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A difference scheme using a parametric spline for differential difference equation with twin layers

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Abstract

In this study, a parametric spline approach is used to evaluate the solution of differential-difference equations with delay and advanced parameters having twin layers. Using the continuity condition of the first-order derivative of the spline at the interior node, the difference scheme is derived. Thenon-standard finite differences for the first derivatives are employed in the scheme to increase the precision of the solution. According to an analysis of the suggested approach, its fourth-order convergence is established. Maximum absolute errors for the examples chosen from the literature are tabulated. To demonstrate the effectiveness of the method, numerical results are shown along with comparisons to other methods.

Keywords: Differential-difference equation, Twin layers, Parametric spline, Maximum error
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1 Introduction

Differential equations are those in which the growth of a present parameters is inconsistently dependent on a particular history. This means that a physical systems rate of change depends on both its present and past history. The layer behaviour differential difference equations have been extensively used in control theory for a number of years. Subsequently, these equations play an important part in predator-prey models [17], thermo-elasticity [2] and population dynamics [12], models of the red blood cell system [16], and models of neuronal variability [26].

Bender and Orszag [1], Doolan et al. [4], Driver [5], El'sgol'ts and Norkin [6], Kokotovic et al. [11], Mickens [18], Miller et al. [19], O'Malley [20] and Kellogg and Tsan [10] are the authors who have produced books explaining various methods for solving a singular perturbed differential-difference equations (SPDDEs). In [8, 9] the authors created an fitted finite difference approach to solve SPDDEs, in which the solution of the problem displays layer profile and arising in a mathematical model of neuronal variability. In [14], authors developed an asymptotic analysis for a class of SPDDEs with negative and positive shifts. In [15], the authors concentrate on problems with solutions that show layer at either one of the boundaries or both of the boundary. The Laplace transforms used to the investigation of the

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layer equation produce new and interesting findings. Devendra Kumar and Kadalbajoo [3] devised a numerical scheme using B-spline collocation method on a piecewise uniform mesh to solve SPDDE. The same authors in [7] suggested a fitted mesh with B-spline collocation to solve SPDDE with a small delay.

The authors in [21] designed non-standard fitted finite difference methods based on the methods given in [18] for SPDDEs with negative and positive shifts. Rai and Sharma [22] developed numerical schemes using some modifications in El-Mistikawy–Werle exponential finite difference scheme. Sirisha et al. [23] devised a mixed difference scheme to solve the SPDDEs. Salama and Al-Amery [24] constructed a mixed asymptotic solution for SPDDE using the composite expansion method. This work deals with constant shifts, which are not affected by the perturbation arguments. Swamy et al. [25] constructed a computational method of order four to solve SPDDE with mixed arguments.

Standard discretization approaches in numerical methods for tackling layer behaviour problems are well recognized to be unstable and fail to produce accurate results when the perturbation parameter is very small. Since the proposed problem involves not only the perturbation parameter but also delay, advanced parameters which impacts the layer behaviour, it is essential to devise numerical approaches for these problems that are accurate regardless of perturbation parameter value. With this motivation, a difference scheme using a parametric spline is devised in the following section.

2 Problem description

Consider a differential equation with small shift terms having twin layers of the form:

$$\varepsilon^2\vartheta''(s) + a(s)\vartheta(s-\delta) + c(s)\vartheta\left(s\right) + b(s)\vartheta\left(s+\eta\right) = f(s), \quad 0 < s < 1 \tag{2.1}$$

subject to the interval conditions

$$\vartheta(s) = \phi(s)\,, \quad -\delta \leqslant s \leqslant 0 \tag{2.2}$$

$$\vartheta(s) = \gamma(s), \qquad 1 \leqslant s \leqslant 1+\eta \tag{2.3}$$

where the functions $a(s),\ b(s),\ c(s),\ f(s),\ \phi(s),$ and $\gamma(s)$ are differentiable over the domain, $\varepsilon(0 < \varepsilon << 1)$ is the perturbed parameter , $\delta(0 < \delta = o(\varepsilon))$ is the delay and $\eta(0 < \eta = o(\varepsilon))$ is the advance parameter. If $a(s)+b(s)+c(s) \leqslant 0$ on $[0, 1]$, then the solution of Eq. (2.1) displays twin layers at both ends of the domain, whereas it shows an oscillatory structure for $a(s) + b(s) + c(s) > 0$.

