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Multivalued operators, data dependence of fixed points and fractals

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Abstract

In the present manuscript, we prove some new fixed point results for multivalued mappings in the setting of b-metric space. Also, we obtain some results for the data dependence of fixed points, the directed graph endowed with a b-metric and for the fractals of an iterated multifunction system. The proven results extend and generalize some of the results in the literature.

Keywords: Multivalued mappings, Fixed points, b-metric space
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1 Introduction and preliminaries

In 1993, Czerwick [10] introduced the concept of the b-metric space which generalized metric spaces. Thereafter, many theorems has been given by many researchers in the framework of b-metric space. In 1969, Nadler [20] has considered multivalued contractions for the study of fixed point theorems. Mizoguchi-Takahashi [18], Reich [21] have generalized the results of fixed point theorems in single-valued metric mappings as well as multivalued mappings in the b-metric space. Thereafter, many authors worked on multivalued mappings in the setting of b-metric space (see [1, 4, 12, 15]).

Fractal is set of points whose fractal dimension exceeds its topological dimension. In mathematics and applied sciences fractals and multivalued fractals has played an important role [3, 7]. Fisher [13] gave Collage Theorems for iterated multivalued systems and for respective continuation principles. Boriceanu et al. [5] presented the study of multivalued fractals in the framework of b-metric spaces. In the same paper, the authors raised some open problems on multivalued fractals in b-metric space assuming continuity of given b-metrics.

The present manuscript has five sections. In the first section, we give introduction of the topic, basic definitions and notations to be used in the sequel. In the second section, we prove some fixed point results for multivalued mappings in the framework of b-metric spaces together with some consequences of the proved results. In the third section, we discuss the data dependance of fixed points for multivalued mappings. In the fourth section, we obtain some results

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for the directed graph endowed with b-metric space. In the last section, we prove some results for the fractals of an iterated multifunction system.

We shall consider the following notion for the families of subsets of a b-metric space (X, d). $P_1(X) = \{Y : Y \subset X\}$, $P(X) = \{Y \in P_1(X) : Y \neq \phi\}$, $P_b(X) = \{Y \in P(X) : Y \text{is bounded}\}\}$, $P_{cl}(X) = \{Y \in P(X) : \text{Y is closed}\}$, $P_{cp}(X) = \{Y \in P(X) : \text{Y is compact}\}$ and $P_{cl,b}(X) = \{Y \in P(X) : \text{Y is closed and bounded}\}$. The following definitions are required in sequel.

Definition 1.1. [8] The gap functional $D : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is defined as

$$D(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

In particular case of $x_0 \in X$, then $D(x_0, B) = D(\{x_0\}, B)$.

Definition 1.2. [8] The excess generalized functional $\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is defined as

$$\rho(A, B) = \sup\{D(a, B) : a \in A\}.$$

Definition 1.3. [8] The Pompeiu-Hausdorff generalized functional $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is defined as

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

Definition 1.4. [8] The generalized diameter functional $\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, is defined as

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

In particular, $\delta(A) = \delta(A, A)$ is diameter of the set A.

Following the result of [20], one can easily obtained the result on b-metric space.

Lemma 1.5. [20] Let A, B be non empty, closed and bounded subsets of a b-metric space (X, d) and $\alpha \in A$, then for each $\epsilon > 0$, there exists $\beta \in B$ such that $d(\alpha, \beta) \leq H_b(A, B) + \epsilon$.

Lemma 1.6. [8] Let (X, d) be a b-metric space with constant s. Then

$$D(x, B) \leq s\big(d(x, y) + D(y, B)\big), \quad \text{for all} \quad x, y \in X, B \subset X.$$

Lemma 1.7. [8] Let (X, d) be a b-metric space with constant s, $B \in P(X)$ and $x \in X$. Then $D(x, B) = 0$ if and only if $x \in \bar{B}$.

2 Fixed point results for multivalued mappings

In this section, we prove some results for multivalued mappings in the setting of b-metric space.

Theorem 2.1. Let (X, d) be a complete b-metric space with constant $s \geq 1$ and Let $T, S : X \to P_{cl,b}(X)$ be multivalued mappings satisfying the conditions;

$$\begin{aligned} H(Tx, Sy) \quad \leq \quad & \alpha_1 D(x, Tx) + \alpha_2 D(y, Sy) + \alpha_3 D(x, Sy) + \alpha_4 D(y, Tx) + \alpha_5 \left(\frac{D(x, Sy) + D(y, Tx)}{2} \right) \\ & + \alpha_6 \frac{D(x, Tx) D(y, Sy)}{1 + d(x, y)} + \alpha_7 d(x, y), \end{aligned} \tag{2.1}$$

for all $x, y \in X$ and $\alpha_i \geq 0$, $1 \leq i \leq 7$, with $(\alpha_1 + \alpha_2)(s + 1) + (s^2 + s)(\alpha_3 + \alpha_4 + \alpha_5) + 2\alpha_6 + 2s\alpha_7 < 2$, and $2(\alpha_2 + \alpha_3) + \alpha_5 < \frac{2}{s}$. Then T and S have a unique common fixed point.

Proof . For any fixed $x \in X$, define $x_0 = x$ and assume that $x_1 \in Tx_0$, $x_2 \in Sx_1$ such that $x_{2n+1} = Tx_{2n}, x_{2n+2} = Sx_{2n+1}$, Using (2.1), we have

$$\begin{aligned} H(Tx_0, Sx_1) &\leq \alpha_1 D(x_0, Tx_0) + \alpha_2 D(x_1, Sx_1) + \alpha_3 D(x_0, Sx_1) + \alpha_4 D(x_1, Tx_0) \\ &\quad + \alpha_5 \left(\frac{D(x_0, Sx_1) + D(x_1, Tx_0)}{2} \right) + \alpha_6 \frac{D(x_0, Tx_0) D(x_1, Sx_1)}{1 + d(x_0, x_1)} + \alpha_7 d(x_0, x_1) \\ &\leq \alpha_1 D(x_0, x_1) + \alpha_2 D(Tx_0, Sx_1) + \alpha_3 s\big(d(x_0, x_1) + D(Tx_0, Sx_1)\big) + \alpha_4 D(x_1, x_1) \\ &\quad + \alpha_5 \left(\frac{s(d(x_0, x_1) + D(Tx_0, Sx_1)) + D(x_1, x_1)}{2} \right) + \alpha_6 \frac{D(x_0, x_1) D(Tx_0, Sx_1)}{1 + d(x_0, x_1)} \\ &\quad + \alpha_7 d(x_0, x_1) \\ &\leq \alpha_1 d(x_0, x_1) + \alpha_2 H(Tx_0, Sx_1) + \alpha_3 s\big(d(x_0, x_1) + H(Tx_0, Sx_1)\big) + \alpha_4 d(x_1, x_1) \\ &\quad + \alpha_5 \left(\frac{s(d(x_0, x_1) + H(Tx_0, Sx_1)) + d(x_1, x_1)}{2} \right) + \alpha_6 H(Tx_0, Sx_1) + \alpha_7 d(x_0, x_1). \end{aligned}$$

On solving, we get

$$(1 - \alpha_2 - s\alpha_3 - \frac{s\alpha_5}{2} - \alpha_6) H(Tx_0, Sx_1) \leq (\alpha_1 + s\alpha_3 + \frac{s\alpha_5}{2} + \alpha_7) d(x_0, x_1). \tag{2.2}$$

By symmetry, we have

$$\begin{aligned} H(Sx_1, Tx_0) &\leq \alpha_1 D(x_1, Sx_1) + \alpha_2 D(x_0, Tx_0) + \alpha_3 D(x_1, Tx_0) + \alpha_4 D(x_0, Sx_1) \\ &\quad + \alpha_5 \left(\frac{D(x_1, Tx_0) + D(x_0, Sx_1)}{2} \right) + \alpha_6 \frac{D(x_1, Sx_1) D(x_0, Tx_0)}{1 + d(x_1, x_0)} + \alpha_7 d(x_1, x_0) \\ &\leq \alpha_1 D(Tx_0, Sx_1) + \alpha_2 D(x_0, x_1) + \alpha_3 D(x_1, x_1) + \alpha_4 s(d(x_0, x_1) + D(Tx_0, Sx_1)) \\ &\quad + \alpha_5 \left(\frac{d(x_1, x_1) + s(d(x_0, x_1) + D(Tx_0, Sx_1))}{2} \right) + \alpha_6 D(Tx_0, Sx_1) + \alpha_7 d(x_1, x_0) \\ &\leq \alpha_1 H(Tx_0, Sx_1) + \alpha_2 d(x_0, x_1) + \alpha_3 d(x_1, x_1) + \alpha_4 s(d(x_0, x_1) + H(Tx_0, Sx_1)) \\ &\quad + \alpha_5 \left(\frac{d(x_1, x_1) + s(d(x_0, x_1) + H(Tx_0, Sx_1))}{2} \right) + \alpha_6 H(Tx_0, Sx_1) + \alpha_7 d(x_1, x_0). \end{aligned}$$

