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Some inequalities for linear canonical curvelet transform

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Abstract

In this paper, we derive some inequalities for linear canonical curvelet transform (LCCT). At the outset, the basic properties of LCCT including the admissibility condition, and Moyal's principle are stated. Thereafter, some notable inequalities and results related to the well-known Heisenberg- type inequalities are derived for linear canonical curvelet transform.

Keywords: Curvelet transform, Linear canonical transform, Convolution, Heisenberg- type inequalities
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1 Introduction

It was in the early 1970s, a promising linear integral transform namely linear canonical transform was independently introduced by Collins [9] in paraxial optics, and Moshinsky and Quesne [21] in quantum mechanics to study the conservation of information and uncertainty under linear maps of phase space. It is taken as the generalization of the classical Fourier transform (FT), Fresnel Transform (FRT) and the fractional Fourier transform (FrFT) [24, 22]. LCT has been extensively used in many fields like signal processing and optics and serves as a magnanimous analysing tool [3, 25, 21]. During the last two decades or so, the areal application for LCT has stimulated a vigorous pace and has been applied in many fields such as time frequency analysis, filter design phase reconstruction, pattern recognition, radar analysis including many more. For more about LCT and their applications, we refer to [13, 30]. The LCT is a three free parameter class of linear integral transforms and includes many well-known single transformations besides signal processing and optics-related mathematical operations as for example the Fourier transform [5], the fractional Fourier transform [1], the Fresnal transform [16], and the scaling operations. Recently, LCT has become a focus of contemporary research in signal processing and considerable attention has also been paid for understanding the mathematical underpinnings of the LCT theory and many relevant theorems, such as sampling theorems [14], convolution theorems [28], uncertainty principles and others have been well established. This transform can also be used in scientific computing and filter design.

In the realm of higher dimensional signal processing the quality of wavelet transform tends to decrease because of the fact that the wavelet transform uses isotropic scaling in dimension $n \geq 2$. These isotropic scalings are rather weak and incompetent to capture the edges and corners in higher dimensional signals appearing due to its spatial occlusion between different objects, as for example, in medical imaging curves separate bones and various other soft

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tissues. Therefore, the key problem in multidimensional signal analysis is to extract and characterize the relevant and directional information regarding the occurrence of boundaries and curves in signals. To address these limitations of wavelet transform, some off-shoots of the wavelet transform, like ridgelet transform [8], curvelet transform [6, 7], stockwell transform [26], contourlet transform [11] and shearlet transform [19], have been introduced.

Curvelet transform is a new two dimensional multi scale integral transform recently introduced by Candes and Donoho [6, 7] to overcome the limitations of ridgelet transform in observing the global straight line singularities in real time applications. This transform is a higher dimensional generalization of the wavelet transform designed to represent images at different scales and different angles. Beside, it overcomes the difficulties in directionality and is widely applied in image processing such as image denoising, imaging in astrophysics, morphological component analysis and seismic imaging. Multiresolution methods are deeply related to image processing, biological and computer vision, scientific computing among others. The curvelet transform is a multiscale directional transform, which allows almost an optimal non-adaptive sparse representation of objects with edges. Curvelets are designed to handle curves using only a small number of coefficients, hence the curvelet handles curve discontinuities well. Curvelets occur at all orientations, scales and locations. The geometric features of curvelets render it is superior to wavelets for this application. Curvelets obey the parabolic scale relation, which helps to resolve structures such as edges in the images. Importantly, it provides for sparsity by reducing redundant information across scale. In addition, the length and width of the ridge obey the anisotropy scaling relation. In this paper, we derive some Inequalities for linear canonical curvelet transform.

This article is organised as follows: In section 2 we discuss preliminaries and some properties of linear canonical curvelet transform (LCCT) including admissibility condition, Moyals principle and also prove some inequalities for linear canonical curvelet transform. In section 3, we prove some results related to Heisenberg- type inequalities associated with LCCT. Section 4 concludes this paper and points out some future research work.

2 Preliminaries

The present section gives the basic background of the linear canonical transform and curvelet transform. For notational convenience, we shall write a 2×2 matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as $M = (A, B, C, D)$ The formal definition of two dimensional LCT is given below [31].

Definition 2.1. For any $f \in L^2(\mathbb{R}^2)$, the two dimensional linear canonical transform with respect to real, unimodular matrix $M = (A, B, C, D)$ is denoted by $\mathcal{L}_M[f]$ and is defined as

$$\mathcal{L}_M[f](\boldsymbol{\omega}) = \begin{cases} \displaystyle\int_{\mathbb{R}^2} f(\mathbf{x})\mathcal{K}_M(\mathbf{x}, \boldsymbol{\omega})d\mathbf{x}, B \neq 0 \\ \\ \sqrt{D} \exp\left\{ \frac{iCD|\boldsymbol{\omega}|^2}{2} \right\} f(D\boldsymbol{\omega}), B = 0 \end{cases}$$

where $\mathbf{x} = (x_1, x_2)^t, \boldsymbol{\omega} = (\omega_1, \omega_2)^t$ and $\mathcal{K}_M(\mathbf{x}, \boldsymbol{\omega})$ denotes the kernel of two dimensional LCT and is given by