Using Taylor series on the terms having small shifts, we have

$$\vartheta\left(s-\delta\right) \approx \vartheta(s) - \delta\vartheta^{'}(s) \tag{2.4}$$

$$\vartheta\left(s+\eta\right) \approx \vartheta(s) + \eta\vartheta^{'}(s) \tag{2.5}$$

Using the above equations in Eq. (2.1), it reduces to an asymptotically similar singular perturbation problem with the form:

$$\varepsilon\vartheta^{"}(s) + \alpha\left(s\right)\vartheta^{'}(s) + \beta\left(s\right)\vartheta\left(s\right) = f(s) \tag{2.6}$$

$$\vartheta(0) = \phi\left(0\right), \vartheta(2.1) = \gamma\left(1\right) \tag{2.7}$$

where $\alpha\left(s\right) = b\left(s\right)\eta - a\left(s\right)\delta.,\quad \beta\left(s\right) = a(s) + c\left(s\right) + b\left(s\right).$

3 Numerical method

The domain $[0, 1]$ is partitioned into L non-overlapping intervals $0 = s_0 < s_1 < ... < s_L = 1$, with each interval having a length h, in order to construct the difference scheme of Eqs. (1) - (2). Then, we have $s_i = s_0 + ih$ for $i = 0, 1, ..., L$

The term parametric cubic-spline function refers to a function $S(s)$ of class $C^2[0,\,1]$ that interpolates $\theta(s)$ at the nodal point s_i depending on a parameter τ and reduces to a cubic spline in $[0,\,1]$ as $\tau \to 0$. The spline function in $[s_i,\, s_{i+1}]$ is of the form

$$S^{''}\left(s\right) + \tau S(s) = \left[S^{''}\left(s_i\right) + \tau S\left(s_i\right)\right]\frac{\left(s_{i+1}-s\right)}{h} + \left[S^{''}\left(s_{i+1}\right) + \tau S\left(s_{i+1}\right)\right]\frac{\left(s-s_i\right)}{h} \tag{3.1}$$

where $S\left(s_i\right) = \vartheta_i$ and $\tau > 0$.

Solving Eq. (3.1) and finding the arbitrary constants using the conditions $S(s_{i+1}) = \theta_{i+1},\ S(s_i) = \vartheta_i$, we get

$$S(s) = -\frac{h^2}{\lambda^2 sin\lambda}\left[M_{i+1} sin\frac{\lambda(s-s_i)}{h} + M_i sin\frac{\lambda(s_{i+1}-s)}{h}\right]$$

$$+\frac{h^2}{\lambda^2}\left[\frac{(s-s_i)}{h}\left(M_{i+1}+\frac{\lambda^2}{h^2}\vartheta_{i+1}\right)+\frac{(s_{i+1}-s)}{h}\left(M_i+\frac{\lambda^2}{h^2}\vartheta_i\right)\right] \tag{3.2}$$

Here $\lambda = h\tau^{1/2}$. Using Eq. (9) and letting s tend to s_i, we obtain

$$S'(s_i+) = \frac{\vartheta_{i+1}-\vartheta_i}{h} + \frac{h}{\lambda^2}\left[\left(1-\frac{h}{sin\lambda}\right)M_{i+1} - (1-\lambda cot\lambda)M_i\right]$$

Proceeding in a similar manner in $(s_{i-1},\ s_i\ )$, we get

$$S'(s_i-) = \frac{\vartheta_i-\vartheta_{i-1}}{h} + \frac{h}{\lambda^2}\left[(1-\lambda cot\lambda)M_i - \left(1-\frac{\lambda}{sin\lambda}\right)M_{i-1}\right]$$

At node s_i, equating the left and right hand derivatives, we obtain

$$\frac{\vartheta_i-\vartheta_{i-1}}{h}+\frac{h}{\lambda^2}\left[(1-\lambda cot\lambda)M_i-\left(1-\frac{\lambda}{sin\lambda}\right)M_{i-1}\right] = \frac{\vartheta_{i+1}-\vartheta_i}{h}+\frac{h}{\lambda^2}\left[\left(1-\frac{\lambda}{sin\lambda}\right)M_{i+1} - (1-\lambda cot\lambda)M_i\right] \tag{3.3}$$

This leads to the system

$$h^2\left(\lambda_1 M_{i-1} + 2\lambda_2 M_i + \lambda_1 M_{i+1}\right) = \vartheta_{i+1} - 2\vartheta_i + \vartheta_{i-1} \tag{3.4}$$

where

$$\lambda_1 = \frac{1}{\lambda^2}\left(\frac{\lambda}{sin\lambda}-1\right), \lambda_2 = \frac{1}{\lambda^2}(1-\lambda cot\lambda), M_i = S''(s_i), \forall i = 1,2,\ldots,L-1.$$