Further, we get

$$(1 - \alpha_1 - s\alpha_4 - \frac{s\alpha_5}{2} - \alpha_6) H(Tx_0, Sx_1) \leq (\alpha_2 + s\alpha_4 + \frac{s\alpha_5}{2} + \alpha_7) d(x_1, x_0). \tag{2.3}$$

On adding (2.2) and (2.3), we get

$$H(Tx_0, Sx_1) \leq k d(x_1, x_0), \quad \text{where} \quad k = \frac{\alpha_1 + \alpha_2 + s\alpha_3 + s\alpha_4 + s\alpha_5 + 2\alpha_7}{2 - (\alpha_1 + \alpha_2 + s\alpha_3 + s\alpha_4 + s\alpha_5 + 2\alpha_6)} < \frac{1}{s}. \tag{2.4}$$

By Lemma (1.5), we can choose $x_2 \in Sx_1$ and $\epsilon = k$, such that

$$d(x_1, x_2) \leq H(Tx_0, Sx_1) + k, \quad \text{where} \quad k = \frac{\alpha_1 + \alpha_2 + s\alpha_3 + s\alpha_4 + s\alpha_5 + 2\alpha_7}{2 - (\alpha_1 + \alpha_2 + s\alpha_3 + s\alpha_4 + s\alpha_5 + 2\alpha_6)} < \frac{1}{s}. \tag{2.5}$$

Now, for $x_3 \in Tx_2$ and using (2.1), we get

$$\begin{aligned} H(Tx_2, Sx_1) &\leq \alpha_1 D(x_2, Tx_2) + \alpha_2 D(x_1, Sx_1) + \alpha_3 D(x_2, Sx_1) + \alpha_4 D(x_1, Tx_2) \\ &\quad + \alpha_5 \left(\frac{D(x_2, Sx_1) + D(x_1, Tx_2)}{2} \right) + \alpha_6 \frac{D(x_2, Tx_2) D(x_1, Sx_1)}{1 + d(x_2, x_1)} + \alpha_7 d(x_2, x_1) \\ &\leq \alpha_1 D(Sx_1, Tx_2) + \alpha_2 D(x_1, x_2) + \alpha_3 D(x_2, x_2) + \alpha_4 s(d(x_1, x_2) + D(Sx_1, Tx_2)) \\ &\quad + \alpha_5 \left(\frac{D(x_2, x_2) + s(d(x_1, x_2) + D(Sx_1, Tx_2))}{2} \right) + \alpha_6 D(x_2, Tx_2) + \alpha_7 d(x_2, x_1) \\ &\leq \alpha_1 H(Sx_1, Tx_2) + \alpha_2 d(x_1, x_2) + \alpha_3 d(x_2, x_2) + \alpha_4 s(d(x_1, x_2) + H(Sx_1, Tx_2)) \\ &\quad + \alpha_5 \left(\frac{d(x_2, x_2) + s(d(x_1, x_2) + H(Sx_1, Tx_2))}{2} \right) + \alpha_6 H(Sx_1, Tx_2) + \alpha_7 d(x_2, x_1). \end{aligned}$$

On solving, we get

$$(1-\alpha_1-s\alpha_4-\frac{s\alpha_5}{2}-\alpha_6)H(Sx_1,Tx_2)\leq(\alpha_2+s\alpha_4+\frac{s\alpha_5}{2}+\alpha_7)d(x_1,x_2). \tag{2.6}$$

By symmetry, we have

$$\begin{aligned}
H(Sx_1,Tx_2) &\leq \alpha_1D(x_1,Sx_1)+\alpha_2D(x_2,Tx_2)+\alpha_3D(x_1,Tx_2)+\alpha_4D(x_2,Sx_1)\\
&\quad+\alpha_5\left(\frac{D(x_1,Tx_2)+D(x_2,Sx_1)}{2}\right)+\alpha_6\frac{D(x_1,Sx_1)D(x_2,Tx_2)}{1+d(x_2,x_1)}+\alpha_7d(x_1,x_2)\\
&\leq \alpha_1D(x_1,x_2)+\alpha_2D(Sx_1,Tx_2)+\alpha_3s(d(x_1,x_2)+D(Sx_1,Tx_2))+\alpha_4D(x_2,x_2)\\
&\quad+\alpha_5\left(\frac{s(d(x_1,x_2)+D(Sx_1,Tx_2))+D(x_2,x_2)}{2}\right)+\alpha_6D(Sx_1,Tx_2)+\alpha_7d(x_1,x_2)\\
&\leq \alpha_1d(x_1,x_2)+\alpha_2H(Sx_1,Tx_2)+\alpha_3s(d(x_1,x_2)+H(Sx_1,Tx_2))+\alpha_4d(x_2,x_2)\\
&\quad+\alpha_5\left(\frac{s(d(x_1,x_2)+H(Sx_1,Tx_2))+d(x_2,x_2)}{2}\right)+\alpha_6H(Sx_1,Tx_2)+\alpha_7d(x_1,x_2).
\end{aligned}$$

Further, we get

$$(1-\alpha_2-s\alpha_3-\frac{s\alpha_5}{2}-\alpha_6)H(Sx_1,Tx_2)\leq(\alpha_1+s\alpha_3+\frac{s\alpha_5}{2}+\alpha_7)d(x_1,x_2). \tag{2.7}$$

On adding (2.6) and (2.7), we get

$$H(Sx_1,Tx_2)\leq kd(x_1,x_2),\quad \text{where}\quad k=\frac{\alpha_1+\alpha_2+s\alpha_3+s\alpha_4+s\alpha_5+2\alpha_7}{2-(\alpha_1+\alpha_2+s\alpha_3+s\alpha_4+s\alpha_5+2\alpha_6)}<\frac{1}{s}. \tag{2.8}$$

By Lemma (1.5), we can choose $x_3\in Tx_2$ and $\epsilon=k^2$, such that

$$d(x_2,x_3)\leq H(Sx_1,Tx_2)+k^2,\quad \text{where}\quad k=\frac{\alpha_1+\alpha_2+s\alpha_3+s\alpha_4+s\alpha_5+2\alpha_7}{2-(\alpha_1+\alpha_2+s\alpha_3+s\alpha_4+s\alpha_5+2\alpha_6)}<\frac{1}{s}. \tag{2.9}$$

On solving, we get

$$\begin{aligned}
d(x_2,x_3) &\leq kd(x_1,x_2)+k^2\\
&\leq k^2d(x_0,x_1)+2k^2,\quad \text{where}\quad k=\frac{\alpha_1+\alpha_2+s\alpha_3+s\alpha_4+s\alpha_5+2\alpha_7}{2-(\alpha_1+\alpha_2+s\alpha_3+s\alpha_4+s\alpha_5+2\alpha_6)}<\frac{1}{s}.
\end{aligned} \tag{2.10}$$

Continuing this process and by induction, we obtain a sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ such that $x_{2n+1}\in Tx_{2n}$, $x_{2n+2}\in Sx_{2n+1}$, using (2.1), we get

$$\begin{aligned}
H(Tx_{2n},Sx_{2n+1}) &\leq \alpha_1D(x_{2n},Tx_{2n})+\alpha_2D(x_{2n+1},Sx_{2n+1})+\alpha_3D(x_{2n},Sx_{2n+1})+\\
&\quad\alpha_4D(x_{2n+1},Tx_{2n})+\alpha_5\left(\frac{D(x_{2n},Sx_{2n+1})+D(x_{2n+1},Tx_{2n})}{2}\right)\\
&\quad+\alpha_6\frac{D(x_{2n},Tx_{2n})D(x_{2n+1},Sx_{2n+1})}{1+d(x_{2n},x_{2n+1})}+\alpha_7d(x_{2n},x_{2n+1})\\
&\leq \alpha_1D(x_{2n},x_{2n+1})+\alpha_2D(Tx_{2n},Sx_{2n+1})+\alpha_3s\big(d(x_{2n},x_{2n+1})+D(Tx_{2n},Sx_{2n+1})\big)\\
&\quad+\alpha_4D(x_{2n+1},x_{2n+1})\\
&\quad+\alpha_5\left(\frac{s(d(x_{2n},x_{2n+1})+D(Tx_{2n},Sx_{2n+1}))+d(x_{2n+1},x_{2n+1})}{2}\right)\\
&\quad+\alpha_6D(Tx_{2n},Sx_{2n+1})+\alpha_7d(x_{2n},x_{2n+1})\\
&\leq \alpha_1d(x_{2n},x_{2n+1})+\alpha_2H(Tx_{2n},Sx_{2n+1})+\alpha_3s\big(d(x_{2n},x_{2n+1})+H(Tx_{2n},Sx_{2n+1})\big)\\
&\quad+\alpha_4d(x_{2n+1},x_{2n+1})\\
&\quad+\alpha_5\left(\frac{s(d(x_{2n},x_{2n+1})+H(Tx_{2n},Sx_{2n+1}))+d(x_{2n+1},x_{2n+1})}{2}\right)\\
&\quad+\alpha_6H(Tx_{2n},Sx_{2n+1})+\alpha_7d(x_{2n},x_{2n+1}).
\end{aligned}$$