$$\begin{aligned} \mathcal{K}_M(\mathbf{x}, \boldsymbol{\omega}) &= \frac{1}{2\pi B} \exp\left\{ \frac{i(A|\mathbf{x}|^2 - 2\mathbf{x}^t\boldsymbol{\omega} + D|\boldsymbol{\omega}|^2)}{2B} \right\}, B \neq 0 \\ &= \frac{1}{2\pi B} \exp\left\{ \frac{i}{2B}[A(x_1^2 + x_2^2) - 2(x_1\omega_1 + x_2\omega_2) + D(\omega_1^2 + \omega_2^2)] \right\}, B \neq 0 \\ &= \frac{1}{2\pi B} \exp\left\{ \frac{i}{2B}[A\mathbf{x}^t\mathbf{x} - 2\mathbf{x}^t\boldsymbol{\omega} + D\boldsymbol{\omega}^t\boldsymbol{\omega}] \right\}, B \neq 0. \end{aligned}$$

We note that for the case $B = 0$, the two dimensional LCT becomes chirp multiplication. Moreover, the case $B < 0$ is also of no particular interest to us. Throughout this article, we only consider the case $B > 0$.

Also the inverse LCT is defined by

$$f(\mathbf{x}) = \mathcal{L}_M^{-1}\big(\mathcal{L}_M(f)\big)(\mathbf{x}) = \int_{\mathbb{R}^2} \mathcal{L}_M(f)(\boldsymbol{\omega})\,\overline{\mathcal{K}_M(\mathbf{x}, \boldsymbol{\omega})}d\boldsymbol{\omega}.$$

Remark 2.2. The Linear canonical transform contains many well-known transforms as special cases, some of which are listed below:

(i) When $M = (0, 1, -1, 0)$, the LCT definition (2.1) reduces to the counterpart of Fourier transform.

(ii) When $M = (cos\alpha, sin\alpha, -sin\alpha, cos\alpha), \alpha \neq n\pi,\ n \in \mathbb{Z}$, The LCT definition reduces to the counterpart of the fractional Fourier transform.

(iii) Putting the matrix $M = (1, B, 0, 1), B \neq 0$ the LCT reduces to the analogue of Fresnel transform.

Definition 2.3. For any pair of functions $f, g \in L^2(\mathbb{R}^2)$, the convolution associated with the above definition is defined as

$$(f \star_M g)(\mathbf{b}) = \int\limits_{\mathbb{R}^2} f(\mathbf{x})g(\mathbf{b}-\mathbf{x}) \exp\left\{\frac{-iA}{B}\mathbf{x}^t(\mathbf{b}-\mathbf{x})\right\} d\mathbf{x}.$$

From [20, 6, 7] the definition of curvelet and curvelet transform are presented as.

Definition 2.4. For $a \in (0, a_0), \mathbf{b} \in \mathbb{R}^2, \theta \in [-\pi, \pi]$ the curvelet $\Gamma_{a,\mathbf{b},\theta}$ generated via scale a, translation b and rotation R_θ, is defined by $\Gamma_{a,b,\theta}(\mathbf{x}) = \Gamma_{a,0,0}(R_\theta(\mathbf{x}-\mathbf{b})), \mathbf{x} \in \mathbb{R}^2$ where, $R_\theta = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & \cos\theta \end{pmatrix}$ is the rotation matrix.

Definition 2.5. For $f \in L^2(\mathbb{R}^2)$, the curvelet transform with respect to curvelet $\Gamma_{a,\mathbf{b},\theta}$ is defined as the integral transform

$$\begin{aligned}(\Gamma f)(a, \mathbf{b}, \theta) &= \langle f, \Gamma_{a,\mathbf{b},\theta}\rangle_2 \\ &= \int\limits_{\mathbb{R}^2} f(\mathbf{x})\overline{\Gamma_{a,0,0}(R_\theta(\mathbf{x}-\mathbf{b}))}d\mathbf{x},\end{aligned} \tag{2.1}$$

where $a \in (0, a_0), a_0 < \pi^2, b \in \mathbb{R}^2,\ \theta \in [-\pi, \pi]$

Remark 2.6. The curvelet transform is a function $\Gamma : \mathbb{R}^2 \to (0, a_o) \times \mathbb{R}^2 \times [-\pi, \pi]$. We can express (2.1) in terms of convolution.

$$\begin{aligned}(\Gamma f)(a, \mathbf{b}, \theta) &= \int\limits_{\mathbb{R}^2} f(\mathbf{x})\overline{\Gamma_{a,0,0}(R_\theta(\mathbf{x}-\mathbf{b})}d\mathbf{x} \\ &= \int\limits_{\mathbb{R}^2} f(\mathbf{x})\overline{\Gamma_{a,0,\theta}(\mathbf{x}-\mathbf{b})}d\mathbf{x}\end{aligned}$$

where,

$$\begin{aligned}\Gamma_{(a,0,\theta)}(\mathbf{x}) &= \Gamma_{a,0,0}(R_\theta \mathbf{x}) \\ &= \int\limits_{\mathbb{R}^2} f(\mathbf{x})\overline{\Gamma_{a,0,\theta}(-(\mathbf{b}-\mathbf{x}))}d\mathbf{x} \\ &= \int\limits_{\mathbb{R}^2} f(\mathbf{x})\check{\bar{\Gamma}}_{a,0,\theta}(\mathbf{b}-\mathbf{x})d\mathbf{x}, \text{where } \check{\bar{\Gamma}}(\mathbf{x}) = \Gamma(-\mathbf{x}) \\ &= \left(f(\mathbf{x}) \star \check{\bar{\Gamma}}_{a,0,\theta}(\mathbf{x})\right)(\mathbf{b}).\end{aligned}$$

Now, we recall the definitioin of Linear canonical curvelet transform as [18].