Using the following first order derivative of ϑ at the grid points $s_1, s_2, ..., s_{L-1}$

$$\vartheta'_{i+1} = \frac{\vartheta_{i-1}-4\vartheta_i+3\vartheta_{i+1}}{2h}, \vartheta'_{i-1} = \frac{-3\vartheta_{i-1}+4\vartheta_i-\vartheta_{i+1}}{2h}$$

$$\vartheta'_i = \left(\frac{1+2\omega h^2\widetilde{\beta}_{i+1}+\omega h[3\widetilde{\alpha}_{i+1}+\widetilde{\alpha}_{i-1}]}{2h}\right)\vartheta_{i+1} - 2\omega\ [\widetilde{\alpha}_{i+1}+\widetilde{\alpha}_{i-1}]\,\vartheta_i$$

$$+\left(\frac{1+2\omega h^2\widetilde{\beta}_{i-1}-\omega h[\ \widetilde{\alpha}_{i+1}+3\widetilde{\alpha}_{i-1}]}{2h}\right)\vartheta_{i-1} + \omega\ h\,[f_{i+1}-f_{i-1}]$$

we get the following three term difference relation

$$E_i\vartheta_{i-1} + F_i\ \theta_i + G_i\vartheta_{i+1} = H_i,\ \forall i = 1,2,...,\ L-1 \tag{3.5}$$

where

$$E_i = \frac{\varepsilon}{h^2} + \frac{3}{2h}\lambda_1\ \widetilde{\alpha}_{i-1} - \lambda_2\ \widetilde{\alpha}_i\omega\,[\widetilde{\alpha}_{i+1}+3\widetilde{\alpha}_{i-1}] + 2\omega\ \widetilde{\alpha}_i\lambda_2\ h\widetilde{\beta}_{i-1} - \frac{\lambda_1}{2h}\widetilde{\alpha}_{i+1} - \lambda_1\ \widetilde{\beta}_{i-1} + \frac{\lambda_2}{h}\ \widetilde{\alpha}_i$$

$$F_i = -\frac{2\varepsilon}{h^2} - \frac{2\lambda_1\ \widetilde{\alpha}_{i-1}}{h} + 4\lambda_2\ \widetilde{\alpha}_i\omega\,[\widetilde{\alpha}_{i+1}+\widetilde{\alpha}_{i-1}]\ + \frac{2\lambda_1\ \widetilde{\alpha}_{i+1}}{h} - 2\lambda_2\ \widetilde{\beta}_i$$

$$G_i = \frac{\varepsilon}{h^2} + \frac{\lambda_1}{2h}\ \widetilde{\alpha}_{i-1} - \lambda_2\ \widetilde{\alpha}_i\omega\,[3\widetilde{\alpha}_{i+1}+\widetilde{\alpha}_{i-1}] - 2\omega h\lambda_2\ \widetilde{\alpha}_i\widetilde{\beta}_{i+1} - \frac{3}{2h}\lambda_1\ \widetilde{\alpha}_{i+1} - \lambda_1\ \widetilde{\beta}_{i+1} - \frac{\lambda_2\ \widetilde{\alpha}_i}{h}$$

$$H_i = (\lambda_1 - 2\omega\lambda_2\ \widetilde{\alpha}_i h)\,\widetilde{f}_{i-1} + 2\lambda_2 f_i + (\lambda_1 + 2\omega\lambda_2\ \widetilde{\alpha}_i h)\,f_{i+1}$$

Thomas algorithm is implemented to solve the tridiagonal system Eq. (12) for the approximations $\theta_{1,}\theta_2, ..., \theta_{L-1}$ of the solution $\theta(s)$ at $s_1, s_2, ...s_{L-1}$.