Further, we obtain

$$(1-\alpha_2-s\alpha_3-\frac{s\alpha_5}{2}-\alpha_6)H(Tx_{2n},Sx_{2n+1})\leq(\alpha_1+s\alpha_3+\frac{s\alpha_5}{2}+\alpha_7)d(x_{2n},x_{2n+1}). \tag{2.11}$$

By symmetry, we have

$$\begin{aligned}
H(Sx_{2n+1},Tx_{2n}) \leq\ & \alpha_1 D(x_{2n+1},Sx_{2n+1})+\alpha_2 D(x_{2n},Tx_{2n})+\alpha_3 D(x_{2n+1},Tx_{2n})+\alpha_4 D(x_{2n},Sx_{2n+1})\\
&+\alpha_5\Big(\frac{D(x_{2n+1},Tx_{2n})+D(x_{2n},Sx_{2n+1})}{2}\Big)\\
&+\alpha_6\frac{D(x_{2n+1},Sx_{2n+1})D(x_{2n},Tx_{2n})}{1+d(x_{2n+1},x_{2n})}+\alpha_7 d(x_{2n+1},x_{2n})\\
\leq\ & \alpha_1 D(x_{2n+1},x_{2n+2})+\alpha_2 D(x_{2n},x_{2n+1})+\alpha_3 D(x_{2n+1},x_{2n+1})+\alpha_4 s\big(d(x_{2n},x_{2n+1})\\
&+D(Tx_{2n},Sx_{2n+1})\big)\\
&+\alpha_5\Big(\frac{s(d(x_{2n},x_{2n+1})+D(Tx_{2n},Sx_{2n+1}))+D(x_{2n+1},x_{2n+1})}{2}\Big)\\
&+\alpha_6 D(Tx_{2n},Sx_{2n+1})+\alpha_7 d(x_{2n+1},x_{2n}),\\
\leq\ & \alpha_1 H(x_{2n+1},x_{2n+2})+\alpha_2 d(x_{2n},x_{2n+1})+\alpha_3 d(x_{2n+1},x_{2n+1})+\alpha_4 s\big(d(x_{2n},x_{2n+1})\\
&+H(Tx_{2n},Sx_{2n+1})\big)\\
&+\alpha_5\Big(\frac{s(d(x_{2n},x_{2n+1})+H(Tx_{2n},Sx_{2n+1}))+d(x_{2n+1},x_{2n+1})}{2}\Big)\\
&+\alpha_6 H(Tx_{2n},Sx_{2n+1})+\alpha_7 d(x_{2n+1},x_{2n}).
\end{aligned}$$

Further, we get

$$(1-\alpha_1-s\alpha_4-\frac{s\alpha_5}{2}-\alpha_6)H(Tx_{2n},Sx_{2n+1})\leq(\alpha_2+s\alpha_4+\frac{s\alpha_5}{2}+\alpha_7)d(x_{2n},x_{2n+1}). \tag{2.12}$$

Adding (2.11) and (2.12), we get

$$H(Tx_{2n},Sx_{2n+1})\leq kd(x_{2n},x_{2n+1}),\quad \text{where}\quad k=\frac{\alpha_1+\alpha_2+s\alpha_3+s\alpha_4+s\alpha_5+2\alpha_7}{2-(\alpha_1+\alpha_2+s\alpha_3+s\alpha_4+s\alpha_5+2\alpha_6)}<\frac{1}{s}.$$

By Lemma (1.5), we can choose $x_{2n+1}\in Tx_{2n}$ and $\epsilon=k^{2n+1}$, such that

$$\begin{aligned}
d(x_{2n+1},x_{2n+2}) &\leq H(Tx_{2n},Sx_{2n+1})+k^{2n+1}\\
&\leq kd(x_{2n},x_{2n+1})+k^{2n+1},
\end{aligned} \tag{2.13}$$

where $k=\frac{\alpha_1+\alpha_2+s\alpha_3+s\alpha_4+s\alpha_5+2\alpha_7}{2-(\alpha_1+\alpha_2+s\alpha_3+s\alpha_4+s\alpha_5+2\alpha_6)}<\frac{1}{s}$. Therefore, we have

$$d(x_n,x_{n+1})\leq kd(x_{n-1},x_n)+k^n,\quad \text{where}\quad k=\frac{\alpha_1+\alpha_2+s\alpha_3+s\alpha_4+s\alpha_5+2\alpha_7}{2-(\alpha_1+\alpha_2+s\alpha_3+s\alpha_4+s\alpha_5+2\alpha_6)}<\frac{1}{s}.$$

Hence,

$$d(x_n,x_{n+1})\leq kd(x_{n-1},x_n)+k^n\leq k(kd(x_{n-2},x_{n-1})+k^{n-1})+k^n\leq ....\leq k^n d(x_0,x_1)+nk^n.$$

For $0<k<1$, $\sum k^n$ and $\sum nk^n$ have same radius of convergence. Then, $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence. Also, since (X,d) is complete b-metric space, there exists $z\in X$ such that $x_n\to z$. Now, we shall show that z is a common fixed point of T and S. Consider

$$\begin{aligned}
D(z,Tz) &\leq s(d(z,x_{2n+2})+D(Sx_{2n+1},Tz))\leq s(d(z,x_{2n+2})+H(Sx_{2n+1},Tz)),\\
D(z,Sz) &\leq s(d(z,x_{2n+1})+D(Tx_{2n},Sz))\leq s(d(z,x_{2n+1})+H(Tx_{2n},Sz)).
\end{aligned} \tag{2.14}$$

Using (2.1), we have

$$\begin{aligned}
H(Tx_{2n},Sz) \leq\ & \alpha_1 D(x_{2n},Tx_{2n})+\alpha_2 D(z,Sz)+\alpha_3 D(x_{2n},Sz)+\alpha_4 D(z,Tx_{2n})\\
&+\alpha_5\Big(\frac{D(x_{2n},Sz)+D(z,Tx_{2n})}{2}\Big)+\alpha_6\frac{D(x_{2n},Tx_{2n})D(z,Sz)}{1+d(x_{2n},z)}+\alpha_7 d(x_{2n},z)\\
\leq\ & \alpha_1 D(x_{2n},x_{2n+1})+\alpha_2 D(z,Sz)+\alpha_3 D(x_{2n},Sz)+\alpha_4 D(z,x_{2n+1})\\
&+\alpha_5\Big(\frac{D(x_{2n},Sz)+D(z,x_{2n+1})}{2}\Big)+\alpha_6\frac{D(x_{2n},x_{2n+1})D(z,Sz)}{1+d(x_{2n},z)}\\
&+\alpha_7 d(x_{2n},z).
\end{aligned} \tag{2.15}$$

Using (2.15) in (2.14) and letting $n \to \infty$, we get

$$\begin{aligned} D(z,Sz) &\leq s\Big(d(z,z)+\alpha_1 d(z,z)+\alpha_2 D(z,Sz)+\alpha_3 D(z,Sz)+\alpha_4 d(z,z) \\ &\quad +\alpha_5\Big(\frac{D(z,Sz)+d(z,z)}{2}\Big)+\alpha_6\frac{d(z,z)D(z,Sz)}{1+d(z,z)}+\alpha_7 d(z,z)\Big). \end{aligned}$$

On solving, we get

$$\big(1-s(\alpha_2+\alpha_3+\frac{\alpha_5}{2})\big)D(z,Sz)\leq 0.$$

But, since $\big(1-s(\alpha_2+\alpha_3+\frac{\alpha_5}{2})\big)>0$. Therefore, we obtain $D(z,Sz)=0$. Also, $S(z)$ is closed. Hence by Lemma (1.7), we have $z\in Sz$. Working on similar lines we can show that $z\in Tz$. Hence, z is a common fixed point of T and S. Next, we shall show that z is a unique common fixed point of T and S. For this consider

$$\begin{aligned} d(z,v) &\leq H(Tz,Sv) \\ &\leq \alpha_1 D(z,Tz)+\alpha_2 D(v,Sv)+\alpha_3 D(z,Sv)+\alpha_4 D(v,Tz)+\alpha_5\Big(\frac{D(z,Sv)+D(v,Tz)}{2}\Big) \\ &\quad +\alpha_6\frac{D(z,Tz)D(v,Sv)}{1+d(z,v)}+\alpha_7 d(z,v) \\ &\leq \alpha_1 d(z,z)+\alpha_2 d(v,v)+\alpha_3 d(z,v)+\alpha_4 d(v,z)+\alpha_5\Big(\frac{d(z,v)+d(v,z)}{2}\Big) \\ &\quad +\alpha_6\frac{d(z,z)d(v,v)}{1+d(z,v)}+\alpha_7 d(z,v). \end{aligned}$$

On solving, we get

$$\big(1-(\alpha_3+\alpha_4+\alpha_5+\alpha_7)\big)d(v,z)\leq 0.$$

But, since $\big(1-(\alpha_3+\alpha_4+\alpha_5+\alpha_7)\big)>0$. Therefore, $d(v,z)=0$ i.e $v=z$. Hence, z is a unique common fixed point of T and S. □

For $T=S$, we have the following result.

