I - J_{\lambda}^{B_2})Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tu_n, \forall\ n \geq 1, \end{cases}$$

where $\gamma \in (0, \frac{2}{\|A\|^2})$, $f : H_1 \to H_1$ is a contraction mapping with constant $\alpha \in (0,1)$ and $\{\alpha_n\}_{n=1}^{\infty} \in (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$.
Very recently, Alofi et al. [5] introduced an algorithm based on Halpern's iteration for solving SIP (1.1) in a uniformly convex and smooth Banach space E. They proposed the following algorithm for given $x_1 \in E$:

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)J_{\lambda_n}^{B_1}(x_n - \lambda_n A^* J_{E_2}^{p}(I - J_{r_n}^{B_2})Ax_n) \\ x_{n+1} = \beta_n(u_n) + (1-\beta_n)y_n, \end{cases}$$

where $\{\alpha_n\}_{n=1}^{\infty} \in (0,1)$, $\{\beta_n\}_{n=1}^{\infty} \in [0,1)$ and $\{\lambda_n\}_{n=1}^{\infty}, \{r_n\}_{n=1}^{\infty} \in (0,\infty)$, satisfying the some additional conditions on parameters and the stepsize λ_n.

In 2018, by employing the idea of Halpern's iteration process Suantai *et al.* [35] proved strong convergence theorem for (1.1) in Banach spaces. For given $x_1 \in E_1$ their sequences generated by the following iterative scheme under some suitable conditions:

$$\begin{cases} z_n = J_{E_1^*}^{q}(J_{E_1}^{p}(x_n) - \lambda_n A^* J_{E_2}^{p}(I - J_{r_n})Ax_n) \\ y_n = J_{E_1^*}^{q}(\alpha_n J_{E_1}^{p}(u_n) + (1-\alpha_n)J_{E_1}^{p} J_{\lambda_n}(z_n)) \\ x_{n+1} = J_{E_1^*}^{q}(\beta_n J_{E_1}^{p}(x_n) + (1-\beta_n)J_{E_1}^{p}(y_n)), \end{cases}$$

where stepsize λ_n is a sequence, chosen in such a way that,

$$0 < a \leq \lambda_n \leq b < \left(\frac{q}{C_q}\|A\|^q\right)^{\frac{1}{q-1}}, \ \text{ for some } a, b \in (0,\infty),$$

where $\{\alpha_n\}_{n=1}^{\infty} \in (0,1)$, $\{\beta_n\}_{n=1}^{\infty} \in [0,1)$ and $\{\lambda_n\}_{n=1}^{\infty}, \{r_n\}_{n=1}^{\infty} \in (0,\infty)$. In recent years, many authors have constructed several iterative methods for solving SIP (see, [5, 7, 14, 27, 33, 38]).
However, in order to achieve the solution of mentioned above problems, one has to obtain the operator norm $\|A\|$, which is not easy to calculate in general. To avoid this computation, López *et al.* [22] find a new way to select the stepsize as follows:

$$\mu_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \ n \geq 1,$$

where $\rho_n \in (0,4), f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$ and $\nabla f(x_n) = A^*(I - P_Q)Ax_n$, for all $n \geq 1$, where P_Q is the metric projection of H_2 onto Q. This method is a modification of the CQ method often called the self-adaptive method, which permits step-size being selected self adaptively, for more see [29, 39]

Motivated by the work of Suantai *et al.* [35] and López *et al.* [22], intention of this paper is to propose an algorithm to study SIP (1.1) and fixed point of relatively nonexpansive mapping in p-uniformly convex and uniformly smooth Banach spaces. Stepsize is being selected without the prior knowledge of operator norm, so it can be more efficiently implemented. Also, this result is applied to find the common fixed points of a family of relatively nonexpansive mappings which ais also the solution of the SIP (1.1).

2 Preliminaries

Let C be a nonempty closed, convex subset of Real Banach space E with dual E^* and $1 < q \leq 2 \leq p$ with $\frac{1}{p}+\frac{1}{q}=1$. The modulus of convexity $\delta_E : [0,2] \to [0,1]$ is defined as

$$\delta_B(\varepsilon) = \inf\{1 - \frac{\|u+v\|}{2} : \|u\| = 1 = \|v\|, \|u-v\| \geq \varepsilon\}.$$

A Banach space E is called uniformly convex [16] if $\delta_E(\varepsilon) > 0$, for $\varepsilon \in (0,2]$ and p-uniformly convex if there exist $C_p > 0$, such that $\delta_E(\varepsilon) \geq C_P \varepsilon^p$ for any $\varepsilon \in (0,2]$. The modulus of smoothness $\rho_E(\varepsilon) : [0,\infty) \to [0,\infty)$ is defined by

$$\rho_E(\tau) = \{\frac{\|u+\tau v\| + \|u-\tau v\|}{2} - 1 : \|u\| = \|v\| = 1\}.$$

A Banach space E is called uniformly smooth [17] if $\lim_{\tau\to 0} \frac{\rho_B(\tau)}{\tau} = 0$; $q-$uniformly smooth if there exist $C_q > 0$ such that $\rho_E(\tau) \leq C_q \tau^q$ for any $\tau > 0$.
A continuous strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a gauge if $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. The mapping $J_\varphi^E : E \to E^*$ associated with a gauge function φ defined by

$$J_\varphi^E(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\varphi(\|x\|), \|f\| = \varphi(\|x\|), \forall x \in E\},$$

is called the duality mapping with gauge φ, where $\langle ., .\rangle$ denotes the duality pairing between E and E^*.

If $\varphi(t) = t$, then $J_\varphi^E = J$ is the normalized duality mapping. In particular, $\varphi(t) = t^{p-1}$, where $p > 1$, the duality mapping $J_\varphi^E = J_p^E$ is called the generalized duality mapping defined by

$$J_p^E(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\},\ x \in E.$$

It is well known that if E is uniformly smooth, the generalized duality mapping J_p^E is norm to norm uniformly continuous on bounded subsets of E (see [31]). Furthermore, J_p^E is one-to-one, single-valued and satisfies $J_p^E = (J_q^{E^*})^{-1}$, where $J_q^{E^*}$ is the generalized duality mapping of E^* (see [30], [15] for more details).
For a gauge φ, the function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\Phi(t) = \int_0^t \varphi(s)ds$$

is a continuous convex strictly increasing differentiable function on $\mathbb{R}^+$ with $\Phi'(t) = \varphi(t)$ and $\lim_{t\to\infty} \frac{\Phi(t)}{t} \to \infty$. Therefore, Φ has a continuous inverse function Φ^{-1}. We next recall the Bregman distance, which was introduced and studied by Bregman in [10].

Definition 2.1. Let E be a real smooth Banach space. The Bregman distance $\Delta_\varphi(x,y)$ between x and y in E is defined by

$$\Delta_\varphi(x,y) = \Phi(\|y\|) - \Phi(\|x\|) - \langle J_\varphi(x), y - x\rangle.$$

We note that the Bregman distance Δ_φ does not satisfy the well-known properties of a metric because Δ_φ is not symmetric and does not satisfy the triangle inequality. Moreover, the Bregman distance has the following important properties:

$$\Delta_\varphi(x,y) = \Delta_\varphi(x,z) + \Delta_\varphi(z,y) + \langle J_\varphi^E x - J_\varphi^E z, z - y\rangle, \tag{2.1}$$

and

$$\Delta_\varphi(x,y) + \Delta_\varphi(y,x) = \langle J_\varphi^E x - J_\varphi^E y, x - y\rangle,\ \ \forall x,y,z \in E.$$