4 Convergence analysis

The matrix form of the system of equations Eq. (12) is

$$(A+J)\,Z+\widetilde{Q}+T(h)=O \tag{4.1}$$

$$A=[-\varepsilon,\quad 2\varepsilon,\quad -\varepsilon]=\begin{bmatrix} 2\varepsilon & -\varepsilon & 0 & 0 & .... & 0\\ -\varepsilon & 2\varepsilon & -\varepsilon & 0 & .... & 0\\ 0 & -\varepsilon & 2\varepsilon & -\varepsilon & .... & 0\\ .. & .. & .. & .. & .. & ..\\ .. & .. & .. & .. & .. & ..\\ 0 & .. & .. & 0 & -\varepsilon & 2\varepsilon \end{bmatrix}$$

$$J=[z_i,v_i,w_i]=\begin{bmatrix} v_1 & w_1 & 0 & 0 & .... & 0\\ z_2 & v_2 & w_2 & 0 & .... & 0\\ 0 & z_3 & v_3 & w_3 & .... & 0\\ .. & .. & .. & .. & .. & ..\\ .. & .. & .. & .. & .. & ..\\ 0 & .. & .. & 0 & z_{N-1} & v_{N-1} \end{bmatrix}$$

$$z_i=-\frac{3}{2h}\lambda_1\ \widetilde{\alpha}_{i-1}\ +\lambda_2\ \widetilde{\alpha}_i\omega\left[\widetilde{\alpha}_{i+1}+3\widetilde{\alpha}_{i-1}\right]-2\omega\ \widetilde{\alpha}_i\lambda_2\ h\widetilde{\beta}_{i-1}+\frac{\lambda_1}{2h}\widetilde{\alpha}_{i+1}+\lambda_1\ \widetilde{\beta}_{i-1}-h\lambda_2\ \widetilde{\alpha}_i$$

$$v_i=\frac{2\lambda_1\ \widetilde{\alpha}_{i-1}}{h}\ -4\lambda_2\ \widetilde{\alpha}_i\omega\left[\widetilde{\alpha}_{i+1}+\widetilde{\alpha}_{i-1}\right]\ -\frac{2\lambda_1\ \widetilde{\alpha}_{i+1}}{h}+2\lambda_2\ \widetilde{q}_i$$

$$w_i=-\frac{\lambda_1}{2h}\ \widetilde{\alpha}_{i-1}\ +\lambda_2\ \widetilde{\alpha}_i\omega\left[3\widetilde{\alpha}_{i+1}+\widetilde{\alpha}_{i-1}\right]+2\omega h\lambda_2\ \widetilde{\alpha}_i\widetilde{\beta}_{i+1}+\frac{3}{2h}\lambda_1\ \widetilde{\alpha}_{i+1}+\lambda_1\ \widetilde{\beta}_{i+1}+\frac{\lambda_2\ \widetilde{\alpha}_i}{h},\forall i=1\ to\ (L-1)$$

$$\widetilde{Q}=\left[\widetilde{\beta}_1+(-\varepsilon+z_1)\gamma_0,\widetilde{\beta}_2,\widetilde{\beta}_3,...,\widetilde{\beta}_{N-1}+(-\varepsilon+w_{N-1})\gamma_1\right]$$

$$\widetilde{\beta}_i=(\lambda_1-2\omega\lambda_2\ \widetilde{\alpha}_ih)\,f_{i-1}+2\lambda_2\ f_i+(\lambda_1+2\omega\lambda_2\ \widetilde{\alpha}_ih)\,f_{i+1}\ ,\ \ i=1,2,...,L-1$$

and $\vartheta=[\vartheta_1,\vartheta_2,...,\vartheta_{L-1}]^T\cong Z$, $T(h)=[\tau_1,\tau_2,...,\tau_{L-1}]^T$, $O=[0,0,...,0]^T$ are the associated vectors with Eq. (13). The local truncation error related with the scheme is

$$T(h)=\left[-1+2(\lambda_1+\lambda_2)\right]\varepsilon h^2\theta''(s_i)$$

$$+\left\{\left[\left(4\omega\varepsilon+\frac{1}{3}\right)\lambda_2-\frac{2\lambda_1}{3}\right]\widetilde{\alpha}(s_i)\theta'''(s_i)+(-1+12\lambda_1)\,\frac{\varepsilon}{12}\theta^{(4)}(s_i)\right\}h^4+O(h^6)$$

i.e.,

$$T(h)=O(h^6)\ \forall\lambda_1=\frac{1}{12},\lambda_2=\frac{5}{12},\omega=-\frac{1}{20\varepsilon}$$