Corollary 2.2. Let (X,d) be a complete b-metric space with constant $s\geq 1$ and $T: X\to P_{cl,b}(X)$ multivalued mapping satisfying the conditions;

$$\begin{aligned} H(Tx,Ty) &\leq \alpha_1 D(x,Tx)+\alpha_2 D(y,Ty)+\alpha_3 D(x,Ty)+\alpha_4 D(y,Tx)+\alpha_5\Big(\frac{D(x,Ty)+D(y,Tx)}{2}\Big) \\ &\quad +\alpha_6\frac{D(x,Tx)D(y,Ty)}{1+d(x,y)}+\alpha_7 d(x,y), \end{aligned} \tag{2.16}$$

for all $x,y\in X$ and $\alpha_i\geq 0$, $1\leq i\leq 7$, with $(\alpha_1+\alpha_2)(s+1)+(s^2+s)(\alpha_3+\alpha_4+\alpha_5)+2\alpha_6+2s\alpha_7<2$ and $2(\alpha_2+\alpha_3)+\alpha_5<\frac{2}{s}$. Then T has unique fixed point.

On the lines of Theorem 2.1, we have the following result.

Theorem 2.3. Let (X,d) be a complete b-metric space with constant $s\geq 1$ and Let $T,S: X\to P_{cl,b}(X)$ be multivalued mappings satisfying the conditions;

$$H(Tx,Sy) \leq \alpha\frac{D(y,Sy)D(x,Tx)}{1+d(x,y)}+\beta d(x,y)+\gamma(D(x,Tx)+D(y,Sy))+\delta(D(y,Tx)+D(x,Sy)),$$

for all $x,y\in X$, α,β,γ and $\delta\geq 0$ with $\alpha+s\beta+(s+1)\gamma+(s^2+s)\delta<1$ and $\gamma+\delta<\frac{1}{s}$. Then T and S have unique common fixed point.

For $T = S$, we have the following result.

Corollary 2.4. Let (X, d) be a complete b-metric space with constant $s \geq 1$ and Let $T : X \to P_{cl,b}(X)$ be multivalued mappings satisfying the conditions

$$H(Tx, Ty) \leq \alpha \frac{D(y, Ty)D(x, Tx)}{1 + d(x, y)} + \beta d(x, y) + \gamma(D(x, Tx) + D(y, Ty)) + \delta(D(y, Tx) + D(x, Ty)),$$

for all $x, y \in X$, α, β, γ and $\delta \geq 0$ with $\alpha + s\beta + (s + 1)\gamma + (s^2 + s)\delta < 1$, and $\gamma + \delta < \frac{1}{s}$. Then T has unique fixed point.

As a consequences of the Theorem (2.1), we have the following result.

Theorem 2.5. Let (X, d) be a complete b-metric space with constant $s \geq 1$ and Let $T, S : X \to P_{cl,b}(X)$ be multivalued mappings satisfying the conditions;

$$\begin{aligned} H(Tx, Sy) \leq\ & \alpha_1 d(x, Tx) + \alpha_2 d(y, Sy) + \alpha_3 d(x, Sy) + \alpha_4 d(y, Tx) + \alpha_5 \left(\frac{d(x, Sy) + d(y, Tx)}{2} \right) \\ & + \alpha_6 \frac{d(x, Tx)d(y, Sy)}{1 + d(x, y)} + \alpha_7 d(x, y), \end{aligned} \tag{2.17}$$

for all $x, y \in X$ and $\alpha_i \geq 0$, $1 \leq i \leq 7$, with $(\alpha_1 + \alpha_2)(s + 1) + (s^2 + s)(\alpha_3 + \alpha_4 + \alpha_5) + 2\alpha_6 + 2s\alpha_7 < 2$ and $2(\alpha_2 + \alpha_3) + \alpha_5 < \frac{2}{s}$. Then T and S have unique common fixed point.

Example 2.6. Let $X = \mathbb{R}$, we define $d : X \times X \to X$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a complete b-metric space. Define $T : X \to P_{cl,b}(X)$ by $Tx = \frac{x}{2}$ for all $x, y \in X$. Then,

$$H(Tx, Ty) \leq \frac{1}{2} d(x, y) \left(\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0 \quad \text{and} \quad \alpha_7 = \frac{1}{2} \right).$$

Therefore, using Theorem 2.5(for the case when $T = S$) we obtain that T has a unique fixed point which is $0 \in X$.

Example 2.7. Let $X = \{0, \frac{1}{2}, 1\}$ and define $d : X \times X \to \mathbb{R}^+$ as

$$d(0, 0) = d(1/2, 1/2) = d(1, 1) = 0$$

and,

$$\begin{aligned} d(0, 1) &= 10 = d(1, 0) \\ d(0, 1/2) &= 1 = d(1/2, 0) \\ d(1, 1/2) &= 8 = d(1/2, 1). \end{aligned}$$

Then (X, d) is a complete b-metric space with $s = 10/9$. Define multivalued map $T, S : X \to P_{cl,b}(X)$ as $T(x) = 1/2$ for all $x \in X$ and

$$S(x) = \begin{cases} \{0\} & for\ x = 1; \\ \{1/2\} & for\ x = 0, 1/2. \end{cases}$$

Clearly from above we have $T \neq S$.

Case(i): When $x = 0$, $y = 1$ we have $Tx = 1/2$ and $Sy = 0$. Then $H(Tx, Sy) = 1$ and

$$\begin{aligned} \alpha_1 d(x, Tx) + \alpha_2 d(y, Sy) + \alpha_3 d(x, Sy) & + \alpha_4 d(y, Tx) + \alpha_5 \left(\frac{d(x, Sy) + d(y, Tx)}{2} \right) \\ & + \alpha_6 \frac{d(x, Tx)d(y, Sy)}{1 + d(x, y)} + \alpha_7 d(x, y) \geq 2, \end{aligned}$$

where $\alpha_5 = 1/2$ and $\alpha_i = 0$ for all $i = 1, 2, 3, 4, 6, 7$.

Case(ii): When $x = 1/2$, $y = 1$ we have $Tx = 1/2$ and $Sy = 0$. Then $H(Tx, Sy) = 1$ and

$$\begin{aligned}\alpha_1 d(x,Tx) + \alpha_2 d(y,Sy) + \quad \alpha_3 d(x,Sy) \quad &+ \alpha_4 d(y,Tx) + \alpha_5 \left(\frac{d(x,Sy) + d(y,Tx)}{2} \right) \\ &+ \alpha_6 \frac{d(x,Tx)d(y,Sy)}{1 + d(x,y)} + \alpha_7 d(x,y) \geq 9/4,\end{aligned}$$

where $\alpha_5 = 1/2$ and $\alpha_i = 0$ for all $i = 1, 2, 3, 4, 6, 7$.

Case(iii): When $x = 1$, $y = 1$ we have $Tx = 1/2$ and $Sy = 0$. Then $H(Tx, Sy) = 1$ and

$$\begin{aligned}\alpha_1 d(x,Tx) + \alpha_2 d(y,Sy) + \quad \alpha_3 d(x,Sy) \quad &+ \alpha_4 d(y,Tx) + \alpha_5 \left(\frac{d(x,Sy) + d(y,Tx)}{2} \right) \\ &+ \alpha_6 \frac{d(x,Tx)d(y,Sy)}{1 + d(x,y)} + \alpha_7 d(x,y) \geq 9/2,\end{aligned}$$

where $\alpha_5 = 1/2$ and $\alpha_i = 0$ for all $i = 1, 2, 3, 4, 6, 7$. For any other values of x and y, we have $H(Tx, Ty) = 0$. Since all conditions of Theorem 2.5 holds, so we have that T and S have unique common fixed point which is $1/2$.

Remark 2.1. For different values of α_is in the inequality (2.16), we can extend the following version of well known results of literature for multivalued mappings in the framework of b-metric space.