Definition 2.7. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{2\times 2}$ be a matrix with parameters satisfying $det(M) = AD - BC = 1$. Then linear canonical curvelet transform of a signal $f \in L^2(\mathbb{R}^2)$, is defined as the integral transform

$$(\Gamma^M f)(a, \mathbf{b}, \theta) = \int\limits_{\mathbb{R}^2} f(\mathbf{x})\overline{\Gamma^M_{a,\mathbf{b},\theta}(\mathbf{x})}d\mathbf{x}, \tag{2.2}$$

where,

$$\Gamma^{M}_{a,\mathbf{b},\theta}(\mathbf{x}) = \Gamma_{a,0,0}(R_\theta(\mathbf{x}-\mathbf{b}))\exp\left\{\frac{iA}{B}\mathbf{x}^t(\mathbf{b}-\mathbf{x})\right\}.$$

Now, we shall write the relationship between linear canonical transform and curvelet transform.

Proposition 2.8. For $f \in L^2(\mathbb{R}^2)$ and $\Gamma^M f(a,\mathbf{b},\theta)$ be the LCCT of a function $f \in L^2(\mathbb{R}^2)$, then we have
$\mathcal{L}_M\left(\Gamma^M f(a,\mathbf{b},\theta)\right)(\boldsymbol{\omega})$

$$= 2\pi B\left(\exp\left\{\tfrac{iD\boldsymbol{\omega}^t\boldsymbol{\omega}}{2B}\right\}\mathcal{L}_M[f](\boldsymbol{\omega})\overline{\mathcal{L}_M\left(\Gamma_{a,0,0}(\mathbf{z})\exp\left\{\tfrac{-iA\mathbf{z}^t\mathbf{z}}{B}\right\}\right)(R_\theta\boldsymbol{\omega})}\right).$$

Proof. For proof see [18]

Now we state some fundamental properties of the linear canonical curvelet transform. In this direction we have the following theorem which assembles some of the basic properties of LCCT.

Theorem 2.9. For any $f, g \in L^2(\mathbb{R}^2)$ and $\alpha, \beta, \in \mathbb{C}$, $\mathbf{k} \in \mathbb{R}^2, \lambda \in \mathbb{R}^+$, then LCCT satisfies the following properties.

(i) Linearity: $\Gamma^M(\alpha f + \beta g)(a,\mathbf{b},\theta) = \alpha[\Gamma^M f](a,\mathbf{b},\theta) + \beta[\Gamma^M g](a,\mathbf{b},\theta)$.

(ii) Translation: $\Gamma^M[f(\mathbf{x}-\mathbf{k})](a,\mathbf{b},\theta) = \exp\left\{\frac{-iA\mathbf{k}(\mathbf{b}-\mathbf{k})}{B}\right\}\Gamma^M[\exp\left\{\frac{iA\mathbf{z}\mathbf{k}}{B}\right\}f(\mathbf{z})]$
$(a,\mathbf{b}-\mathbf{k},\theta)$.

(iii) Scaling : $\Gamma^M[f(\lambda\mathbf{x})](a,\mathbf{b},\theta) = [\Gamma^M f](\frac{a}{\lambda},\mathbf{b}\lambda,\theta)$ Provided $M = (A, \lambda^2 B, C, D)$ is the unimodular matrix.

(iv) Parity: $\Gamma^M[f(-\mathbf{x})](a,\mathbf{b},\theta) = \Gamma^M[f(\mathbf{x})](-a,-\mathbf{b},\theta)$.

Proof. The proof of above theorem is quite simple, hence is omitted here.

In the remaining part of the section, we state some important theorems including Moyal's principles pertaining to the linear canonical curvelet transform.

Definition 2.10. (Admissibility condition). A given function $\Gamma \in L^2(\mathbb{R}^2)$ is said to be admissible if and only if it satisfies the following condition

$$C_\Gamma = \int_0^{a_0}\int_{-\pi}^{\pi}\left|\mathcal{L}_M\left(\Gamma_{a,0,0}(\mathbf{z})\exp\left\{\frac{-iA\mathbf{z}^t\mathbf{z}}{B}\right\}\right)(R_\theta\boldsymbol{\omega})\right|^2 d\theta da < \infty \quad a.e$$

Now we are in a position to state the orthogonality relation for the linear canonical curvelet transform.

Theorem 2.11. (Moyal's principle). If $[\Gamma^M f](a,\mathbf{b},\theta)$ is the LCCT of f, then for $f, g \in L^2(\mathbb{R}^2)$, we have

$$\int_0^{a_0}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}[\Gamma^M f](a,\mathbf{b},\theta)\overline{[\Gamma^M g](a,\mathbf{b},\theta)}d\theta d\mathbf{b}da \quad = \quad 4\pi^2 B^2\, C_\Gamma\langle f, g\rangle, \tag{2.3}$$

where C_Γ is the admissibility condition.
Proof. The proof of above theorem is quite simple, hence is omitted here.