Let $\vartheta=[\vartheta_1,\vartheta_2,...,vartheta_{L-1}]^T\cong Z$ satisfies the equation

$$(A+J)\,\vartheta+\widetilde{Q}=0 \tag{4.2}$$

Let the discretization error be $e_i=\vartheta_i-Z_i,\ i=1,2,...L-1$ so that

$$E=[e_1,e_2,...,e_{L-1}]^T=\vartheta-Z.$$

Using Eq. (13) and Eq. (14), we have the error equation as

$$(A+J)\,E=T(h) \tag{4.3}$$

Let $|\widetilde{\alpha}(s)|\le C_1$ and $|\widetilde{\alpha}(s)|\le C_2$ where C_1,C_2 are positive constants. If $J_{i,j}$ be the $(i,j)^{th}$ element of J, then

$$|J_{i,i+1}|=|w_i|\le\left(h(\lambda_1+\lambda_2)C_1+h^2\lambda_1C_2+4\lambda_2\omega h^2C_1^2+2h^3\lambda_2\omega C_1C_2\right),\quad i=1,2,...,L-2$$

$$|J_{i,i-1}|=|z_i|\le\left(h(\lambda_1+\lambda_2)C_1+h^2\lambda_1C_2+4\lambda_2\omega h^2C_1^2+2h^3\lambda_2\omega C_1C_2\right),\quad i=2,3,...,L-1$$

Thus for a small h, we have

$$|J_{i,i+1}|<\varepsilon,\quad i=1,2,...,L-2$$

$$|J_{i,i-1}| < \varepsilon,\ i = 2, 3, ..., L-1$$

Therefore $(A+J)$ is irreducible. Let $\widetilde{S}_i$ be the sum of the elements of the i^{th} row of the matrix $(A+J)$, then we have

$$\widetilde{S}_i = \varepsilon + \frac{\lambda_1 h}{2}\left(3\widetilde{\alpha}_{i-1} - \widetilde{\alpha}_{i+1}\right) - h\lambda_2\widetilde{\alpha}_i + h^2\left(\lambda_1\ \widetilde{\alpha}_{i+1} + 2\lambda_2\ \widetilde{\alpha}_i\right)$$
$$-h^2\lambda_2\omega\ \widetilde{\alpha}_i\left(\widetilde{\alpha}_{i+1} + 3\widetilde{\alpha}_{i-1}\right) + 2h^3\lambda_2\omega\widetilde{\alpha}_i\widetilde{\beta}_{i+1}, \forall i = 1$$
$$\widetilde{S}_i = h^2\left(\lambda_1\widetilde{\beta}_{i-1} + 2\lambda_2\widetilde{\beta}_i + \lambda_1\widetilde{\beta}_{i+1}\right) + 2h^3\lambda_2\widetilde{\alpha}_i\omega\left(\widetilde{\alpha}_{i+1} - \widetilde{\alpha}_{i-1}\right), \forall i = 2, 3, ..., L-2$$
$$\widetilde{S}_i = \varepsilon + \frac{\lambda_1 h}{2}\left(\widetilde{\alpha}_{i-1} - 3\widetilde{\alpha}_{i+1}\right) - h\lambda_2\widetilde{\alpha}_i + h^2\left(\lambda_1\widetilde{\beta}_{i-1} + 2\lambda_2\widetilde{\beta}_i\right)$$
$$-h^2\lambda_2\omega\widetilde{\alpha}_i\left(\widetilde{\alpha}_{i+1} + \widetilde{\alpha}_{i-1}\right) - 2h^3\lambda_2\omega\widetilde{\alpha}_i\widetilde{\beta}_{i-1}\forall i = L-1$$

Let $C_{1^*} = \min_{1\le i\le N}|\widetilde{\alpha}(s)|$, $C_1^* = \max_{1\le i\le N}|\widetilde{\alpha}(s)|$, $C_{2^*} = \min_{1\le i\le N}\left|\widetilde{\beta}(s)\right|$ and $C_2^* = \max_{1\le i\le N}\left|\widetilde{\beta}(s)\right|$. Since $0 < \varepsilon << 1$ and $\varepsilon \propto O(h)$, it is verified that for h, $(A+J)$ is monotone. Hence $(A+J)^{-1}$ exists and $(A+J)^{-1} \ge 0$. Thus from Eq. (15), we have

$$||E|| \le ||(A+J)^{-1}||\ ||T|| \tag{4.4}$$

Let $(i,k)^{th}$ element of $(A+J)^{-1}$ be $(A+J)^{-1}_{i,k}$ and define

$$||(A+J)^{-1}|| = \max_{1\le i\le L-1}\sum_{k=1}^{L-1}(A+J)^{-1}_{i,k},\quad ||T(h)|| = \max_{1\le i\le L-1}|T(h)|. \tag{4.5}$$