In the case $\varphi(t) = t^{p-1}$, where $p > 1$, the distance $\Delta_\varphi = \Delta_p$ is called the p-Lyapunov function which was studied in [9] and it is given by

$$\Delta_p(x, y) = \frac{1}{q}\|x\|^p - \langle J_\varphi^E x, y\rangle + \frac{1}{p}\|y\|^p, \tag{2.2}$$

where p, q are conjugate exponents. For the p-uniformly convex space, the Bregman distance has the following relation, see [32]:

$$\tau\|x-y\|^p \le \Delta_p(x, y) \le \langle J_\varphi^E x - J_\varphi^E y, x - y\rangle,$$

where $\tau > 0$ is some fixed number. If $p = 2$, we get

$$\Delta_2(x, y) = \phi(x, y) = \|x\|^2 - 2\langle Jx, y\rangle + \|y\|^2,$$

where ϕ is called the Lyapunov function which was introduced by Alber ([1], [2]). The function $V_p : E \times E^* \to [0, +\infty)$ is defined by,

$$V_p(\bar{x}, x) = \frac{1}{q}\|\bar{x}\|^q - \langle \bar{x}, x\rangle + \frac{1}{p}\|x\|^p, \qquad \forall x \in E, \bar{x} \in E^*.$$

Then $V_p \ge 0$ and also satisfy following property [28],

$$V_p(\bar{x}, x) = \Delta_p(J_E^q(\bar{x}), x), \ \ \forall\, x \in E,\ \bar{x} \in E^*. \tag{2.3}$$

Moreover,

$$V_p(\bar{x}, x) + \langle \bar{y}, J_E^q(\bar{x}) - x\rangle \le V_p(\bar{x} + \bar{y}, x),\ \forall x \in E \text{ and } \bar{x}, \bar{y} \in E^*.$$

Lemma 2.2. [26] Let E be a p-uniformly convex and uniformly smooth real Banach space. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. Then $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$ if and only if $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3. [40] Let $x, y \in E$. If E is q-uniformly smooth, then there is a $C_q > 0$ so that

$$\|x - y\|^q \le \|x\|^q - q\langle y, J_E^q(x)\rangle + C_q\|y\|^q.$$

Let C be a closed and convex subset of E, a point $x^* \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to x^* and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Similarly a point $x^* \in C$ is a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to x^* and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Set of strong asymptotic fixed points and asymptotic fixed point of T is denoted by $\tilde{F}(T)$ and $\hat{F}(T)$, respectively.

Definition 2.4. [24] A mapping T from C to C is said to be Bregman relatively nonexpansive if $F(T) \ne \emptyset, \hat{F}(T) = F(T)$ and

$$\Delta_p(x^*, Ty) \le \Delta_p(x^*, y),\ \forall y \in C, x^* \in F(T).$$

For more detail, see [31]. Let $B : E \to 2^{E^*}$ be a mapping, The effective domain of B is denoted by $D(B)$, such that, $D(B) = \{x \in E : Bx \ne \emptyset\}$. Mapping B is monotone if,

$$\langle u - v, x - y\rangle \ge 0,\ \forall x, y \in D(B),\ u \in Bx \text{ and } v \in By.$$

A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E.

Let E be a p-uniformly convex and uniformly smooth Banach space and B is a monotone operator on E, then for $\lambda > 0$ and $x \in E$, consider the metric resolvent of $M_\lambda^B : E \to D(B)$ of B, defined as,

$$M_\lambda^B(x) = (I + \lambda(J_E^p)^{-1}B)^{-1}(x),\ \forall\, x \in E.$$

Set of null point of B is defined by $B^{-1}(0) = \{z \in E : 0 \in Bz\}$. Since $B^{-1}(0)$ is closed and convex, Then we have

$$0 \in J_E^P(M_\lambda^B(x) - x) + \lambda B M_\lambda^B(x).$$

Next, $F(M_\lambda^B) = B^{-1}(0)$ for $\lambda > 0$, from [21] we also have for all $x, y \in E$,

$$\langle M_\lambda^B(x) - M_\lambda^B(y), J_E^p(x - M_\lambda^B(x)) - J_E^p(y - M_\lambda^B(y))\rangle \ge 0,$$

if $B^{-1}(0) \neq \emptyset$, then

$$\langle J_E^p(x - M_\lambda^B(x)) - (M_\lambda^B(x) - z)\rangle \geq 0, \quad \forall z \in B^{-1}(0).$$

The monotonicity of B implies that M_λ^B is a firmly nonexpansive-like mapping. Now, we can define a mapping $N_\lambda^B : E_1 \to D(B)$ called the relative resolvent of B [20], for $\lambda > 0$, as

$$N_\lambda^B = (J_E^p + \lambda B)^{-1} J_E^p(x), \; \forall \; x \in E.$$

It is known that N_λ^B is relatively nonexpansive mapping and $F(N_\lambda^B) = B^{-1}(0)$, for $\lambda > 0$.

Lemma 2.5. [20] Let $B : E \to 2^{E^*}$ be a maximal monotone operator with $B^{-1} \neq \emptyset$ and let N_λ^B be a resolvent operator of B for $\lambda > 0$. Then

$$\Delta_p(N_\lambda^B(x), z) + \Delta_p(N_\lambda^B(x), x) \leq \Delta_p(x, z), \;\; \forall x \in E \text{ and } z \in B^{-1}(0).$$

Lemma 2.6. [36] Let E_1, E_2 be two p-uniformly convex and uniformly smooth Banach spaces with duals E_1^*, E_2^*, respectively. $N_{\lambda_1}^{E_1}$ is the resolvent operator of a maximal monotone E_1 for $\lambda_1 > 0$ and $M_{\lambda_2}^{E_2}$ is the metric resolvent operator of a maximal monotone E_2 for $\lambda_2 > 0$. Assume $\Omega \neq \emptyset$, $\lambda > 0$ and $x^* \in E_1$. Then x^* is a solution of problem (1.1) if and only if

$$x^* = N_{\lambda_1}^{E_1}(J_{E_1^*}^q(J_{E_1}^p(x^*) - \lambda A^* J_{E_2}^p(I - M_{\lambda_2}^{E_2})Ax^*)).$$

Lemma 2.7. [23] Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

1. $\tau(n_0) \leq \tau(n_0 + 1) \leq \ldots$ and $\tau(n) \to \infty$;
2. $\Gamma_{\tau_n} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

Proposition 2.8. Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E. Let $x_0 \in C$ and $x \in E$, then there exists a unique element x_0 in C such that

$$\Delta_\varphi(x_0, x) = \inf\{\Delta_\varphi(z, x) : z \in C\}.$$

In this case, we denote the generalized projection from E onto C by $\Pi_C^\varphi(x) = x_0$. When $\varphi(t) = t$, we have $\Pi_C^\varphi(x)$ coincides with the generalized projection studied in [1]. Let $p > 1$ and $\varphi(t) = t^{p-1}$, then Π_C^φ becomes the generalized projection with respect to p and is denoted by Π_C.

Proposition 2.9. [21] Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E. Let $x_0 \in C$ and $x \in E$, then the following assertions are equivalent:

(a) $x_0 = \Pi_C^\varphi(x)$;

(b) $\langle z - x_0, J_\varphi(x_0) - J_\varphi(x)\rangle \geq 0$, $\forall z \in C$.

Also, we have

$$\Delta_\varphi(y, \Pi_C^\varphi) + \Delta_\varphi(\Pi_C^\varphi, x) \leq \Delta_\varphi(y, x), \; \forall y \in C.$$

3 Algorithm and their convergence

For rest of the paper, let

- E_1 be a p-uniformly convex and uniformly smooth Banach space and E_2 be a uniformly convex and smooth Banach space with duals E_1^*, E_2^*, respectively,
- $B_1 : E_1 \to 2^{E_1{}^*}$ and $B_2 : E_2 \to 2^{E_2{}^*}$ be maximal monotone operators, such that $B_1^{-1}(0) \neq 0$, $B_2^{-1}(0) \neq 0$,
- $N_{\lambda_1}^{B_1}$ is the resolvent operator of B_1 for $\lambda_1 > 0$ and $M_{\lambda_2}^{B_2}$ is the metric resolvent operator of B_2 for $\lambda_2 > 0$.