Remark 2.12. For $f = g$, the orthogonality relation (2.3) gives

$$||\Gamma^M f(a,\mathbf{b},\theta)||^2 = 4\pi^2 B^2\, C_\Gamma||f||_2^2 \tag{2.4}$$

In the following theorem, we show that the linear canonical curvelet transform satisfies some notable inequalities.

Theorem 2.13. Let $f \in L^2(\mathbb{R}^2)$, then the linear canonical curvelet transform satisfy the following inequality:

$$\left\|\Gamma^M[f](a,\mathbf{b},\theta)\right\|_p \leq \|\Gamma_{a,0,0}\|_p \|f\|_1$$

Proof . By integral form of Minkowski inequality, we have

$$\begin{aligned}
\left\|\Gamma^M[f](a,\mathbf{b},\theta)\right\|_p &= \left\{\int\limits_{\mathbb{R}^2}\left|\int\limits_{\mathbb{R}^2} f(\mathbf{x})\overline{\Gamma_{a,0,0}(R_\theta(\mathbf{x}-\mathbf{b}))}\exp\left\{\frac{-iA}{B}\mathbf{x}^t(\mathbf{b}-\mathbf{x})\right\}d\mathbf{x}\right|^P d\mathbf{b}\right\}^{\frac{1}{p}} \\
&\leq \int\limits_{\mathbb{R}^2}\left\{\int\limits_{\mathbb{R}^2}\left|f(\mathbf{x})\overline{\Gamma_{a,0,0}(R_\theta(\mathbf{x}-\mathbf{b}))}\exp\left\{\frac{-iA}{B}\mathbf{x}^t(\mathbf{b}-\mathbf{x})\right\}\right|^P d\mathbf{b}\right\}^{\frac{1}{p}} d\mathbf{x} \\
&= \int\limits_{\mathbb{R}^2}\left\{\int\limits_{\mathbb{R}^2}\left|f(\mathbf{x})\overline{\Gamma_{a,0,0}(R_\theta(\mathbf{x}-\mathbf{b}))}\right|^P d\mathbf{b}\right\}^{\frac{1}{p}} d\mathbf{x} \\
&= \int\limits_{\mathbb{R}^2}\left\{\int\limits_{\mathbb{R}^2}\left|\overline{\Gamma_{a,0,0}(\mathbf{z})}\right|^P d\mathbf{z}\right\}^{\frac{1}{p}} |f(\mathbf{x})|\, d\mathbf{x}
\end{aligned}$$

By Fubini theorem, the above inequality can be written as

$$\begin{aligned}
\left\|\Gamma^M[f](a,\mathbf{b},\theta)\right\|_p &= \left\{\int\limits_{\mathbb{R}^2}\left|\overline{\Gamma_{a,0,0}(\mathbf{z})}\right|^P d\mathbf{z}\right\}^{\frac{1}{p}} \int\limits_{\mathbb{R}^2}|f(\mathbf{x})|\, d\mathbf{x} \\
&= \|\Gamma_{a,0,0}(\mathbf{z})\|_p \|f\|_1
\end{aligned}$$

This completes the proof of the theorem. □

Theorem 2.14. The linear canonical curvelet transform of any $f \in L^2(\mathbb{R}^2)$ with respect to analysing function $\Gamma \in L^2(\mathbb{R}^2)$ satisfies:

$$\int\limits_0^{a_o}\int\limits_{-\pi}^{\pi}\left[N^M(a)\right]^2 d\theta da = 4\pi^2 B^2 C_\Gamma \|f\|_2^2$$

where

$$N^M(a) = \left\{\int\limits_{\mathbb{R}^2}|\Gamma_M[f](a,\mathbf{b},\theta)|^2 d\mathbf{b}\right\}^{\frac{1}{2}}$$

Proof . Let $f \in L^2(\mathbb{R}^2)$, we have

$$\left[N^M(a)\right]^2 = \int\limits_{\mathbb{R}^2}|\Gamma_M[f](a,\mathbf{b},\theta)|^2 d\mathbf{b}$$

integrating with respect to the measure $dad\theta$, we have

$$\int\limits_0^{a_o}\int\limits_{-\pi}^{\pi}\left[N^M(a)\right]^2 d\theta da = \int\limits_0^{a_o}\int\limits_{\mathbb{R}^2}\int\limits_{-\pi}^{\pi}\left|\Gamma^M[f](a,\mathbf{b},\theta)\right|^2 d\theta d\mathbf{b} da$$

By using orthogonality relation (2.3), we obtain

$$\int_{0}^{a_o}\int_{-\pi}^{\pi}\left[N^M(a)\right]^2 d\theta da = 4\pi^2 B^2\, C_\Gamma\, \|f\|_2^2$$

This completes the proof of the theorem. □

3 Heisenberg-type inequalities for linear canonical curvelet transform

In this section, we shall establish Heisenberg-type inequality for linear canonical curvelet transform. The classical uncertainty principle states that a non-zero function and its Fourier transform cannot be both sharply localized. It has been a fundamental principle in mathematics and physics and plays an important role in signal processing.