1. (Banach type, see [2]) There exists a number $r \in (0, \frac{1}{s})$ such that for each $x, y \in X$
$$H(Tx, Ty) \leq rd(x, y).$$
2. (Kannan type, see [16, 17]) There exists a number $r \in (0, \frac{1}{s+1})$ such that for each $x, y \in X$
$$H(Tx, Ty) \leq r(D(x, Tx) + D(y, Ty)).$$
3. (Chatterjea type, see [6]) There exists a number $r \in (0, \frac{1}{s(s+1)})$ such that for each $x, y \in X$
$$H(Tx, Ty) \leq r(D(x, Ty) + D(y, Tx)).$$
4. (Hardy and Roger type, see [14]) There exits non-negative α_i (for $i = 1, 2, 3, 4, 5$) satisfying $(\alpha_1 + \alpha_2)(s + 1) + (s^2 + s)(\alpha_3 + \alpha_4) + 2s\alpha_5 < 2$ and $\alpha_2 + \alpha_3 < \frac{1}{s}$, such that for each $x, y \in X$
$$H(Tx, Ty) \leq \alpha_1 D(x, Tx) + \alpha_2 D(y, Ty) + \alpha_3 D(x, Ty) + \alpha_4 D(y, Tx) + \alpha_5 d(x, y).$$
5. (Reich Type, see [22]) There exit non-negative α_i (for $i = 1, 2, 3, 4, 5$) satisfying $(\alpha_1 + \alpha_2)(s + 1) + 2s\alpha_3 < 2$ and $\alpha_2 < \frac{1}{s}$, such that for each $x, y \in X$
$$H(Tx, Ty) \leq \alpha_1 D(x, Tx) + \alpha_2 D(y, Ty) + \alpha_3 d(x, y).$$

3 Data Dependence of Fixed Points for Multivalued Mappings

In this section, we discuss the data dependence of fixed points for multivalued mappings in the setting of b-metric space.

Theorem 3.1. Let (X, d) be a complete b-metric space with constant $s \geq 1$, $S_1, S_2 : X \to P_{cl,b}(X)$ be two multivalued mappings satisfying the following conditions;

1. there exists $\xi > 0$ such that $H(S_1(x), S_2(x)) \leq \xi$ for all $x \in X$;
2. there exists $\alpha_{ji} \in \mathbb{R}_+$ for $j = 1, 2, .., 7$, $(\alpha_{1i} + \alpha_{2i})(s + 1) + (s^2 + s)(\alpha_{3i} + \alpha_{4i} + \alpha_{5i}) + 2\alpha_{6i} + 2s\alpha_{7i} < 2$ and $2(\alpha_{2i} + \alpha_{3i}) + \alpha_{5i} < \frac{2}{s}$ such that
$$\begin{aligned} H(S_i x, S_i y) \quad \leq \quad & \alpha_{1i} D(x, S_i x) + \alpha_{2i} D(y, S_i y) + \alpha_{3i} D(x, S_i y) + \alpha_{4i} D(y, S_i x) \\ & + \alpha_{5i} \left(\frac{D(x, S_i y) + D(y, S_i x)}{2} \right) \\ & + \alpha_{6i} \frac{D(x, S_i x) D(y, S_i y)}{1 + d(x, y)} + \alpha_{7i} d(x, y) \quad \text{for all} \quad x, y \in X, \quad i \in \{1, 2\}. \end{aligned}$$

In these conditions, we have

$$H(FixS_1, FixS_2) \leq \frac{s\xi}{1 - s\max\{k_1, k_2\}} + \frac{s^2 \max\{k_1, k_2\}}{(1 - s\max\{k_1, k_2\})^2},$$

where $k_i = \frac{\alpha_{1i}+\alpha_{2i}+s\alpha_{3i}+s\alpha_{4i}+s\alpha_{5i}+2\alpha_{7i}}{2-(\alpha_{1i}+\alpha_{2i}+s\alpha_{3i}+s\alpha_{4i}+s\alpha_{5i}+2\alpha_{6i})} < \frac{1}{s}$ for $i \in \{1, 2\}$.

Proof . We shall prove that for every $y_1^* \in FixS_1$, there exists $y_2^* \in FixS_2$ such that

$$d(y_1^*, y_2^*) \leq \frac{s\xi}{1 - sk_2} + \frac{s^2 k_2}{(1 - sk_2)^2}.$$

Let $y_1^* \in FixS_1$ be chosen arbitrarily and assume that $k_2 = \frac{\alpha_{12}+\alpha_{22}+s\alpha_{32}+s\alpha_{42}+s\alpha_{52}+2\alpha_{72}}{2-(\alpha_{12}+\alpha_{22}+s\alpha_{32}+s\alpha_{42}+s\alpha_{52}+2\alpha_{62})} < \frac{1}{s}$. On the lines of the Theorem (2.1), we can construct a sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ of successive approximations of S_2 with $y_0 = y_1^*$ and $y_1 \in S_2(y_1^*)$ with

$$d(y_n, y_{n+1}) \leq k_2^n d(y_0, y_1) + nk_2^n \qquad \text{for each} \qquad n \in \mathbb{N}.$$

If we consider that the sequence $\{y_n\}_{n\in N}$ converges to y_2^*, we have $y_2^* \in FixS_2$. Moreover, for each $n \geq 0$, we have

$$d(y_n, y_{n+p}) \leq sk_2^n\left(\frac{1-(sk_2)^p}{1-sk_2}\right)d(x_0, x_1) + nsk_2^n\left(\frac{1-(sk_2)^p}{1-sk_2}\right) + sk_2^n\sum_{t=1}^{p-1} t(sk_2)^t, \quad p \in \mathbb{N}.$$

Letting $p \to \infty$, we get

$$d(y_n, y_2^*) \leq \left(\frac{sk_2^n}{1-sk_2}\right)d(x_0, x_1) + \left(\frac{nsk_2^n}{1-sk_2}\right) + \frac{s^2 k_2^{n+1}}{(1-sk_2)^2}, \quad n \in \mathbb{N}. \tag{3.1}$$

Choosing $n = 0$ in (3.1), we get

$$\begin{aligned} d(y_1^*, y_2^*) &\leq \frac{s}{(1-sk_2)}d(y_1^*, y_1) + \frac{s^2k_2}{(1-sk_2)^2} \\ &\leq \frac{s}{(1-sk_2)}H(S_1y_1^*, S_2y_1^*) + \frac{s^2k_2}{(1-sk_2)^2} \\ &\leq \frac{s\xi}{(1-sk_2)} + \frac{s^2k_2}{(1-sk_2)^2}. \end{aligned}$$

Interchanging the role of S_1 and S_2, we have that for every $\mu \in FixS_2$, there exists $\nu \in FixS_1$ such that

$$d(\mu, \nu) \leq \frac{s\xi}{1-sk_1} + \frac{s^2k_1}{(1-sk_1)^2}, \quad \text{where} \quad k_1 = \frac{2-(\alpha_{11}+\alpha_{21}+s\alpha_{31}+s\alpha_{41}+s\alpha_{51}+2\alpha_{61})}{\alpha_{11}+\alpha_{21}+s\alpha_{31}+s\alpha_{41}+s\alpha_{51}+2\alpha_{71}} < \frac{1}{s}.$$

Thus,

$$H(FixS_1, FixS_2) \leq \frac{s\xi}{1 - s\max\{k_1, k_2\}} + \frac{s^2 \max\{k_1, k_2\}}{(1 - s\max\{k_1, k_2\})^2}.$$

□

4 Some fixed point results for the directed graph endowed with *b*-metric space

Let (X, d) be a b-metric space and Δ be the diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, $E(G)$ being the set of edges of the graph. Assuming that G has no parallel edges we will have that G can be identified with the pair $(V(G), E(G))$.

If x and y are the vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_n\}_{n\in\{0,1,2,...,k\}}$ of vertices such that $x_0 = x, x_k = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i \in \{1, 2, ..., k\}$.

Let the undirected graph obtained from G by ignoring the direction of edges be $\tilde{G}$. We know that a graph G is connected if there is a path between any two vertices and it is weakly connected if $\tilde{G}$ is connected. Suppose that the graph obtained by reversing the direction of edges be represented by G^{-1}. Thus,

$$E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}.$$

Since it is more convenient to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention, we have

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Definition 4.1. [8] Let (X,d) be a complete b-metric space with constant $s \geq 1$ and G be a directed graph. We say that the triple (X,d,G) has a property (A) if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ with $x_n \to x$, as $n \to \infty$, and $(x_n, x_{n+1}) \in E(G)$, for $n \in \mathbb{N}$, we have $(x_n, x) \in E(G)$.

In this section, we prove some fixed point results for a multivalued mapping satisfying a more general contractive of Hardy and Roger type with respect to functional H. Define a set $X_T = \{x \in X :$ there exists $y \in T(x)$ such that $(x,y) \in E(G)\}$.