- $J^p_{E_1}$ and $J^p_{E_2}$ represent the duality mappings of E_1 and E_2, respectively and $J^p_{E_1} = (J^q_{E_1^*})^{-1}$, where $J^q_{E_1^*}$ is the duality mapping of E_1^*,
- $T : E_1 \to E_1$ be a Bregman relatively nonexpansive mapping and $A : E_1 \to E_2$ be a bounded linear operator with its adjoint $A^* : E_2^* \to E_1^*$, and
- $\{\alpha_n\}_{n=1}^{\infty} \in (0,1)$, $\{\beta_n\}_{n=1}^{\infty} \in [0,1)$ and $\{u_n\}_{n=1}^{\infty}$ be a sequence such that $u_n \to u \in E$.

Algorithm 3.1. Select $x_1 \in E_1$ and let sequence $\{x_n\}_{n=1}^{\infty}$ be generated by,

$$\begin{cases} z_n = N^{B_1}_{\lambda_1}(J^q_{E_1^*}(J^p_{E_1}(x_n) - \rho_n \frac{f^{p-1}(x_n)}{\|g(x_n)\|^p} g(x_n)) \\ y_n = J^q_{E_1^*}(\alpha_n J^p_{E_1}(u_n) + (1-\alpha_n) J^p_{E_1} T(z_n)) \\ x_{n+1} = J^q_{E_1^*}(\beta_n J^p_{E_1}(x_n) + (1-\beta_n) J^p_{E_1}(y_n)), \end{cases} \tag{3.1}$$

where $f(x_n) := \frac{1}{p}\|(I - M^{B_2}_{\lambda_2})Ax_n\|^p$, $f^{p-1}(x_n) := \left(\frac{1}{p}\|(I - M^{B_2}_{\lambda_2})Ax_n\|^p\right)^{p-1}$, $g(x_n) := A^* J^p_{E_2}(I - M^{B_2}_{\lambda_2})Ax_n$ and $\{\rho_n\} \in (0,\infty)$ satisfies $\liminf_{n\to\infty} \rho_n(pq - C_q \rho_n^{q-1}) > 0$. If $g(x_n) = 0$, then $z_n = x_n$ and the iterative process stops, x_n is a solution. Otherwise, we set n := n + 1 and go to (3.1).

Lemma 3.1. Sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ generated by Algorithm 3.1 are bounded.

Proof . Since $g(x_n) = A^* J^p_{E_2}(I - M^{B_2}_{\lambda_2})Ax_n$,

$$\begin{aligned} \langle g(x_n), u^* - x_n\rangle &= \langle A^* J^p_{E_2}(I - M^{B_2}_{\lambda_2})Ax_n, u^* - x_n\rangle \\ &= \langle J^p_{E_2}(I - M^{B_2}_{\lambda_2})Ax_n, Au^* - Ax_n\rangle \\ &= \langle J^p_{E_2}(I - M^{B_2}_{\lambda_2})Ax_n, M^{B_2}_{\lambda_2}Ax_n - Ax_n\rangle \\ &\quad + \langle J^p_{E_2}(I - M^{B_2}_{\lambda_2})Ax_n, Au^* - M^{B_2}_{\lambda_2}Ax_n\rangle \\ &\leq -\|Ax_n - M^{B_2}_{\lambda_2}Ax_n\|^p = -pf(x_n). \end{aligned} \tag{3.2}$$

Let $u^* \in \Gamma \cap F(T)$, from Lemma 2.3 and (2.2), we have

$$\begin{aligned} \Delta_p(z_n, u^*) &= \Delta_p(J^q_{E_1^*}[J^p_{E_1}(x_n) - \rho_n \frac{f^{p-1}(x_n)}{\|g(x_n)\|^p} g(x_n)], u^*) \\ &= \frac{\|u^*\|^p}{p} + \frac{1}{q}\|J^p_{E_1}(x_n) - \rho_n \frac{f^{p-1}(x_n)}{\|g(x_n)\|^p} g(x_n)\|^q - \langle J^p_{E_1}(x_n), u^*\rangle \\ &\quad + \rho_n \frac{f^{p-1}(x_n)}{\|g(x_n)\|^p}\langle u^*, g(x_n)\rangle \\ &\leq \frac{\|u^*\|^p}{p} + \frac{1}{q}\|x_n\|^p - \rho_n \frac{f^{p-1}(x_n)}{\|g(x_n)\|^p}\langle x_n, g(x_n)\rangle + \frac{C_q}{q}\rho_n^q \frac{f^p(x_n)}{\|g(x_n)\|^p} \\ &\quad - \langle u^*, J^p_{E_1}x_n\rangle + \rho_n \frac{f^{p-1}(x_n)}{\|g(x_n)\|^p}\langle u^*, g(x_n)\rangle \\ &\leq \frac{1}{p}\|u^*\|^p + \frac{1}{q}\|x_n\|^p - \langle u^*, J^p_{E_1}x_n\rangle + \rho_n \frac{f^{p-1}(x_n)}{\|g(x_n)\|^p}\langle u^* - x_n, g(x_n)\rangle \\ &\quad + \frac{C_q}{q}\rho_n^q \frac{f^p(x_n)}{\|g(x_n)\|^p} \\ &= \Delta_p(x_n, u^*) + \rho_n \frac{f^{p-1}(x_n)}{\|g(x_n)\|^p}\langle u^* - x_n, g(x_n)\rangle + \frac{C_q}{q}\rho_n^q \frac{f^p(x_n)}{\|g(x_n)\|^p}. \end{aligned} \tag{3.3}$$

Using (3.2) and (3.3),

$$\begin{aligned} \Delta_p(z_n, u^*) &\leq \Delta_p(x_n, u^*) - \rho_n p \frac{f^p(x_n)}{\|g(x_n)\|^p} + \frac{C_q}{q}\rho_n^q \frac{f^p(x_n)}{\|g(x_n)\|^p} \\ &= \Delta_p(x_n, u^*) - (\rho_n p - \frac{C_q}{q}\rho_n^q)\frac{f^p(x_n)}{\|g(x_n)\|^p}. \end{aligned} \tag{3.4}$$

Since $\liminf_{n\to\infty} \rho_n(pq - C_q\rho_n^{q-1}) > 0$, thus

$$\Delta_p(z_n, u^*) \leq \Delta_p(x_n, u^*) \;\; n \geq 1. \tag{3.5}$$

Thus,

$$\begin{aligned}
\Delta_p(u^*, y_n) &= \Delta_p(u^*, J^q_{E_1^*}(\alpha_n J^p_{E_1}(u_n) + (1-\alpha_n)J^p_{E_1}(Tz_n))) \\
&= \frac{\|u^*\|^p}{p} + \frac{1}{q}\|J^q_{E_1^*}\left(\alpha_n J^p_{E_1}(u_n) + (1-\alpha_n)J^p_{E_1}(Tz_n)\right)\|^p \\
&\quad - \langle u^*, \alpha_n J^p_{E_1}(u_n) - (1-\alpha_n)J^p_{E_1}(Tz_n)\rangle \\
&= \frac{\|u^*\|^p}{p} + \frac{1}{q}\|\alpha_n J^p_{E_1}(u_n) + (1-\alpha_n)J^p_{E_1}(Tz_n)\|^q \\
&\quad - \alpha_n\langle u^*, J^p_{E_1}(u_n)\rangle - (1-\alpha_n)\langle u^*, J^p_{E_1}(Tz_n)\rangle \\
&\leq \frac{\|u^*\|^p}{p} + \frac{1}{q}\left(\alpha_n\|J^p_{E_1}(u_n)\|^q + (1-\alpha_n)\|J^p_{E_1}(Tz_n)\|^q\right) \\
&\quad - \alpha_n\langle u^*, J^p_{E_1}(u_n)\rangle - (1-\alpha_n)\langle u^*, J^p_{E_1}(Tz_n)\rangle \\
&= \frac{\|u^*\|^p}{p} + \alpha_n\frac{\|u_n\|^p}{q} + (1-\alpha_n)\frac{\|Tz_n\|^p}{q} \\
&\quad - \alpha_n\langle u^*, J^p_{E_1}(u_n)\rangle - (1-\alpha_n)\langle u^*, J^p_{E_1}(Tz_n)\rangle \\
&= \alpha_n\left(\frac{\|u^*\|^p}{p} + \frac{\|u_n\|^p}{q}\langle u^*, J^p_{E_1}(u_n)\rangle\right) \\
&\quad + (1-\alpha_n)\left(\frac{\|u^*\|^p}{p} + \frac{\|Tz_n\|^p}{q} - \langle u^*, J^p_{E_1}(Tz_n)\rangle\right) \\
&= \alpha_n\Delta_p(u^*, u_n) + (1-\alpha_n)\Delta_p(u^*, Tz_n) \\
&\leq \alpha_n\Delta_p(u^*, u_n) + (1-\alpha_n)\Delta_p(u^*, z_n) \\
&\leq \alpha_n\Delta_p(u^*, u_n) + (1-\alpha_n)\Delta_p(u^*, x_n).
\end{aligned} \tag{3.6}$$