The linear canonical transform is a generalization of the ordinary Fourier transform and the fractional Fourier transform. It has been recognized that this transform is an effective tool for chirp signal analysis. Various applications have been found in signal analysis and optics [29]. Different kinds of uncertainty inequalities associated with the linear canonical transform have been studied by many researchers in the last two decades [4, 29, 27, 12, 33]. Especially, the Donoho-Stark uncertainty principle for the linear canonical transform was studied in [32]. Recently, the linear canonical transform is further generalized to the quaternion domain, which has found several interesting applications in colour image processing.

Theorem 3.1. If $[\Gamma^M f](a,\mathbf{b},\theta)$ is the linear canonical curvelet transform of any function $f \in L^2(\mathbb{R}^2)$, then the following uncertainty inequality holds:

$$\left\{\int_{\mathbb{R}^2}\int_{0}^{a_o}\int_{-\pi}^{\pi}|\mathbf{b}|^2|[\Gamma^M f](a,\mathbf{b},\theta)|^2 d\theta da d\mathbf{b}\right\}^{\frac{1}{2}}\left\{\int_{\mathbb{R}^2}|\boldsymbol{\omega}|^2|\mathcal{L}_M[f](\boldsymbol{\omega})|^2 d\boldsymbol{\omega}\right\}^{\frac{1}{2}}$$

$$\geq \pi B^2\sqrt{C_\Gamma}||f||^2$$

Following the idea of M. G. Cowling and J. F. Price [10], we shall derive the generalization of theorem (3.1) for the space $L^p(\mathbb{R}^2), 1 \leq p \leq 2$ and $p \geq 2$ in the following theorems.

Theorem 3.2. Let $\Gamma \in L^2(\mathbb{R}^2)$ be the analysing function which satisfies the admissibility condition. Then for arbitrary $f \in L^2(\mathbb{R}^2)$, we have

$$\left\{\int_{0}^{a_o}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}|\mathbf{b}\Gamma^M f(a,\mathbf{b},\theta)|^p d\theta d\mathbf{b} da\right\}^{\frac{1}{p}}\left\{\int_{\mathbb{R}^2}|\boldsymbol{\omega}\mathcal{L}_M[f](\boldsymbol{\omega})|^p d\boldsymbol{\omega}\right\}^{\frac{1}{p}}$$
$$\geq \pi B^2\sqrt{C_\Gamma}||f||_2^2,\ 1 \leq p \leq 2.$$

For proof of these theorems see [18].
Next, we derive the generalized inequality for the linear canonical curvelet transform (LCCT) for the case $p \geq 2$.

Theorem 3.3. Let $\Gamma \in L^2(\mathbb{R}^2)$ be the analysing function which satisfies the admissibility condition. Then for arbitrary $f \in L^2(\mathbb{R}^2)$, We have

$$\left\{\int_{0}^{a_o}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}|\mathbf{b}|^p\left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta d\mathbf{b} da\right\}^{\frac{1}{p}}\left\{\int_{\mathbb{R}^2}|\boldsymbol{\omega}|^p\left|\mathcal{L}_M[f](\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\right\}^{\frac{1}{p}}$$
$$\geq (4\pi^2 B^2)^{\frac{1}{p}-\frac{1}{2}}\pi B^2 C_\Gamma^{\frac{1}{p}}||f||_2^{\frac{4}{p}}$$

Proof . By virtue of Holder's inequality, we have

$$\left\{\int_0^{a_o}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}|\mathbf{b}|^p\left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta d\mathbf{b}da\right\}^{\frac{2}{p}}\left\{\int_0^{a_o}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}\left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta d\mathbf{b}da\right\}^{1-\frac{2}{p}}$$

$$\begin{aligned}
&=\left\{\int_0^{a_o}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}\left(|\mathbf{b}|^2\left|\Gamma^M f(a,\mathbf{b},\theta)\right|^{\frac{4}{p}}\right)^{\frac{p}{2}} d\theta d\mathbf{b}da\right\}^{\frac{2}{p}}\left\{\int_0^{a_o}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}\left(\left|\Gamma^M f(a,\mathbf{b},\theta)\right|^{2-\frac{4}{p}}\right)^{\frac{1}{1-\frac{2}{p}}} d\theta d\mathbf{b}da\right\}^{1-\frac{2}{p}}\\
&\geq \int_0^{a_o}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}\left|(|\mathbf{b}|^2\left(\Gamma^M f(a,\mathbf{b},\theta)\right)^{\frac{4}{p}})\left(\Gamma^M f(a,\mathbf{b},\theta)\right)^{2-\frac{4}{p}}\right| d\theta d\mathbf{b}da\\
&=\int_0^{a_o}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}\left|\mathbf{b}\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta d\mathbf{b}da
\end{aligned}$$

Therefore ,we have

$$\left\{\int_0^{a_o}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}|\mathbf{b}|^p\left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta d\mathbf{b}da\right\}^{\frac{1}{p}} \geq \frac{\left\{\int_0^{a_o}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}\left|\mathbf{b}\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta d\mathbf{b}da\right\}^{\frac{1}{2}}}{\left\{\int_0^{a_o}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}\left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta d\mathbf{b}da\right\}^{\frac{1}{2}-\frac{1}{p}}} \tag{3.1}$$