Definition 4.2. Let (X,d) be a complete b-metric space with constant $s \geq 1$, G be a directed graph and $T : X \to P_b(X)$ a multivalued mapping. The mapping T is said to be a multivalued G-contraction with constant k if $0 < k < \frac{1}{s}$, where $k = \frac{\alpha_1+\alpha_2+s(\alpha_3+\alpha_4+\alpha_5)+2\alpha_7}{2-\alpha_1-\alpha_2-s(\alpha_3+\alpha_4+\alpha_5)-2\alpha_6}$ and

(a) for all $(x,y) \in E(G)$,

$$\begin{aligned} H(Tx,Ty) \leq\ & \alpha_1 D(x,Tx) + \alpha_2 D(y,Ty) + \alpha_3 D(x,Ty) + \alpha_4 D(y,Tx) + \\ & \alpha_5 \left(\frac{D(x,Ty)+D(y,Tx)}{2} \right) + \alpha_6 \frac{D(x,Tx)D(y,Ty)}{1+d(x,y)} + \alpha_7 d(x,y); \end{aligned}$$

(b) for $(x,y) \in E(G)$, if $u \in T(x)$ and $v \in T(y)$ are such that $d(u,v) \leq ad(u,v) + \alpha$, for some $\alpha > 0$, then $(u,v) \in E(G)$.

Theorem 4.3. Let (X,d) be a complete b-metric space with constant $s \geq 1$ and G be a directed graph such that the triple (X,d,G) has the property (A). If $T : X \to P_{b,cl}$ is a multivalued G-contraction defined by (4.2), then;

1. For any $x \in X_T$, $T|_{[x]_{\tilde{G}}}$ has a fixed point;
2. If $Y = \cup\{[x]_{\tilde{G}}; x \in X_T\}$, then $T|_Y$ has a fixed point in Y;
3. $Fix(T) \neq \phi$ if and only if $X_T \neq \phi$.

Proof . Let $x_0 \in X_T$, there exist $x_1 \in T(x_0)$ such that $(x_0,x_1) \in E(G)$. On the line of Theorem (2.1), we have the following result

$$H(Tx_0,Tx_1) \leq kd(x_0,x_1), \quad \text{where} \quad k = \frac{\alpha_1+\alpha_2+s\alpha_3+s\alpha_4+s\alpha_5+2\alpha_7}{2-(\alpha_1+\alpha_2+s\alpha_3+s\alpha_4+s\alpha_5+2\alpha_6)}. \tag{4.1}$$

By Lemma (1.5), for $\epsilon = k$, there exists $x_2 \in T(x_1)$ such that

$$d(x_1,x_2) \leq H(Tx_0,Tx_1) + k \leq kd(x_0,x_1) + k.$$

We have $(x_0,x_1) \in E(G)$, $x_1 \in Tx_0, x_2 \in Tx_1$ and $d(x_1,x_2) \leq kd(x_0,x_1) + k$. Using definition (4.2), we get $(x_1,x_2) \in E(G)$. Working on the lines, we get $H(Tx_1,Tx_2) \leq kd(x_1,x_2) \leq k^2 d(x_0,x_1) + k^2$. Using Lemma (1.5), for $\epsilon = k^2$, there exists $x_3 \in Tx_2$ such that

$$d(x_2,x_3) \leq H(Tx_1,Tx_2) + k^2 \leq k^2 d(x_0,x_1) + 2k^2.$$

Continuing the same process, we get $x_{n+1} \in Tx_n$ such that $(x_n, x_{n+1}) \in E(G)$ and

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1) + nk^n \quad \text{for each} \quad n \in N.$$

Now,

$$\begin{aligned} d(x_n, x_{n+p}) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + ... + s^p d(x_{n+p-1}, x_{n+p}) \\ &\leq (sk^n d(x_0, x_1) + snk^n) + (s^2 k^{n+1} d(x_0, x_1) + s^2(n+1)k^{n+1}) \\ &\quad +... + (s^p k^{n+p-1} d(x_0, x_1) + s^p(n+p-1)k^{n+p-1}) \\ &= sk^n\big(1 + (sk) + ... + (sk)^{p-1}\big)d(x_0, x_1) + nsk^n\big(1 + sk + (sk)^2 + ... + (sk)^{p-1}\big) \\ &\quad +sk^n\big((sk) + 2(sk)^2 + ... + (p-1)(sk)^{p-1}\big). \end{aligned}$$

Therefore, $d(x_n, x_{n+p}) \to 0$ if $n \to \infty$. Thus the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a complete b-metric space. Hence, there exists $x \in X$ such that $x_n \to x$, as $n \to \infty$. Therefore, from property (A) we conclude that $(x_n, x) \in E(G)$, for each $n \in \mathbb{N}$. Hence, by using the Definition (4.2) and the above property, we get

$$0 \leq \lim_{n\to\infty} H(Tx_n, Tx) \leq \lim_{n\to\infty} d(x_n, x) = 0.$$

Next, we shall prove that $x \in Tx$. We have,

$$D(x, Tx) \leq s(d(x, x_{n+1}) + D(x_{n+1}, x)) \leq s(d(x, x_{n+1}) + H(x_{n+1}, x)) \to 0, \quad \text{as} \quad n \to \infty.$$

Therefore, $D(x, Tx) = 0$, this implies $x \in Tx$. Also $(x_n, x) \in E(G)$, for $n \in \mathbb{N}$, we conclude that $(x_0, x_1, ..., x_{k_n}, x)$ is a path in G, and thus $x \in [x_0]_{\tilde{G}}$.

(2) It can be obtain as the consequences of the above case.

(3) If $FixT \neq \phi$, then there exists $x \in Tx$. As $\Delta \subset E(G)$ and we have $(x, x) \in E(G)$, thus $x \in X_T$. If $X_T \neq \phi$, using case (i) we conclude that $FixT \neq \phi$. □

Example 4.4. Let $X = \{0, 1, 2, 3, 4\}$ and define $d : X \times X \to \mathbb{R}^+$ as $d(x, y) = |x - y|$. Then (X, d) is a complete b-metric space. Define multivalued map $T : X \to P_{b,cl}$ as:

$$d(x, y) = \begin{cases} \{0\} & for\ x \in \{0, 1\}; \\ \{0, 1\} & for\ x \in \{2, 3, 4\}. \end{cases}$$

Let $V(G) = \{0, 1, 2, 3, 4\}$ and $E(G) = \{(0, 1), (0, 2), (0, 3), (0, 4)\}$, where graph $G = (V(G), E(G))$.

Case(i): If $(x, y) = (0, 1)$. Then $H(Tx, Ty) = 0$ and

$$\alpha_1 D(x, Tx) + \alpha_2 D(y, Ty) + \alpha_3 D(x, Ty) + \alpha_4 D(y, Tx) + \alpha_5\left(\frac{D(x, Ty) + D(y, Tx)}{2}\right) + \alpha_6 \frac{D(x, Tx)D(y, Ty)}{1 + d(x, y)} + \alpha_7 d(x, y) \geq 1/2,$$

for all $\alpha_i = 0$ where $i = 1, 2, 3, 5, 6, 7$ and $\alpha_4 = 1/2$.

Case(ii): If $(x, y) = (0, 2)$. Then $H(Tx, Ty) = 1$ and

$$\alpha_1 D(x, Tx) + \alpha_2 D(y, Ty) + \alpha_3 D(x, Ty) + \alpha_4 D(y, Tx) + \alpha_5\left(\frac{D(x, Ty) + D(y, Tx)}{2}\right) + \alpha_6 \frac{D(x, Tx)D(y, Ty)}{1 + d(x, y)} + \alpha_7 d(x, y) \geq 1,$$

for all $\alpha_i = 0$ where $i = 1, 2, 3, 5, 6, 7$ and $\alpha_4 = 1/2$.

Case(iii): If $(x, y) = (0, 3)$. Then $H(Tx, Ty) = 1$ and

$$\alpha_1 D(x, Tx) + \alpha_2 D(y, Ty) + \alpha_3 D(x, Ty) + \alpha_4 D(y, Tx) + \alpha_5\left(\frac{D(x, Ty) + D(y, Tx)}{2}\right) + \alpha_6 \frac{D(x, Tx)D(y, Ty)}{1 + d(x, y)} + \alpha_7 d(x, y) \geq 3/2,$$

for all $\alpha_i = 0$ where $i = 1,2,3,5,6,7$ and $\alpha_4 = 1/2$.

Case(iv): If $(x, y) = (0, 4)$. Then $H(Tx, Ty) = 1$ and

$$\alpha_1 D(x,Tx) + \alpha_2 D(y,Ty) + \alpha_3 D(x,Ty) + \alpha_4 D(y,Tx) + \alpha_5\left(\frac{D(x,Ty)+D(y,Tx)}{2}\right) + \alpha_6\frac{D(x,Tx)D(y,Ty)}{1+d(x,y)} + \alpha_7 d(x,y) \geq 2,$$

for all $\alpha_i = 0$ where $i = 1,2,3,5,6,7$ and $\alpha_4 = 1/2$. Since all conditions of Theorem 4.3 holds, so we obtain that T has a fixed point which is $x = 0$.