We can also show that,

$$\Delta_p(u^*, x_{n+1}) \leq \beta_n\Delta_p(u^*, x_n) + (1-\beta_n)\Delta_p(u^*, y_n). \tag{3.7}$$

Since $\{u_n\}$ is bounded, there exists a constant $K > 0$ such that $\Delta_p(u^*, u_n) \leq K, \;\; \forall\, n \geq 1$ and from (3.6) and (3.7), we have

$$\begin{aligned}
\Delta_p(u^*, x_{n+1}) &\leq \beta_n\Delta_p(u^*, x_n) + (1-\beta_n)\Delta_p(u^*, y_n) \\
&\leq \beta_n\Delta_p(u^*, x_n) + (1-\beta_n)\left(\alpha_n\Delta_p(u^*, u_n) + (1-\alpha_n)\Delta_p(u^*, x_n)\right) \\
&= (1-\alpha_n(1-\beta_n))\,\Delta_p(u^*, x_n) + \alpha_n(1-\beta_n)\Delta_p(u^*, u_n) \\
&\leq (1-\alpha_n(1-\beta_n))\,\Delta_p(u^*, x_n) + \alpha_n(1-\beta_n)K \\
&\leq \max\{K, \Delta_p(u^*, x_n)\} \\
&\;\;\vdots \\
&\leq \max\{K, \Delta_p(u^*, x_1)\}.
\end{aligned}$$

By induction, we have that $\Delta_p(u^*, x_n)$ is bounded, So are $\{y_n\}$, $\{z_n\}$ and $\{Tz_n\}$. □

Theorem 3.2. If $\{\alpha_n\} \to 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\limsup_{n\to\infty}\beta_n < 1$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated by Algorithm 3.1 converges strongly to $x^* \in \Gamma \cap F(T)$, where $x^* = \Pi_{\Gamma\cap F(T)}u$.

Proof . Let $x^* = \Pi_{\Gamma\cap F(T)}u$. Then, by using (2.3), we get that

$$\begin{aligned}
\Delta_p(x^*, x_{n+1}) &\le \beta_n \Delta_p(x^*, x_n) + (1-\beta_n)\Delta_p(x^*, y_n) \\
&= \beta_n \Delta_p(x^*, x_n) + (1-\beta_n)\Delta_p\left(x^*, J^q_{E_1^*}(\alpha_n, J^p_{E_1}(x_n) + (1-\alpha_n)J^p_{E_1}(Tz_n))\right) \\
&= \beta_n \Delta_p(x^*, x_n) + (1-\beta_n)V_p(x^*, \alpha_n J^p_{E_1}(x_n) + (1-\alpha_n)J^p_{E_1}(Tz_n)) \\
&\le \beta_n \Delta_p(x^*, x_n) + (1-\beta_n)[V_p(x^*, \alpha_n J^p_{E_1}(x_n) \\
&\quad + (1-\alpha_n)J^p_{E_1}(Tz_n) - \alpha_n(J^p_{E_1}(x_n) - J^p_{E_1}(x^*)))] \\
&\quad + (1-\beta_n)\langle y_n - x^*, \alpha_n \left(J^p_{E_1}(x_n) - J^p_{E_1}(x^*)\right)\rangle \\
&= \beta_n \Delta_p(x^*, x_n) + (1-\beta_n)[V_p\left(x^*, (1-\alpha_n)J^p_{E_1}(Tz_n) + \alpha_n J^p_{E_1}(x^*)\right)] \\
&\quad + \alpha_n(1-\beta_n)\langle y_n - x^*, J^p_{E_1}(x_n) - J^p_{E_1}(x^*)\rangle \\
&\le \beta_n \Delta_p(x^*, x_n) + (1-\beta_n)[(1-\alpha_n)V_p\left(x^*, J^p_{E_1}(Tz_n) + \alpha_n V_p(x^*, J^p_{E_1}(x^*))\right)] \\
&\quad + \alpha_n(1-\beta_n)\langle y_n - x^*, J^p_{E_1}(x_n) - J^p_{E_1}(u)\rangle \\
&\quad + \alpha_n(1-\beta_n)\langle y_n - x^*, J^p_{E_1}(u) - J^p_{E_1}(x^*)\rangle \\
&= \beta_n \Delta_p(x^*, x_n) + (1-\beta_n)[(1-\alpha_n)\Delta_p(x^*, Tz_n) + \alpha_n \Delta_p(x^*, x^*)] \\
&\quad + \alpha_n(1-\beta_n)\langle y_n - x^*, J^p_{E_1}(x_n) - J^p_{E_1}(u)\rangle \\
&\quad + \alpha_n(1-\beta_n)\langle y_n - x^*, J^p_{E_1}(u) - J^p_{E_1}(x^*)\rangle \\
&\le \beta_n \Delta_p(x^*, x_n) + (1-\beta_n)[(1-\alpha_n)\left(\Delta_p(x^*, z_n) - \Delta_p(z_n, Tz_n)\right)] \\
&\quad + \alpha_n(1-\beta_n)\langle y_n - x^*, J^p_{E_1}(x_n) - J^p_{E_1}(u)\rangle \\
&\quad + \alpha_n(1-\beta_n)\langle y_n - x^*, J^p_{E_1}(u) - J^p_{E_1}(x^*)\rangle.
\end{aligned}$$

From (3.4) we obtain

$$\begin{aligned}
\Delta_p(x^*, x_{n+1}) &\le \beta_n \Delta_p(x^*, x_n) + (1-\beta_n)(1-\alpha_n)\Delta_p(x^*, x_n) \\
&\quad - (1-\beta_n)(1-\alpha_n)(\rho_n p - \frac{C_q}{q}\rho_n^q)\frac{f^p(x_n)}{\|g(x_n)\|^p} \\
&\quad - (1-\beta_n)(1-\alpha_n)\Delta_p(z_n, Tz_n) + \alpha_n(1-\beta_n)\langle y_n - x^*, J^p_{E_1}(x_n) - J^p_{E_1}(u)\rangle \\
&\quad + \alpha_n(1-\beta_n)\langle y_n - x^*, J^p_{E_1}(u) - J^p_{E_1}(x^*)\rangle \\
&= \left(1 - (1-\beta_n)\alpha_n\right)\Delta_p(x^*, x_n) \\
&\quad - (1-\beta_n)(1-\alpha_n)(\rho_n p - \frac{C_q}{q}\rho_n^q)\frac{f^p(x_n)}{\|g(x_n)\|^p} \\
&\quad - (1-\beta_n)(1-\alpha_n)\Delta_p(z_n, Tz_n) + \alpha_n(1-\beta_n)\langle y_n - x^*, J^p_{E_1}(x_n) - J^p_{E_1}(u)\rangle \\
&\quad + \alpha_n(1-\beta_n)\langle y_n - x^*, J^p_{E_1}(u) - J^p_{E_1}(x^*)\rangle. \qquad (3.8)
\end{aligned}$$