By virtue of orthogonality relation (2.3), and in analogy with above, we have

$$\begin{aligned}
\left\{\int_{\mathbb{R}^2}|\boldsymbol{\omega}|^p\left|\mathcal{L}_M[f](\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\right\}^{\frac{1}{p}} &\geq \frac{\left\{\int_{\mathbb{R}^2}\left|\boldsymbol{\omega}\mathcal{L}_M[f](\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\right\}^{\frac{1}{2}}}{\left\{\int_{\mathbb{R}^2}|\mathcal{L}_M[f](\boldsymbol{\omega})|^2 d\boldsymbol{\omega}\right\}^{\frac{1}{2}-\frac{1}{p}}}\\
&=(4\pi^2B^2\,C_\Gamma)^{\frac{1}{2}-\frac{1}{p}}\frac{\left\{\int_{\mathbb{R}^2}\left|\boldsymbol{\omega}\mathcal{L}_M[f](\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\right\}^{\frac{1}{2}}}{\left\{4\pi^2B^2\,C_\Gamma\int_{\mathbb{R}^2}\left|\mathcal{L}_M[f](\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\right\}^{\frac{1}{2}-\frac{1}{p}}}\\
&=(4\pi^2B^2\,C_\Gamma)^{\frac{1}{2}-\frac{1}{p}}\frac{\left\{\int_{\mathbb{R}^2}\left|\boldsymbol{\omega}\mathcal{L}_M[f](\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\right\}^{\frac{1}{2}}}{\left\{\int_0^{a_o}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}\left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta d\mathbf{b}da\right\}^{\frac{1}{2}-\frac{1}{p}}}
\end{aligned} \tag{3.2}$$

Multiplying (3.1) and (3.2) and using theorem (3.1), we get

$$\left\{\int_0^{a_o}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}|\mathbf{b}|^p\left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta d\mathbf{b}da\right\}^{\frac{1}{p}}\left\{\int_{\mathbb{R}^2}|\boldsymbol{\omega}|^p\left|\mathcal{L}_M[f](\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\right\}^{\frac{1}{p}}$$
$$\geq (4\pi^2B^2\,C_\Gamma)^{\frac{1}{2}-\frac{1}{p}}\frac{\left\{\int_0^{a_o}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}\left|\mathbf{b}\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta d\mathbf{b}da\right\}^{\frac{1}{2}}\left\{\int_{\mathbb{R}^2}\left|\boldsymbol{\omega}\mathcal{L}_M[f](\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\right\}^{\frac{1}{2}}}{\left\{\int_0^{a_o}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}\left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta d\mathbf{b}da\right\}^{1-\frac{2}{p}}}$$
$$= (4\pi^2B^2\,C_\Gamma)^{\frac{1}{2}-\frac{1}{p}}\pi B^2\frac{\sqrt{C_\Gamma}||f||^2}{(4\pi^2B^2C_\Gamma||f||^2)^{1-\frac{2}{p}}}$$
$$= (4\pi^2B^2)^{\frac{1}{p}-\frac{1}{2}}\pi B^2 C_\Gamma^{\frac{1}{p}}||f||_2^{\frac{4}{p}}$$

This completes the proof of the theorem. □

For $p=2$ Theorem (3.3) boils down to Theorem (3.1). In our next theorem, we prove the logarithmic uncertainty inequality associated with LCCT.

Theorem 3.4. For any $f\in L^2(\mathbb{R}^2)$, the linear canonical curvelet transform satisfies the following logarithm estimate of uncertainty inequality:

$$\int_{\mathbb{R}^2}\int_0^{a_o}\int_{-\pi}^{\pi}\ln|\mathbf{b}|\left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta da d\mathbf{b}+4\pi^2B^2\,C_\Gamma\int_{\mathbb{R}^2}\ln|\boldsymbol{\omega}|\,|\mathcal{L}_M[f](\boldsymbol{\omega})|^2 d\boldsymbol{\omega}$$
$$\geq 4\pi^2B^2\,C_\Gamma||f||_2^2\left[\frac{\Gamma'(1/2)}{\Gamma(1/2)}-\ln\pi+\ln|B|\right]$$

Where $\Gamma(t)$ denotes the Euler's Gamma function.

Proof . The logarithmic uncertainty principle for LCT is given by [2]

$$\int_{\mathbb{R}^2}\ln|\mathbf{b}|\,|f(\mathbf{b})|^2 d\mathbf{b}+\int_{\mathbb{R}^2}\ln|\boldsymbol{\omega}|\,|\mathcal{L}_M[f](\boldsymbol{\omega})|^2 d\boldsymbol{\omega}$$
$$\geq\left[\frac{\Gamma'(1/2)}{\Gamma(1/2)}-\ln\pi+\ln|B|\right]\int_{\mathbb{R}^2}|f(\mathbf{b})|^2 d\mathbf{b} \tag{3.3}$$