5 Some fixed point results for multivalued fractals in *b*-metric spaces

Let (X, d) be a b-metric space with constant $s \geq 1$ and $S_1, S_2, ..., S_m : X \to P(X)$ be multivalued mappings. The system $S = (S_1, S_2, ..., S_m)$ is called an iterated multivalued system (IMS). If $S = (S_1, S_2, ..., S_m)$ is such that $S_i : X \to P_{cp}(X), i = 1,2,...,m$ are upper semi-continuous, then the mapping T_S defined as

$$T_S(Y) = \bigcup_{i=1}^{m} S_i(Y), \quad \text{for each} \quad Y \in P_{cp}(X),$$

is called the multi-fractal mapping generated by the iterated multifunction system $S = (S_1, S_2, ..., S_m)$. Since the mappings $S_i : X \to P_{cp}(X), i = 1,2,...,m$ are upper semi continuous, then $T_S : P_{cp}(X) \to P_{cp}(X)$. A nonempty compact subset $B^* \subset X$ is said to be a multivalued fractals with respect to the iterated multifunction system $S = (S_1, S_2, ..., S_m)$ if and only if it is a fixed point for the associated multi-fractal mapping, i.e $T_S(B^*) = B^*$.

Theorem 5.1. Let (X, d) be a b-metric space with constant $s \geq 1$ and $T : X \to X$ be a self mapping defined as

$$\begin{aligned} d(Tx,Ty) \quad \leq \quad & \alpha_1 d(x,Tx) + \alpha_2 d(y,Ty) + \alpha_3 d(x,Ty) + \alpha_4 d(y,Tx) + \alpha_5\left(\frac{d(x,Ty)+d(y,Tx)}{2}\right) \\ & +\alpha_6\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + \alpha_7 d(x,y) \quad \text{for all} \quad x,y \in X, \end{aligned} \tag{5.1}$$

with constant $k \in (0, \frac{1}{s})$ where $k = \frac{\alpha_1+\alpha_2+s(\alpha_3+\alpha_4+\alpha_5)+2\alpha_7}{2-\alpha_1-\alpha_2-s(\alpha_3+\alpha_4+\alpha_5)-2\alpha_6}$. Then

1. Fix T=$\{x^*\}$;
2. For every $x \in X$ the sequence $\{T^n(x)\}_{n\in\mathbb{N}} \xrightarrow{\text{d}} x^*$, as $n \to \infty$.

Proof . The existence of the fixed point and result (2) follows from Theorem (4.3). For uniqueness assume that $x^*, y^* \in FixT$, $x^* \neq y^*$, then

$$\begin{aligned} d(x^*,y^*) &= d(Tx^*,Ty^*) \\ &\leq \alpha_1 d(x^*,Tx^*) + \alpha_2 d(y^*,Ty^*) + \alpha_3 d(x^*,Ty^*) + \alpha_4 d(y^*,Tx^*) \\ &\quad +\alpha_5\left(\frac{d(x^*,Ty^*)+d(y^*,Tx^*)}{2}\right) \\ &\quad +\alpha_6\frac{d(x^*,Tx^*)d(y^*,Ty^*)}{1+d(x^*,y^*)} + \alpha_7 d(x^*,y^*) \quad \text{for all} \quad x^*,y^* \in X. \end{aligned}$$

On solving, we get

$$\big(1 - (\alpha_3+\alpha_4+\alpha_5+\alpha_7)\big)d(x^*,y^*) \leq 0.$$

But, $\sum_{i=1}^{7} \alpha_i < 1$. Therefore, $d(x^*, y^*) = 0$ and hence $x^* = y^*$. This implies uniqueness. □

Example 5.2. Let $X = [0, 1/2]$ and $d(x, y) = |x - y|^2$. Then (X, d) is a complete b-metric space with $s = 2$. Define $T : X \to X$ as:

$$g(x) = \begin{cases} 0 & for\ x \in [0,1/4]; \\ \frac{1}{8} & for\ x \in (1/4,1/2]. \end{cases}$$

Case(i): When $x, y \in [0, 1/4]$, we have $d(Tx, Ty) = 0$ and,

$$\alpha_1 d(x,Tx) + \alpha_2 d(y,Ty) + \alpha_3 d(x,Ty) + \alpha_4 d(y,Tx) + \alpha_5\left(\frac{d(x,Ty)+d(y,Tx)}{2}\right) + \alpha_6\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + \alpha_7 d(x,y) \geq 0,$$

for all $\alpha_i = 0$ where $i = 1, 3, 4, 5, 6, 7$ and $\alpha_2 = 1/2$.

Case(ii): When $x, y \in (1/4, 1/2]$, we have $d(Tx, Ty) = 0$ and,

$$\alpha_1 d(x,Tx) + \alpha_2 d(y,Ty) + \alpha_3 d(x,Ty) + \alpha_4 d(y,Tx) + \alpha_5\left(\frac{d(x,Ty)+d(y,Tx)}{2}\right) + \alpha_6\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + \alpha_7 d(x,y) \geq 0,$$

for all $\alpha_i = 0$ where $i = 1, 3, 4, 5, 6, 7$ and $\alpha_2 = 1/2$.

Case(iii): When $x \in [0, 1/4]$ and $y \in (1/4, 1/2]$, we have $d(Tx, Ty) = 1/64$ and,

$$\alpha_1 d(x,Tx) + \alpha_2 d(y,Ty) + \alpha_3 d(x,Ty) + \alpha_4 d(y,Tx) + \alpha_5\left(\frac{d(x,Ty)+d(y,Tx)}{2}\right) + \alpha_6\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + \alpha_7 d(x,y) \geq 1/32,$$

for all $\alpha_i = 0$ where $i = 1, 3, 4, 5, 6, 7$ and $\alpha_2 = 1/2$.

Case(iv): When $x \in (1/4, 1/2]$ and $y \in [0, 1/4]$, we have $d(Tx, Ty) = 1/64$ and,

$$\alpha_1 d(x,Tx) + \alpha_2 d(y,Ty) + \alpha_3 d(x,Ty) + \alpha_4 d(y,Tx) + \alpha_5\left(\frac{d(x,Ty)+d(y,Tx)}{2}\right) + \alpha_6\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + \alpha_7 d(x,y) \geq 1/32,$$

for all $\alpha_i = 0$ where $i = 1, 3, 4, 5, 6, 7$ and $\alpha_2 = 1/2$. Since the hypothesis of Theorem 5.1 holds, thus T has fixed point which is $x = 0$.

Corollary 5.3. Let (X, d) be a complete b-metric space with constant $s \geq 1$ and Let $T : X \to X$ be self mappings satisfying the conditions

$$d(Tx,Ty) \leq \alpha\frac{d(y,Ty)d(x,Tx)}{1+d(x,y)} + \beta d(x,y) + \gamma(d(x,Tx)+d(y,Ty)) + \delta(d(y,Tx)+d(x,Ty))$$

for all $x, y \in X$, α, β, γ and $\delta \geq 0$ with $\alpha + s\beta + (s+1)\gamma + (s^2+s)\delta < 1$, $\gamma + \delta < \frac{1}{s}$. Then T has a unique fixed point.

Theorem 5.4. Let (X, d) be a complete b-metric space with constants $s \geq 1$, such that $d : X \times X \to R_+$ is continuous b-metric. Let $S_i : X \to P_{cp}(X), i = 1, .., m$, be upper semi continuous multivalued mappings of the type

$$\begin{aligned} H(S_ix, S_iy) \leq\ & \alpha_{1i}D(x,S_ix) + \alpha_{2i}D(y,S_iy) + \alpha_{3i}D(x,S_iy) + \alpha_{4i}D(y,S_ix) \\ & + \alpha_{5i}\left(\frac{D(x,S_iy)+D(y,S_ix)}{2}\right) + \alpha_{6i}\frac{D(x,S_ix)D(y,S_iy)}{1+d(x,y)} \\ & + \alpha_{7i}d(x,y) \quad \text{for all} \quad (x,y) \in X \quad \text{with constant} \quad k_i \in (0, \frac{1}{s}), \end{aligned} \tag{5.2}$$

where $k_i = \frac{\alpha_{1i}+\alpha_{2i}+s(\alpha_{3i}+\alpha_{4i}+\alpha_{5i})+2\alpha_{7i}}{2-\alpha_{1i}-\alpha_{2i}-s(\alpha_{3i}+\alpha_{4i}+\alpha_{5i})-2\alpha_{6i}}$. Then, the multivalued mapping T_S generated by the iterated multifunction system $S = (S_1, S_2, ..., S_m)$, by the relation $T_S(Y) = \bigcup_{i=1}^{m} S_i(Y)$, for each $Y \in P_{cp}(X)$, verify the following conditions:

1. $T_S : (P_{cp}(X), H) \to (P_{cp}(X), H)$;
2. T_S is a multivalued mapping of the type (5.2), in the sense that

$$\begin{aligned} H(T_S(Y_1), T_S(Y_2)) \leq\ & \alpha_1 H(Y_1, T_S(Y_1)) + \alpha_2 H(Y_2, T_S(Y_2)) + \alpha_3 H(Y_1, T_S(Y_2)) + \alpha_4 H(Y_2, T_S(Y_1)) + \\ & \alpha_5\frac{(H(Y_1,T_S(Y_2)) + H(Y_2,T_S(Y_1)))}{2} + \alpha_6\frac{H(Y_1,T_S(Y_1))H(Y_2,T_S(Y_2))}{1+H(Y_1,Y_2)} + \\ & \alpha_7 H(Y_1, Y_2); \end{aligned}$$

3. There exists a unique multivalued fractals $B^*_{T_S}$, such that $(T^n_S(B))_{n\in N} \overset{H}{\to} B^*_{T_S}$, as $n\to\infty$, for every $B \in P_{cp}(X)$.