We now divide the proof into following two cases:
Case 1: Suppose there is an $n_0 \in \mathbb{N}$ such that $\{\Delta_p(x^*, x_n)\}$ is nonincreasing. Then

$$\Delta_p(x^*, x_n) - \Delta_p(x^*, x_{n+1}) \to 0.$$

From (3.8), we obtain

$$\begin{aligned}
(1-\beta_n)(1-\alpha_n)[(\rho_n p - \frac{C_q}{q}\rho_n^q)\frac{f^p(x_n)}{\|g(x_n)\|^p} + \Delta_p(z_n, Tz_n)] \\
\le (\Delta_p(x^*, x_n) - \Delta_p(x^*, x_{n+1})) \\
+ \alpha_n(1-\beta_n)(\langle y_n - x^*, J^p_{E_1}(x_n) - J^p_{E_1}(u)\rangle \\
+ \langle y_n - x^*, J^p_{E_1}(x_n) - J^p_{E_1}(x^*)\rangle - \Delta_p(x^*, x_n)).
\end{aligned}$$

On taking $n \to \infty$, we have by assumption,

$$\|(I - M^{B_2}_{\lambda_2})Ax_n)\| = \|Ax_n - M^{B_2}_{\lambda_2}Ax_n\| \to 0 \text{ and } \Delta_p(z_n, Tz_n) \to 0.$$

This implies that by Proposition 2.2

$$\|z_n - Tz_n\| \to 0. \tag{3.9}$$

Since $J^p_{E_1}$ is norm to norm uniformly continuous on bounded subsets of E_1, we get $\|J^p_{E_1}(Tz_n) - J^p_{E_1}(z_n)\| \to 0$. By the boundedness of $\{x_n\}$ and the reflexivity of E_1, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_n\} \to \hat{x}$. From (3.9), we get $\hat{x} \in F(\hat{T}) = F(T)$.
Also

$$\begin{aligned}\|J^p_{E_1}(z_n) - J^p_{E_1}(x_n)\| &= \lambda_n \|A^* J^p_{E_2}(Ax_n - M^{B_2}_{\lambda_2} Ax_n)\| \\ &\le \lambda_n \|A^*\| \|J^p_{E_2}(Ax_n - M^{B_2}_{\lambda_2} Ax_n)\| \\ &= \lambda_n \|A\| \|Ax_n - M^{B_2}_{\lambda_2} Ax_n\|^{p-1} \\ &\to 0. \end{aligned} \tag{3.10}$$

Since $J^p_{E_1^*}$ is norm to norm uniformly continuous on bounded subsets of E_1^*, we obtain $\|z_n - x_n\| \to 0$. Moreover,

$$\begin{aligned}\|J^p_{E_1}(y_n) - J^p_{E_1}(x_n)\| &\le \alpha_n \|J^p_{E_1}(u_n) - J^p_{E_1}(x_n)\| + (1-\alpha_n)\|J^p_{E_1}(Tz_n) - J^p_{E_1}(x_n)\| \\ &\le \alpha_n \|J^p_{E_1}(u_n) - J^p_{E_1}(x_n)\| + (1-\alpha_n)\|J^p_{E_1}(Tz_n) - J^p_{E_1}(z_n)\| \\ &\quad + (1-\alpha_n)\|J^p_{E_1}(z_n) - J^p_{E_1}(x_n)\|. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} \|J^p_{E_1}(y_n) - J^p_{E_1}(x_n)\| = 0,$$

yields

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Thus, we have

$$\|J^p_{E_1}(x_{n+1}) - J^p_{E_1}(x_n)\| = (1-\beta_n)\|J^p_{E_1}(y_n) - J^p_{E_1}(x_n)\| \to 0. \tag{3.11}$$

Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to \hat{x} \in E$. From $\|Ax_n - M^{B_2}_{\lambda_2} Ax_n\| \to 0$ and by the boundedness and the linearity of A, we have $Ax_{n_i} \to Aw$ and $M^{B_2}_{\lambda_2} Ax_{n_i} \to A\hat{x}$. Since $M^{B_2}_{\lambda_2}$ is a resolvent metric of B_2 for $r_n > 0$, we have

$$\frac{J^p_{E_2}(Ax_n - M^{B_2}_{\lambda_2} Ax_n)}{r_n} \in B_2 M^{B_2}_{\lambda_2} Ax_n, \quad \forall n \in \mathbb{N}.$$

So we obtain

$$0 \le \langle v - M^{B_2}_{\lambda_2} Ax_{n_i}, v^* - \frac{J^p_{E_2}(Ax_{n_i} - M^{B_2}_{\lambda_2} Ax_{n_i})}{r_{n_i}} \rangle, \quad \forall (v, v^*) \in B_2.$$

It follows that

$$0 \le \langle v - A\hat{x}, v^* - 0\rangle, \quad \forall v, v^* \in E_2.$$

Since B_2 is maximal monotone, $A\hat{x} \in F(M^{B_2}_{\lambda_2}) = B_2^{-1}0$ and hence $\hat{x} \in A^{-1}(B_2^{-1}0)$.
Since, $v_n := J^q_{E_1^*}[J^p_{E_1}(x_n) - t_n A^* J^p_{E_2}(Ax_n M^{B_2}_{\lambda_2}(Ax_n))], \forall n \ge 1$. By Lemma 2.5 and (3.4), we have

$$\begin{aligned}\Delta_p(z_n, v_n) &= \Delta_p(N^{B_1}_{\lambda_1} v_n, v_n) \\ &\le \Delta_p(v_n, u^*) - \Delta_p(z_n, u^*) \\ &\le \Delta_p(x_n, u^*) - \Delta_p(z_n, u^*) \ \to \ 0 \text{ as } n \to \ \infty. \end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} \|N^{B_1}_{\lambda_1} v_n - v_n\| = \lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{3.12}$$

Since $x_{n_j} \to \hat{x} \in E_1$, we also have $v_{n_j} \to \hat{x} \in E_1$. From (3.12), we have $\hat{x} \in F(N^{B_1}_{\lambda_1}) \in B_1^{-1}0$. This concludes that $\hat{x} \in B_1^{-1}0 \cap A^{-1}(B_2^{-1}0)$.
Proposition 2.9 implies that

$$\begin{aligned}\limsup_{n\to\infty} \langle y_n - x^*, J^p_{E_1}(u) - J^p_{E_1}(x^*)\rangle &= \lim_{n\to\infty} \langle y_{n_i} - x^*, J^p_{E_1}(u) - J^p_{E_1}(x^*)\rangle \\ &= \langle w - x^*, J^p_{E_1}(u) - J^p_{E_1}(x^*)\rangle \le 0. \end{aligned} \tag{3.13}$$

We note that $x_n \to u$ implies $J^p_{E_1}(x_n) \to J^p_{E_1}(u)$ and consequently, $\lim_{n\to\infty}\langle y_n - x^*, J^p_{E_1}(x_n) - J^p_{E_1}(u)\rangle = 0$. Combining $\Sigma_{n=1}^{\infty}(1-\beta_n)\alpha_n = \infty$ and (3.13), we have by using Lemma 2.3, $\Delta_p(x^*, x_n) \to 0$. Thus, by Lemma 2.2, we have $\|x_n - x^*\| \to 0$ as $n \to \infty$.