Since $(A+J)^{-1}_{i,k} \ge 0$ and $\sum_{k=1}^{L-1}(A+J)^{-1}_{i,k}.\widetilde{S}_k = 1\ \ for\ all\ i = 1, 2, ..., L-1$.

$$(A+J)^{-1}_{i,k} \le \frac{1}{S_i} < \frac{1}{h^2\left[\left(\lambda_1 + 2\lambda_2\right)C_{2^*} - 4\lambda_2\omega C_{1^*}^2\right]},\quad i = 1 \tag{4.6}$$

$$(A+J)^{-1}_{i,k} \le \frac{1}{S_i} < \frac{1}{h^2\left[\left(\lambda_1 + 2\lambda_2\right)C_{2^*} - 4\lambda_2\omega C_{1^*}^2\right]},\quad i = L-1 \tag{4.7}$$

Furthermore,

$$\sum_{k=1}^{L-1}(A+J)^{-1}_{i,k} \le \frac{1}{\min\limits_{2\le i\le L-2} S_i} \le \frac{1}{h^2\left(2\left(\lambda_1 + \lambda_2\right)C_{2^*}\right)}. \tag{4.8}$$

Using the Eqs. (18) - (20), from Eq. (16), we get $||E|| \le O(h^4)$. Therefore, the scheme is fourth order convergent for $\lambda_1 = \frac{1}{12},\ \lambda_2 = \frac{5}{12},\ \omega = -\frac{1}{20\varepsilon}$.

5 Numerical Examples

The exact solution of the proposed equation with constant coefficients (i.e., $a(s) = a, b(s) = b,\ c(s) = c, f(s) = f, \phi(s) = \phi$ and $\gamma(s) = \gamma$ are constants) is given by [11]

$$\vartheta(s) = \frac{f}{(a+b+c)} + \frac{\left(\left[(-a-b-c+1)e^{(m_2)} - 1\right]e^{(m_1 s)} - \left[(-a-b-c+1)e^{(m_1)}\right]e^{(m_2 s)}\right)}{\left[(a+b+c)\left(\exp\left(m_1\right) - \exp\left(m_2\right)\right)\right]}$$

where

$$m_1 = \frac{\left[(a\delta - b\eta) + \sqrt{(b\eta - a\delta)^2 - 4\varepsilon^2(a+b+c)}\right]}{2\varepsilon^2}, m_2 = \frac{\left[(a\delta - b\eta) - \sqrt{(b\eta - a\delta)^2 - 4\varepsilon^2(a+b+c)}\right]}{2\varepsilon^2}$$

Example 5.1.

$$\varepsilon^2\theta''(s) - 2\vartheta(s-\delta) - \vartheta\left(s\right) - 2\vartheta(s+\eta) = 1$$

with

$$\vartheta(s) = 1, -\delta \le s \le 0, \vartheta(s) = 0,\ 1 \le s \le 1+\eta.$$

Example 5.2.

$$\varepsilon^2\vartheta''(s) - 0.25\vartheta(s-\delta) - \vartheta\left(s\right) + 0.25\vartheta(s+\eta) = 1$$

with

$$\theta(s) = 1, -\delta \le s \le 0, \vartheta(s) = 0,\ 1 \le s \le 1+\eta.$$

6 Discussions and Conclusion

For differential-difference equations exhibiting twin-layer behavior, a finite difference approach based on a parametric spline is proposed. Test examples have been solved for a range of values of δ, η, and ε to demonstrate the scheme's versatility. The numerical errors are tabulated and compared with results in [[8], [13]]. From the error tables (Tables 1-4), it is observed that the error decreases with the decrease in mesh size h, for different values of δ and η, which implies the convergence of the method. Furthermore, using the Figure 1, it has been observed that, when the coefficients of delay and advance terms are of O(1), by increasing the value of δ for a fixed η value, the thickness of the boundary layer decreases at left end and that of the boundary layer increases at right end. Figure 2 shows that the thickness of the left end layer increases and right end layer decreases by increasing the value of η for a fixed value of δ. Using the Figure 3, it has been noticed that, when the coefficients of delay and advance terms are of $o(1)$, the thickness of the boundary layer increases at left end and that of the right boundary layer decreases by increasing the value of δ for a fixed value of η. From the Figure 4, it is observed that for a fixed δ by increasing the value of η, the width of the left boundary layer decreases and right boundary layer increases.