Proof . (1) By the upper semi continuous mapping S_i, $i = 1, 2, ..., m$, we have that $T_S : (P_{cp}(X), H) \to (P_{cp}(X), H)$.

(2) We will firstly prove that for any $Y_1, Y_2 \in P_{cp}(X)$, we have

$$\begin{aligned} H(S_i(Y_1), S_i(Y_2)) \leq\ & \alpha_{1i}H(Y_1, S_i(Y_1)) + \alpha_{2i}H(Y_2, S_i(Y_2)) + \alpha_{3i}H(Y_1, S_i(Y_2)) + \alpha_{4i}H(Y_2, S_i(Y_1)) + \\ & \alpha_{5i}\left(\frac{H(Y_1, S_i(Y_2)) + H(Y_2, S_i(Y_1))}{2}\right) + \alpha_{6i}\frac{H(Y_1, S_i(Y_1))H(Y_2, S_i(Y_2))}{1 + H(Y_1, Y_2)} + \\ & \alpha_{7i}H(Y_1, Y_2) \quad \text{for all} \quad i = 1, 2, ..., m. \end{aligned}$$

For this assume that $Y_1, Y_2 \in P_{cp}(X)$. For each $i = 1, 2, ..., m$, we have

$$\begin{aligned} \rho(S_i(Y_1), S_i(Y_2)) &= \sup_{x\in Y_1} \rho(S_i(x), S_i(Y_2)) \\ &= \sup_{x\in Y_1} (\inf_{y\in Y_2} (\rho(S_i(x), S_i(y)))) \\ &\leq \sup_{x\in Y_1} (\inf_{y\in Y_2} (H(S_i(x), S_i(y)))) \\ &= \sup_{x\in Y_1} \left(\inf_{y\in Y_2} \left(\alpha_{1i}D(x, S_i(x)) + \alpha_{2i}D(y, S_i(y)) + \alpha_{3i}D(x, S_i(y)) + \alpha_{4i}D(y, S_i(x)) \right.\right. \\ &\quad \left.\left. +\alpha_{5i}\left(\frac{D(x, S_i(y)) + D(y, S_i(x))}{2}\right) + \alpha_{6i}\frac{D(x, S_i(x))D(y, S_i(y))}{1 + d(x, y)} + \alpha_{7i}d(x, y)\right)\right) \\ &= \alpha_{1i}\rho(Y_1, S_i(Y_1)) + \alpha_{2i}\rho(Y_2, S_i(Y_2)) + \alpha_{3i}\rho(Y_1, S_i(Y_2)) + \alpha_{4i}\rho(Y_2, S_i(Y_1)) \\ &\quad +\alpha_{5i}\left(\frac{\rho(Y_1, S_i(Y_2)) + \rho(Y_2, S_i(Y_1))}{2}\right) + \alpha_{6i}\frac{\rho(Y_1, S_i(Y_1))\rho(Y_2, S_i(Y_2))}{1 + H(Y_1, Y_2)} \\ &\quad +\alpha_{7i}H(Y_1, Y_2) \\ &\leq \alpha_{1i}H(Y_1, S_i(Y_1)) + \alpha_{2i}H(Y_2, S_i(Y_2)) + \alpha_{3i}H(Y_1, S_i(Y_2)) + \alpha_{4i}H(Y_2, S_i(Y_1)) \\ &\quad +\alpha_{5i}\left(\frac{H(Y_1, S_i(Y_2)) + H(Y_2, S_i(Y_1)}{2}\right) + \alpha_{6i}\frac{H(Y_1, S_i(Y_1))H(Y_2, S_i(Y_2))}{1 + H(Y_1, Y_2)} \\ &\quad +\alpha_{7i}H(Y_1, Y_2). \end{aligned}$$

Hence, for each $i = 1, 2, ..m$, we have

$$\begin{aligned} H(S_i(Y_1), S_i(Y_2)) \leq\ & \alpha_{1i}H(Y_1, S_i(Y_1)) + \alpha_{2i}H(Y_2, S_i(Y_2)) + \alpha_{3i}H(Y_1, S_i(Y_2)) + \alpha_{4i}H(Y_2, S_i(Y_1)) \\ & +\alpha_{5i}\left(\frac{H(Y_1, S_i(Y_2)) + H(Y_2, S_i(Y_1)}{2}\right) + \alpha_{6i}\frac{H(Y_1, S_i(Y_1))H(Y_2, S_i(Y_2))}{1 + H(Y_1, Y_2)} \\ & +\alpha_{7i}H(Y_1, Y_2). \end{aligned}$$

Using the property,

$$H\left(\bigcup_{i=1}^{m} S_i(Y_1), \bigcup_{i=1}^{m} S_i(Y_2)\right) \leq \max\left\{H(S_1(Y_1), S_1(Y_2)), ..., H(S_m(Y_1), S_m(Y_2))\right\},$$

we get,

$$H(T_S(Y_1), T_S(Y_2)) \leq \max_{i\in\{1,2,...,m\}} \left\{H(S_i(Y_1), S_i(Y_2))\right\},$$

$$\begin{aligned} H(T_S(Y_1), T_S(Y_2)) \leq\ & \max_{i\in\{1,2,...,m\}} \left\{\alpha_{1i}H(Y_1, S_i(Y_1)) + \alpha_{2i}H(Y_2, S_i(Y_2)) + \alpha_{3i}H(Y_1, S_i(Y_2)) \right. \\ & +\alpha_{4i}H(Y_2, S_i(Y_1)) + \alpha_{5i}\frac{(H(Y_1, S_i(Y_2)) + H(Y_2, S_i(Y_1)))}{2} \\ & \left. +\alpha_{6i}\frac{H(Y_1, S_i(Y_1))H(Y_2, S_i(Y_2))}{1 + H(Y_1, Y_2)} + \alpha_{7i}H(Y_1, Y_2)\right\}, \end{aligned}$$

$$\leq \max_{i\in\{1,2,...,m\}}(\alpha_{1i})H\Big(Y_1,\bigcup_{i=1}^m S_i(Y_1)\Big)+\max_{i\in\{1,2,...,m\}}(\alpha_{2i})H\Big(Y_2,\bigcup_{i=1}^m S_i(Y_2)\Big)$$
$$+\max_{i\in\{1,2,...,m\}}(\alpha_{3i})H\Big(Y_1,\bigcup_{i=1}^m S_i(Y_2)\Big)+\max_{i\in\{1,2,...,m\}}(\alpha_{4i})H\Big(Y_2,\bigcup_{i=1}^m S_i(Y_1)\Big)+$$
$$+\max_{i\in\{1,2,...,m\}}(\alpha_{5i})\left(\frac{H\Big(Y_1,\bigcup_{i=1}^m S_i(Y_2)\Big)+H\Big(Y_2,\bigcup_{i=1}^m S_i(Y_1)\Big)}{2}\right)$$
$$+\max_{i\in\{1,2,...,m\}}(\alpha_{6i})\frac{H\Big(Y_1,\bigcup_{i=1}^m S_i(Y_1)\Big)H\Big(Y_2,\bigcup_{i=1}^m S_i(Y_2)\Big)}{1+H(Y_1,Y_2)}$$
$$+\max_{i\in\{1,2,...,m\}}(\alpha_{7i})H(Y_1,Y_2).$$

Let $\alpha_j=\max_{i\in\{1,2,...,m\}}(\alpha_{ji})$, where $j=1,2,...,7$. Therefore, we get

$$\begin{aligned}H(T_S(Y_1),T_S(Y_2)) \leq\ & \alpha_1H(Y_1,T_S(Y_1))+\alpha_2H(Y_2,T_S(Y_2))+\alpha_3H(Y_1,T_S(Y_2))+\alpha_4H(Y_2,T_S(Y_1))\\ &+\alpha_5\frac{(H(Y_1,T_S(Y_2))+H(Y_2,T_S(Y_1)))}{2}+\alpha_6\frac{H(Y_1,T_S(Y_1))H(Y_2,T_S(Y_2))}{1+H(Y_1,Y_2)}\\ &+\alpha_7H(Y_1,Y_2).\end{aligned}$$

(3) From (2), we have T_S is a single valued mapping on the complete b-metric space $(P_{cp}(X),H)$ and by the Theorem (5.1), we get $Fix(T_S)=\{B^*_{T_S}\}$ and $T^n_S\to B^*_{T_S}$ and $n\to\infty$, for each $B\in P_{cp}(X)$. □

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