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$, for all $n \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by $\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}$. Then, by Lemma 2.7, we obtain $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$. Put Γ_n for all $n \in \mathbb{N}$. So, by 3.11, we have $\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0$. As in the proof of Case 1, we also can show that

$$\lim_{n\to\infty}\|z_\tau(n) - Tz_\tau(n)\| \to 0,$$

$$\lim_{n\to\infty}\|(I - M^{B_2}_{\lambda_2})Ax_\tau(n))\| = \|Ax_\tau(n) - M^{B_2}_{\lambda_2}Ax_\tau(n)\| \to 0,$$

$$\lim_{n\to\infty}\|z_\tau(n) - x_\tau(n)\| \to 0,$$

and

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - x_\tau(n)\| \to 0.$$

Also,

$$\limsup_{n\to\infty}\langle y_{\tau(n)} - x^*, J^p_{E_1}(u) - J^p_{E_1}(x^*)\rangle \le 0.$$

Since $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$, by (3.8) we have

$$\begin{aligned}(1-\beta_{\tau(n)})\Delta_p(x^*, x_{\tau(n)}) \le{}& (1-\beta_{\tau(n)})\alpha_{\tau(n)}\left(\langle y_{\tau(n)} - x^*, J^p_{E_1}(u_{\tau(n)}) - J^p_{E_1}(u)\rangle\right)\\ &+ \langle y_{\tau(n)} - x^*, J^p_{E_1}(u) - J^p_{E_1}(x^*)\rangle,\end{aligned}$$

which yields

$$\Delta_p(x^*, x_{\tau(n_i)}) \le \langle y_{\tau(n)} - x^*, J^p_{E_1}(u_{\tau(n)}) - J^p_{E_1}(u)\rangle + \langle y_{\tau(n)} - x^*, J^p_{E_1}(u) - J^p_{E_1}(x^*)\rangle.$$

Thus we have

$$\limsup_{n\to\infty}\Delta_p(x^*, x_{\tau(n)}) \le 0.$$

So $\lim_{n\to\infty}\Delta_p(x^*, x_{\tau(n)}) = 0$. From (2.1) we have

$$\begin{aligned}&\Delta_p(x^*, x_{\tau(n)+1}) + \Delta_p(x_{\tau(n)+1}, x_{\tau(n)}) - \Delta_p(x^*, x_{\tau(n)})\\ &\quad= \langle x^* - x_{\tau(n)+1}, J^p_{E_1}(x_{\tau(n)}) - J^p_{E_1}(x_{\tau(n)+1})\rangle.\end{aligned}$$

Thus

$$\begin{aligned}\Delta_p(x^*, x_{\tau(n)+1}) &\le \Delta_p(x^*, x_{\tau(n)}) + \langle x^* - x_{\tau(n)+1}, J^p_{E_1}(x_{\tau(n)}) - J^p_{E_1}(x_{\tau(n)+1})\rangle\\ &\to 0,\end{aligned}$$

by Lemma 2.7, we have $\Delta_p(x^*, x_n) \le \Delta_p(x^*, x_{\tau(n)+1}) \to 0$. Hence $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

4 A countable family of relatively nonexpansive mappings

A family of mappings $\{T_n\}_{n=1}^{\infty}$ is said to be countable family of relatively nonexpansive mappings(see, for example [37]) if the following conditions are satisfied:

1. $F(\{T_n\}_{n=1}^{\infty}) \neq \emptyset$,
2. $\Delta_p(x^*, T_n x) \le \Delta_p(x^*, x)$, for all $x \in C, x^* \in F(T_n), n \ge 1$,
3. $\cap_{n=1}^{\infty} F(T_n) = \hat{F}(\{T_n\}_{n=1}^{\infty})$.

The set of asymptotic fixed points of $\{T_n\}_{n=1}^{\infty}$ is denoted by $\hat{F}(\{T_n\}_{n=1}^{\infty})$.

Definition 4.1. [6] Let C be a subset of a real p-uniformly convex and uniformly smooth Banach space E. Let $\{Tn\}_{n=1}^{\infty}$ be a sequence of mappings of C in to E such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{Tn\}_{n=1}^{\infty}$ is said to satisfy the $AKTT$-condition if, for any bounded subset B of C

$$\sum_{n=1}^{\infty}\sup_{x\in B}\{\|J^E_p(T_{n+1}x) - J^E_p(T_n x)\|\} < \infty.$$

As in [34], we prove the following Proposition:

Proposition 4.2. Let C be a nonempty, closed and convex subset of a real p-uniformly convex and uniformly smooth Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{T_n\}_{n=1}^{\infty}$ satisfies the $AKTT$-condition. Suppose that for any bounded subset B of C. Then there exists the mapping $T : B \to E$ such that

$$Tx = \lim_{n\to\infty} T_n x, \forall x \in B, \tag{4.1}$$

and

$$\lim_{n\to\infty} \sup_{x\in B} \|J_p^E(Tx) - J_p^E(T_n x)\| = 0.$$

Proof . To complete the proof we show that $\{T_n x\}$ is cauchy sequence for each $x \in C$. Let $\epsilon > 0$ be given and by the $AKKT$-condition $\exists\ l_0 \in \mathbb{N}$, such that

$$\sum_{l_0}^{\infty} \sup\{\|T_{n+1}y - T_n y\| : y \in C\} < \epsilon.$$

Let $k > l \geq l_0$, then

$$\begin{aligned}
\|T_k x - T_l x\| &\leq \sup\{\|T_k y - T_l y\| : y \in C\} \\
&\leq \sup\{\|T_k y - T_{k-1} y\| : y \in C\} + \sup\{\|T_{k-1} y - T_l y\| : y \in C\} \\
&\ \ \vdots \\
&\leq \sum_{l}^{k-1} \sup\{\|T_{n+1}y - T_n y\| : y \in C\} \\
&\leq \sum_{l_0}^{\infty} \sup\{\|T_{n+1}y - T_n y\| : y \in C\} < \epsilon. \qquad (4.2)
\end{aligned}$$

Therefore we have that $\{T_n x\}$ is Cauchy sequence, moreover (4.2) implies that,

$$\|Tx - T_l x\| = \lim_{k\to\infty} \|T_k x - T_l x\| \leq \sum_{l_0}^{\infty} \sup\{\|T_{n+1}y - T_n y\| : y \in C\},$$

for all $x \in C$. So,

$$\sup \|Tx - T_l x\| \leq \sum_{l_0}^{\infty} \sup\{\|T_{n+1}y - T_n y\| : y \in C\},$$

therefore we conclude that $\lim_{l_0\to\infty} \sup \|Tx - T_{l_0} x\| = 0$. □

In the sequel, we say that $(\{T_n\}, T)$ satisfies the $AKTT$-condition if $\{T_n\}_{n=1}^{\infty}$ satisfies the $AKTT$-condition and T is defined by (4.1) with $\bigcap_{n=1}^{\infty} F(T_n) = F(T)$.

Algorithm 4.1. Select $x_1 \in E_1$ and let sequence $\{x_n\}_{n=1}^{\infty}$ be generated by,

$$\begin{cases}
z_n = N_{\lambda_1}^{B_1}(J_{E_1^*}^q(J_{E_1}^p(x_n) - \rho_n \frac{f^{p-1}(x_n)}{\|g(x_n)\|^p} g(x_n)) \\
y_n = J_{E_1^*}^q(\alpha_n J_{E_1}^p(u_n) + (1-\alpha_n) J_{E_1}^p T_n(z_n)) \\
x_{n+1} = J_{E_1^*}^q(\beta_n J_{E_1}^p(x_n) + (1-\beta_n) J_{E_1}^p(y_n)),
\end{cases} \tag{4.3}$$

where $f(x_n) := \frac{1}{p}\|(I - M_{\lambda_2}^{B_2})Ax_n\|^p$, $f^{p-1}(x_n) := \left(\frac{1}{p}\|(I - M_{\lambda_2}^{B_2})Ax_n\|^p\right)^{p-1}$, $g(x_n) := A^* J_{E_2}^p (I - M_{\lambda_2}^{B_2})Ax_n$ and $\{\rho_n\} \in (0, \infty)$ satisfies $\liminf \rho_n(pq - C_q \rho_n^{q-1}) > 0$. If $g(x_n) = 0$, then $z_n = x_n$ and the iterative process stops, x_n is a solution. Otherwise, we set n := n + 1 and go to (4.3).