Table 1. Maximum absolute errors in Example 1 for ε = 0.01 and δ =0.007

N	10^2	10^3	10^4	10^5
η	Results by Proposed Method			
0.000	8.7021e-07	8.7394e-11	2.6974e-11	2.0077e-11
0.003	8.7021e-07	8.7394e-11	2.6974e-11	2.0077e-11
0.006	8.7021e-07	8.7394e-11	2.6974e-11	2.0077e-11
0.009	8.7021e-07	8.7394e-11	2.6974e-11	2.0077e-11
	Results in Kadalbajoo and Sharma [8]			
0.000	0.02473511	0.00389701	0.00041008	0.00004121
0.003	0.00608203	0.00223159	0.00024367	0.00002457
0.006	0.01493783	0.00041870	0.00006168	0.00000637
0.009	0.03309705	0.00157562	0.00013880	0.00001369

Table 2. Maximum absolute errors in Example 1 for ε =0.01 and η = 0.005

N	10^2	10^3	10^4	10^5
δ	Results in Proposed Method			
0.000	9.1792e-07	9.1956e-11	2.7249e-11	2.0188e-11
0.003	8.9976e-07	9.0227e-11	2.7406e-11	1.9868e-11
0.006	8.7819e-07	8.8158e-11	2.6940e-11	1.9754e-11
0.009	8.5311e-07	8.5759e-11	2.7169e-11	2.0022e-11
	Results in Kadalbajoo and Sharma [8]			
0.000	0.01226843	0.00628765	0.00069355	0.00006998
0.003	0.02913693	0.00219962	0.00028233	0.00002885
0.006	0.05492486	0.00194291	0.00014674	0.00001420
0.009	0.07571602	0.00551711	0.00052442	0.00005217

Table 3. Maximum absolute errors in Example 2 for ε = 0.01 and δ =0.007.

N	10^2	10^3	10^4	10^5
η	Results in Proposed Method			
0.000	5.9260e-08	5.2773e-08	2.7013e-10	2.1796e-12
0.003	5.9011e-08	5.1630e-08	2.7104e-10	2.1767e-12
0.006	5.8744e-08	5.1572e-08	2.7595e-10	2.1704e-12
0.009	5.8466e-08	5.1807e-08	2.7068e-10	2.1664e-12
	Results in [13]			
0.000	2.1146e-02	2.2009e-04	2.2030e-06	2.2352e-08
0.003	1.7718e-02	1.8893e-04	1.8900e-06	1.9201e-08
0.006	1.9622e-02	2.1782e-04	2.1801e-06	2.1802e-08
0.009	2.3539e-02	2.5531e-04	2.5548e-06	2.5547e-08

Table 4. Maximum absolute errors in Example 2 for ε =0.01 and η = 0.005.

N	10^2	10^3	10^4	10^5
δ	Results in Proposed Method			
0.000	5.8181e-08	5.1520e-08	2.7125e-10	2.1744e-12
0.003	5.8466e-08	5.1807e-08	2.7068e-10	2.1664e-12
0.006	5.8744e-08	5.1572e-08	2.7595e-10	2.1704e-12
0.009	5.9011e-08	5.1630e-08	2.7104e-10	2.1767e-12
	Results in [13]			
0.000	2.8211e-02	2.9840e-04	2.9847e-06	2.9846e-08
0.003	2.3539e-02	2.5531e-04	2.5548e-06	2.5547e-08
0.006	1.9622e-02	2.1782e-04	2.1801e-06	2.1802e-08
0.009	1.7718e-02	1.8893e-04	1.8900e-06	1.9201e-08

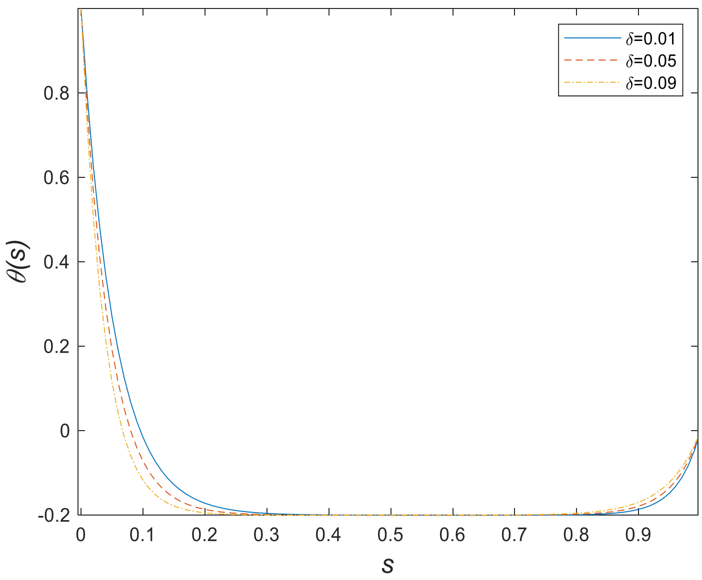

Figure 1. Solution profile in Example 1 for ε = 0.1 and η =0.05.