Replace the function f by $\Gamma^M f(a,\mathbf{b},\theta)$ so that after integration with respect to measure $d\theta da$, the above inequality becomes

$$\int_{\mathbb{R}^2}\int_0^{a_o}\int_{-\pi}^{\pi}\ln|\mathbf{b}|\left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta da d\mathbf{b}+\int_{\mathbb{R}^2}\int_0^{a_o}\int_{-\pi}^{\pi}\ln|\boldsymbol{\omega}|\left|\mathcal{L}_M[\Gamma^M f(a,\mathbf{b},\theta)](\boldsymbol{\omega})\right|^2 d\theta da d\boldsymbol{\omega}$$
$$\geq\left[\frac{\Gamma'(1/2)}{\Gamma(1/2)}-\ln\pi+\ln|B|\right]\int_{\mathbb{R}^2}\int_0^{a_o}\int_{-\pi}^{\pi}\left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta da d\mathbf{b} \tag{3.4}$$

Using proposition (2.8), the second integral on the left of (3.4) becomes

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_0^{a_o}\int_{-\pi}^{\pi} \ln|\boldsymbol{\omega}| \left|\mathcal{L}_M[\Gamma^M f(a,\mathbf{b},\theta)](\boldsymbol{\omega})\right|^2 d\theta da d\boldsymbol{\omega}\\
&=\int_{\mathbb{R}^2}\int_0^{a_o}\int_{-\pi}^{\pi} \ln|\boldsymbol{\omega}|\, \mathcal{L}_M[\Gamma^M f(a,\mathbf{b},\theta)](\boldsymbol{\omega})\overline{\mathcal{L}_M[\Gamma^M f(a,\mathbf{b},\theta)](\boldsymbol{\omega})} d\theta da d\boldsymbol{\omega}\\
&=4\pi^2 B^2\int_{\mathbb{R}^2}\int_0^{a_o}\int_{-\pi}^{\pi} \ln|\boldsymbol{\omega}| \exp\left\{\frac{iD\boldsymbol{\omega}^t\boldsymbol{\omega}}{2B}\right\}\mathcal{L}_M[f](\boldsymbol{\omega})\overline{\mathcal{L}_M\left(\Gamma_{a,0,0}(\mathbf{z})\exp\left\{\frac{-iA\mathbf{z}^t\mathbf{z}}{B}\right\}\right)(R_\theta\boldsymbol{\omega})}\\
&\qquad\times\exp\left\{\frac{-iD\boldsymbol{\omega}^t\boldsymbol{\omega}}{2B}\right\}\overline{\mathcal{L}_M[f](\boldsymbol{\omega})}\mathcal{L}_M\left(\Gamma_{a,0,0}(\mathbf{z})\exp\left\{\frac{-iA\mathbf{z}^t\mathbf{z}}{B}\right\}\right)(R_\theta\boldsymbol{\omega}) d\theta da d\boldsymbol{\omega}\\
&=4\pi^2 B^2\int_{\mathbb{R}^2} \ln|\boldsymbol{\omega}|\, |\mathcal{L}_M[f](\boldsymbol{\omega})|^2 \left\{\int_0^{a_o}\int_{-\pi}^{\pi}\left|\mathcal{L}_M\left(\Gamma_{a,0,0}(\mathbf{z})\exp\left\{\frac{-iA\mathbf{z}^t\mathbf{z}}{B}\right\}\right)(R_\theta\boldsymbol{\omega})\right|^2 d\theta da\right\} d\boldsymbol{\omega}\\
&=4\pi^2 B^2 C_\Gamma\int_{\mathbb{R}^2} \ln|\boldsymbol{\omega}|\, |\mathcal{L}_M[f](\boldsymbol{\omega})|^2 d\boldsymbol{\omega}
\end{aligned}$$

Using this in (3.4) and noting that Γ is admissible, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_0^{a_o}\int_{-\pi}^{\pi} \ln|b| \left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta da d\mathbf{b} + 4\pi^2 B^2\, C_\Gamma\int_{\mathbb{R}^2} \ln|\boldsymbol{\omega}|\, |\mathcal{L}_M[f](\boldsymbol{\omega})|^2 d\boldsymbol{\omega}\\
&\geq 4\pi^2 B^2\, C_\Gamma ||f||_2^2\left[\frac{\Gamma'(1/2)}{\Gamma(1/2)} - \ln\pi + \ln|B|\right]
\end{aligned}$$

This completes the proof of theorem. □

The classical Heisenberg Uncertainty principle measures the localization in terms of dispersions of the respective function. By considering alternate criterion of localization , i.e., smallness of support, in 1993 Nazarov's Uncertainty principle was first proposed by F. L Nazarov [23]. It states that what happens if a non-zero function and its Fourier transform are small outside a compact set?

Motivated by Nazarov's UP in the classical Fourier domain [17, 23] and offset linear canonical transform domain [15]. We extend Nazarov's UP to the LCCT domain. If E_1, E_2 are two subsets of $\mathbb{R}^2$ with finite measure, then for any $f \in L^2(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2} |f(\mathbf{b})|^2 d\mathbf{b} \leq Ce^{C|E_1||E_2|}\left\{\int_{\mathbb{R}^2\setminus E_1} |f(\mathbf{b})|^2 d\mathbf{b} + \int_{\mathbb{R}^2\setminus E_2} |\mathcal{L}_M[f](\boldsymbol{\omega})|^2 d\boldsymbol{\omega}\right\} \tag{3.5}$$

Where C is a positive constant and $|E_1|$and $|E_2|$ denotes the measure of E_1 and E_2 respectively.