Theorem 4.3. Suppose that $\{T_n\}$ be a countable family Bregman relatively nonexpansive mapping on E_1 such that $F(T_n) = \hat{F}(T_n)$, Assume that $\Omega = \bigcap_{n=1}^{\infty} F(T_n) \cap \Gamma \neq \emptyset$ and satisfying following condition:

1. $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty$,
2. $\limsup_{n\to\infty}\beta_n < 1$,
3. u_n be a sequence in E such that $u_n \to u$,
4. $(\{T_n\}_{n=1}^{\infty}, T)$ satisfy AKTT-Condition.

Then the sequence x_n generated by 4.3 converges strongly to $x^* \in \Omega$, where $x^* = \Pi_\Omega u$

Proof . To this end, it suffices to show that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. By following the method of proof in Theorem 3.2, we can show that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - T_nx_n\| = 0$. Since $J_p^{E_1}$ is uniformly continuous on bounded subsets of E_1, we have

$$\lim_{n\to\infty}\|J_p^{E_1}(x_n) - J_p^{E_1}(T_nx_n)\| = 0.$$

By Proposition 4.2, we see that

$$\begin{aligned}\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\| &\leq \|J_p^{E_1}(x_n) - J_p^{E_1}(T_nx_n)\| + \|J_p^{E_1}(T_nx_n) - J_p^{E_1}(Tx_n)\| \\ &\leq \|J_p^{E_1}(x_n) - J_p^{E_1}(T_nx_n)\| + \sup_{x\in\{x_n\}}\|J_p^{E_1}(T_nx) - J_p^{E_1}(Tx)\| \\ &\to 0 \text{ as } n\to\infty.\end{aligned}$$

Since $J_p^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*,

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

This completes the proof. □

5 A semigroup of relatively nonexpansive mappings

Definition 5.1. Let C be a subset of a real p-uniformly convex and uniformly smooth Banach space E. A family of mappings S:= $\{T(t)\}_{t\geq 0}$ from C into C is said to be a nonexpansive semigroup, if it satisfies the following conditions:

(S_1) $T(0)x = x$, for all $x \in C$;
(S_2) $T(s+t) = T(s)T(t)$, for all $s, t \geq 0$;
(S_3) for each $x \in C$ the mapping $t \mapsto T(t)x$ is continuous;
(S_4) for each $t \geq 0$, $T(t)$ is nonexpansive, i.e.

$$\|T(t)x - T(t)y\| \leq \|x - y\|, \forall x, y \in C.$$

We denote by $F(S)$ the set of all common fixed points of S, i.e., $F(S) = \bigcap_{t\geq 0} F(T(t))$.
The following classical examples were one of the main sources for the development of semigroup theory (see Engel and Nagel [18]). The theory of semigroup is very important in theory of differential equations. Let $E = R^n$ and let $L(E)$ be the space of all bounded linear operators on E. Consider the the following initial value problem for a system of homogeneous linear first-order differential equations with constant coeffcients:

$$\begin{cases} {x_1}' = a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n,\ x_1(0) = u_1 \\ {x_2}' = a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n,\ x_2(0) = u_2 \\ \vdots \\ {x_n}' = a_{n1}x_1 + a_{n2}x_2 + \ldots a_{nn}x_n,\ x_n(0) = x_n, \end{cases} \tag{5.1}$$

which can be written in a matrix form as

$$\begin{cases} x'(t) = Ax(t),\ t \geq 0 \\ x(0) = u, \end{cases} \tag{5.2}$$

where $A \in L(E)$ is bounded linear operator and $A = (a_{ij})$ is an $n \times n$ matrix with $a_{ij} \in R$, for $i, j = 1, 2, \ldots n$ and $u = (u_1, u_2, \ldots x_n)^T \in R^n$ is a given initial vector with $u_i \in R$, for all i = 1, 2, ...n. It is well-known that the problem

(5.2) has a unique solution given by explicit formula $x(t) = e^{tA}u,\ t \geq 0$, where e^{tA} is a matrix exponential of the linear differential system (5.2) defined by

$$e^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = I + \frac{tA}{1} + \frac{t^2 A^2}{2!} + \dots.$$

We can check that the operator $\{T(t) : e^{tA},\ t \geq 0\}$ is a semigroup on E. Then, we can write the solution of the problem (5.2) as $x(t) = T(t)u, t \geq 0$.

Example 5.2. Let $E = L^p(R_n)$, $1 \leq p < \infty$. Consider the initial value problem for the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= Du, \qquad x \in \mathbb{R}^n \text{ and } t > 0, \\ u(x,0) &= f(x), \qquad x \in \mathbb{R}^n, \end{aligned} \tag{5.3}$$

where $D = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator on E. We can solve the heat equation using Fourier transform and the solution (5.3) can be written as follows:

$$u(x,t) = \frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} e^{\frac{-\|s-\xi\|^2}{4t}} f(\xi) d\xi,$$

where $t > 0$, $s \in \mathbb{R}^n$ and $f \in E$. Then, we can write the solution $u(x,t)$ in the form of convolution integral as follows:

$$u(x,t) = (K_t * f)(x),$$

where K_t is heat kernel given by $K_t = \frac{1}{\sqrt{(4\pi t)^n}} e^{\frac{-\|x\|^2}{4t}}$. Then the solution of (5.3) can be written as follows:

$$T_t f(x) = u(x,t) = (K_t * f)(x),$$

we can check that the operator $T_t f(x)$ is a semigroup on E.

Definition 5.3. A one-parameter family $S = \{T(t)\}_{t\geq 0} : E \to E$ is said to be a family of uniformly Lipschitzian mappings if there exists a bounded measurable function $L(t) : (0,\infty) \to [0,\infty)$ such that

$$\|T(t)x - T(t)y\| \leq L(t)\|x - y\|, \qquad x, y \in E.$$

We now first give the following definition:

Definition 5.4. A one-parameter family $S = \{T(t)\}_{t\geq 0} : E \to E$ is said to be a Bregman relatively nonexpansive semigroup if it satisfies $(S_1), (S_2), (S_3)$ and the following conditions:

(a) $F(S) = \hat{F}(S) \neq \emptyset$,
(b) $\Delta_p(T(t)x, z) \leq \Delta_p(x, z), \quad \forall x \in E, z \in F(S)$ and $t \geq 0$.

Using idea in Aleyner and Censor [3], Aleyner and Reich [4] and Benavides *et al.* [8], we define the following concept:

Definition 5.5. A continuous operator semigroup $S = \{T(t)\}_{t\geq 0} : E \to E$ is said to be uniformly asymptotically regular (in short, u.a.r.) if for all $S \leq 0$ and any bounded subset B of E such that

$$\lim_{t\to\infty} \sup_{x\in B} \|J_p^E(T(t)x) - J_p^E(T(s)T(s)x)\| = 0.$$

Algorithm 5.1. Select $x_1 \in E_1$ and let sequence $\{x_n\}_{n=1}^{\infty}$ be generated by,

$$\begin{cases} z_n = N_{\lambda_1}^{B_1}(J_{E_1^*}^q(J_{E_1}^p(x_n) - \rho_n \frac{f^{p-1}(x_n)}{\|g(x_n)\|^p} g(x_n)) \\ y_n = J_{E_1^*}^q(\alpha_n J_{E_1}^p(u_n) + (1-\alpha_n) J_{E_1}^p T(t_n) z_n) \\ x_{n+1} = J_{E_1^*}^q(\beta_n J_{E_1}^p(x_n) + (1-\beta_n) J_{E_1}^p(y_n)), \end{cases} \tag{5.4}$$

where $f(x_n) := \frac{1}{p}\|(I - M_{\lambda_2}^{B_2})Ax_n\|^p$, $f^{p-1}(x_n) := \left(\frac{1}{p}\|(I - M_{\lambda_2}^{B_2})Ax_n\|^p\right)^{p-1}$, $g(x_n) := A^* J_{E_2}^p (I - M_{\lambda_2}^{B_2})Ax_n$ and $\{\rho_n\} \in (0,\infty)$ satisfies $\liminf \rho_n(pq - C_q \rho_n^{q-1}) > 0$. If $g(x_n) = 0$, then $z_n = x_n$ and the iterative process stops, x_n is a solution. Otherwise, we set n := n + 1 and go to (5.4).