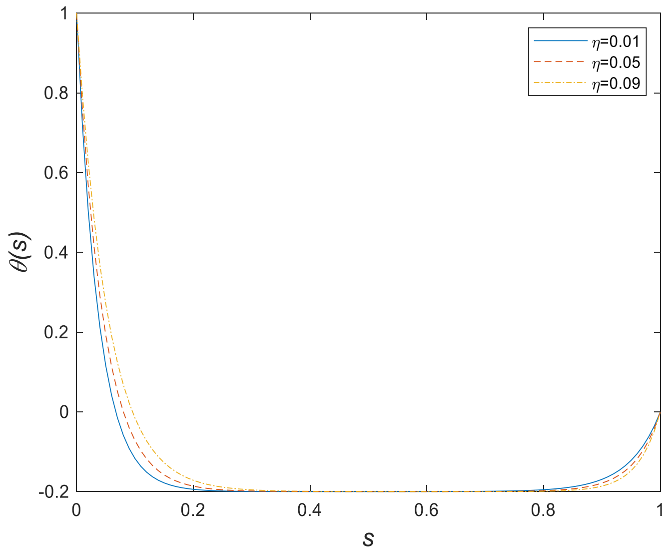

Figure 2. Solution profile in Example 1 for ε = 0.1 and δ =0.05.

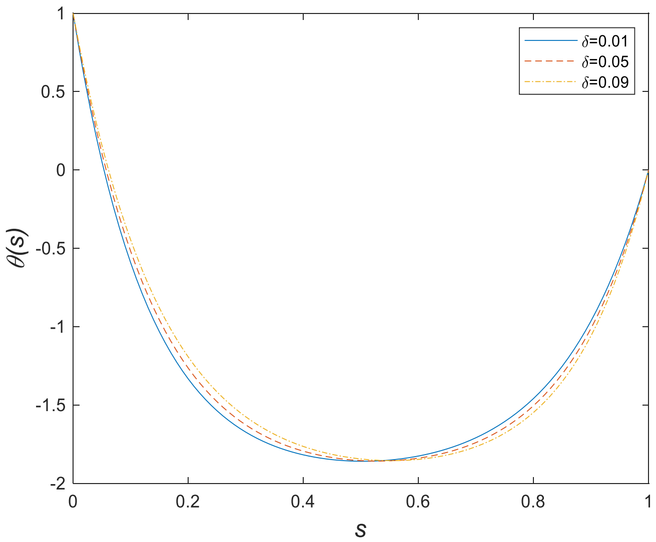

Figure 3. Solution profile in Example 2 for ε = 0.1 and η =0.05.

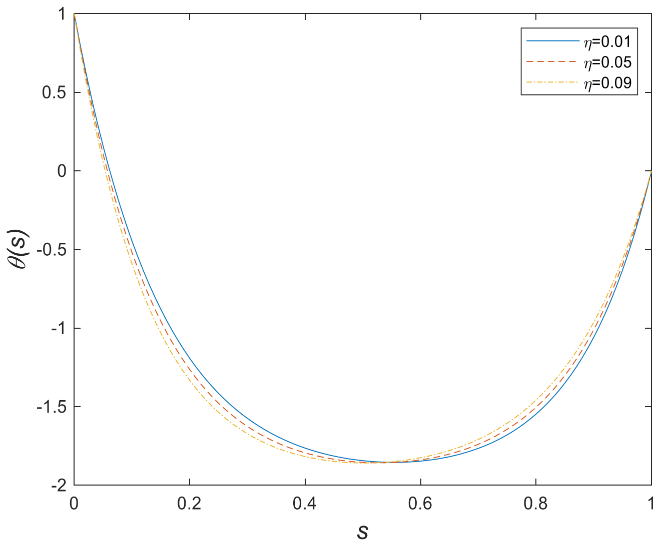

Figure 4. Solution profile in Example 2 for ε = 0.1 and δ =0.05.

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