Theorem 3.5. For any $f \in L^2(\mathbb{R}^2)$, the linear canonical curvelet transform satisfies the following uncertainty inequality:

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus E_1}\int_0^{a_o}\int_{-\pi}^{\pi} \left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta da d\mathbf{b} + 4\pi^2 B^2 C_\Gamma\int_{\mathbb{R}^2\setminus E_2} |\mathcal{L}_M[f](\boldsymbol{\omega})|^2 d\boldsymbol{\omega}\\
&\qquad\geq \frac{4\pi^2 B^2 C_\Gamma ||f||_2^2}{Ce^{C|E_1||E_2|}}
\end{aligned}$$

Where E_1, E_2 are two subsets of $\mathbb{R}^2$ with finite measure and C is a positive constant.

Proof . Replace the function f in (3.5) by $\Gamma^M f(a,\mathbf{b},\theta)$, we obtain

$$\int_{\mathbb{R}^2} \left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\mathbf{b} \leq Ce^{C|E_1||E_2|} \left\{ \int_{\mathbb{R}^2\setminus E_1} \left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\mathbf{b} + \int_{\mathbb{R}^2\setminus E_2} \left|\mathcal{L}_M[\Gamma^M f(a,\mathbf{b},\theta)](\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \right\} \tag{3.6}$$

Integrating (3.6) with respect to measure $d\theta da$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}\int_0^{a_o}\int_{-\pi}^{\pi} \left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta da d\mathbf{b} &\leq Ce^{C|E_1||E_2|} \\ &\times \left\{ \int_{\mathbb{R}^2\setminus E_1}\int_0^{a_o}\int_{-\pi}^{\pi} \left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta da d\mathbf{b} \right. \\ &\left. + \int_{\mathbb{R}^2\setminus E_2}\int_0^{a_o}\int_{-\pi}^{\pi} \left|\mathcal{L}_M[\Gamma^M f(a,\mathbf{b},\theta)](\boldsymbol{\omega})\right|^2 d\theta da d\boldsymbol{\omega} \right\}\end{aligned}$$

Using proposition (2.8), together with (2.4), the above inequality becomes

$$\begin{aligned}&\int_{\mathbb{R}^2\setminus E_1}\int_0^{a_o}\int_{-\pi}^{\pi} \left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta da d\mathbf{b} + 4\pi^2 B^2 \\ &\times \int_{\mathbb{R}^2\setminus E_2}\int_0^{a_o}\int_{-\pi}^{\pi} \left|\exp\left\{\frac{iD\boldsymbol{\omega}^t\boldsymbol{\omega}}{2B}\right\}\mathcal{L}_M[f](\boldsymbol{\omega})\overline{\mathcal{L}_M\left(\Gamma_{a,0,0}(\mathbf{z})\exp\left\{\frac{-iA\mathbf{z}^t\mathbf{z}}{B}\right\}\right)}(R_\theta\boldsymbol{\omega})\right|^2 d\theta da d\boldsymbol{\omega} \\ &\quad \geq \frac{4\pi^2B^2C_\Gamma||f||_2^2}{Ce^{C|E_1||E_2|}} \\ &= \int_{\mathbb{R}^2\setminus E_1}\int_0^{a_o}\int_{-\pi}^{\pi} \left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta da d\mathbf{b} + 4\pi^2 B^2 \int_{\mathbb{R}^2\setminus E_2} \left|\mathcal{L}_M[f](\boldsymbol{\omega})\right|^2 \\ &\quad \times \left\{\int_0^{a_o}\int_{-\pi}^{\pi}\left|\mathcal{L}_M\left(\Gamma_{a,0,0}(\mathbf{z})\exp\left\{\frac{-iA\mathbf{z}^t\mathbf{z}}{B}\right\}\right)(R_\theta\boldsymbol{\omega})\right|^2 d\theta da\right\} d\boldsymbol{\omega} \\ &\quad \geq \frac{4\pi^2B^2C_\Gamma||f||_2^2}{Ce^{C|E_1||E_2|}} \\ &= \int_{\mathbb{R}^2\setminus E_1}\int_0^{a_o}\int_{-\pi}^{\pi} \left|\Gamma^M f(a,\mathbf{b},\theta)\right|^2 d\theta da d\mathbf{b} + 4\pi^2 B^2 C_\Gamma \int_{\mathbb{R}^2\setminus E_2} \left|\mathcal{L}_M[f](\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \\ &\quad \geq \frac{4\pi^2B^2C_\Gamma||f||_2^2}{Ce^{C|E_1||E_2|}}\end{aligned}$$

This completes the proof of the theorem. □

4 Conclusion and Future Work

In the present study, first we state some properties of linear canonical curvelet transform like admissibility condition and Moyals principle. Using these properties, we proved some notable inequalities for linear canonical curvelet transform. Finally, we proved some Heisenberg-type inequalities for linear canonical curvelet transform. These results are of great importance in signal and image processing. In our future works, we will generalize linear canonical curvelet transform to quaternion domain and establish these results for quaternion linear canonical curvelet transform.

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