Theorem 5.6. Let $S = \{T(t)\}_{t\geq 0}$ be a u.a.r. Bregman relatively nonexpansive semigroup of uniformly Lipschitzian mappings on E_1 into E_1 with a bounded measurable function $L_t : (0,\infty) \to [0,\infty)$ such that $F(S) := \cap_{h\geq 0} F(T_h) \neq \emptyset$ and Let $\Gamma \cap F(S) \neq \emptyset$. Suppose that the following condition hold:

1. $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
2. u_n be a sequence in E such that $u_n \to u$,
3. $\{t_n\} \in (0,\infty)\ with\ \lim_{n\to\infty} t_n = 0$,
4. $\limsup_{n\to\infty} \beta_n < 1$.

Then the sequence generated by x_n converges strongly to $x^* \in \Gamma \cap F(S)$, where $x^* = \Pi_{\Gamma\cap F(S)} u$.

Proof . We only have to show that $\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0$ for all $t \geq 0$. By following the method of proof in Theorem 3.2, we can show that $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - T(t_n)x_n\| = 0. \tag{5.5}$$

Since $\{T(t)\}_{t\geq 0}$ is a uniformly of Lipschitzian mappings with a bounded measurable function L_t. Then, we have

$$\begin{aligned} \|T(t)T(t_n)x_n - T(t)x_n\| &\leq L_t \|T(t_n)x_n - x_n\| \\ &\leq \sup_{t\geq 0}\{L_t\} \|T(t_n)x_n - x_n\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $J_p^{E_1}$ is uniformly norm-to-norm continuous on bounded subsets of E_1, then we also have

$$\lim_{n\to\infty} \|J_p^{E_1}(T(t)T_(t_n)x_n) - J_p^{E_1}(T(t)x_n)\| = 0. \tag{5.6}$$

For each $t \geq 0$, we note that

$$\begin{aligned} \|J_p^{E_1}(x_n) - J_p^{E_1}(T(t)x_n)\| &\leq \|J_p^{E_1}(x_n) - J_p^{E_1}(T(t_n)x_n)\| + \|J_p^{E_1}(T(t_n)x_n) - J_p^{E_1}(T(t)T(t_n)x_n)\| \\ &\quad \|J_p^{E_1}(T(t)T(t_n)x_n) - J_p^{E_1}(T(t)x_n)\| \\ &\leq \|J_p^{E_1}(x_n) - J_p^{E_1}(T(t_n)x_n)\| + \|J_p^{E_1}(T(t)T(t_n)x_n) - J_p^{E_1}(T(t)x_n)\| \\ &\quad \sup_{x\in\{x_n\}} \|J_p^{E_1}(T(t_n)x) - J_p^{E_1}(T(t)T(t_n)x)\|. \end{aligned}$$

Since $\{T(t)\}_{t\geq 0}$ is a u.a.r. Bregman relatively nonexpansive semigroup with $\lim_{n\to\infty} t_n = \infty$, then from (5.5) and (5.6), we get

$$\lim_{n\to\infty} \|J_p^{E_1}(x_n) - J_p^{E_1}(T(t)x_n)\| = 0,$$

for all $t \geq 0$. Since $J_q^{E_1^*}$ is uniformly norm-to-norm continuous on bounded subsets of E_1^*, we get

$$\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0.$$

This completes the proof. □

6 Numerical Example

We now give a numerical example of the Algorithm 3.1.

Example 6.1. Let $E_1 = E_2 = l_2(\mathbb{R})$, where $l_2(\mathbb{R}) := \{r = (r_1, r_2, \ldots, r_i, \ldots), r_i \in \mathbb{R} : \sum_{i=1}^{\infty} |r_i|^2 < \infty\}$, $\|r\|_2 = (\sum_{i=1}^{\infty} |r_i|^2)^{\frac{1}{2}}$, $\forall r \in E_1$ and $\langle x, y\rangle := \sum_{i=1}^{\infty} x_i y_i$. We define $B_1 : E_1 \to E_1$ and $B_2 : E_2 \to E_2$ be maximal monotone operators such that $B_1 x = 3x$ and $B_2 x = 5x$, respectively. Let $T : E_1 \to E_1$ be defined by $Tx = \frac{x}{2}$, $\forall x \in E_1$ and $A : E_1 \to E_2$ is a bounded linear operator defined by $Ax = \frac{2x}{3}, \forall x \in E_1$. We choose $\alpha_n = \frac{1}{2n}$, $\beta_n = \frac{2n-1}{3n}$ and $u_n = \frac{1}{6n}$. Furthermore, it can be verified that for $\lambda_1, \lambda_2 \geq 0$,

$$N_{\lambda_1}^{B_1} x = (I + \lambda_1 B_1)^{-1} x = \frac{x}{1 + 3\lambda_1}, \ \ \forall x \in E_1,$$

and

$$M_{\lambda_2}^{B_2} y = (I + \lambda_2 B_2)^{-1} y = \frac{y}{1 + 5\lambda_2}, \ \ \forall y \in E_2.$$

Using MATLAB R2016(a), we now study the convergence behavior of Algorithm 3.1 at different initial values x_1 and different $\{\rho_n\}$. We plot the graphs of errors $= \|x_{n+1} - x_n\|$ against number of iterations with the following choices:

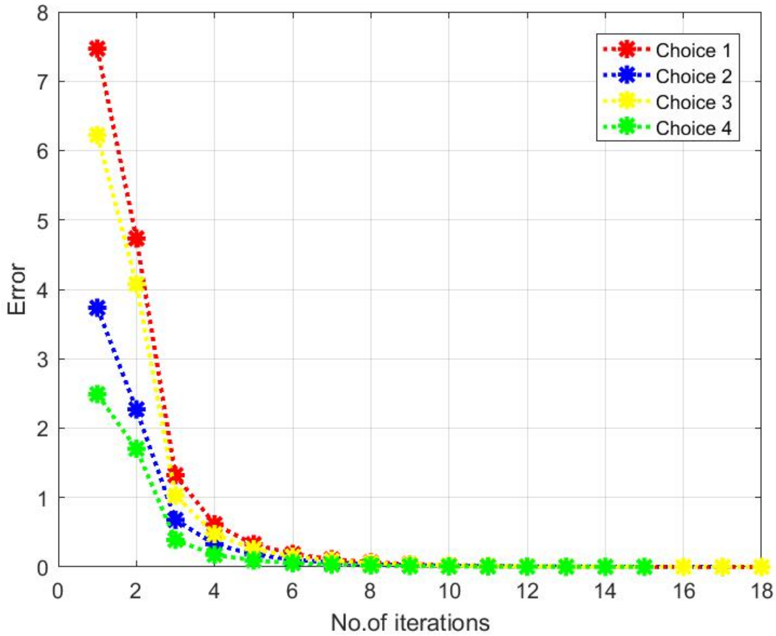

Figure 1: Convergence of Algorithm 3.1 for different x_1 and $\{\rho_n\}$

.

1. $x_1 = (6, \frac{6}{2}, \frac{6}{3}, \dots)$ and $\rho_n = \frac{3n}{n+1}$,
2. $x_1 = (3, \frac{3}{2}, 1, \dots)$ and $\rho_n = \frac{2n}{n+1}$,
3. $x_1 = (-5, \frac{-5}{2}, \frac{-5}{3}, \dots)$ and $\rho_n = \frac{n}{n+1}$,
4. $x_1 = (-2, -1, \frac{-2}{3}, \dots)$ and $\rho_n = \frac{0.5n}{n+1}$.

We observed that different choices of x_1 have no large effect in terms of number of iterations for the convergence of our Algorithm 3.1, also we see that sequences generated by our Algorithm 3.1 converges to $0 \in \Gamma \cap F(T)$. Moreover, the number of iterations significantly decreasing from choice 1 to choice 4. The error plotting for each choices is shown in Figure 1.

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