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Fractional-order Bernstein wavelets for solving stochastic fractional integro-differential equations

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Abstract

In this study, we construct the fractional-order Bernstein wavelets for solving stochastic fractional integro-differential equations. Fractional-order Bernstein wavelets and their properties are presented for the first time. The fractional integral operator of fractional-order Bernstein wavelets together with the Gaussian integration method is applied to reduce stochastic fractional integro-differential equations to the solution of algebraic equations which can be simply solved to obtain the solution of the problem. Also, an error estimation for our approach is introduced. The numerical results demonstrate that our scheme is simply applicable, efficient, powerful and very precise at the small number of basis functions.

Keywords: Fractional-order Bernstein wavelets, Fractional integro-differential equations, Fractional integral operator, Numerical method
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1 Introduction

Fractional calculus has a long history [25] and it has found numerous applications to model viscoelastic materials [3], fluid-dynamic traffic [10], solid mechanics [35], colored noise [19], signal processing [26], economics [4] and control theory [5]. So, many fractional differential and integral equations have been studied by a lot of researchers for finding their approximate solutions ([41], [28], [27]).

Recently, stochastic functional equations arises in different problems ([42], [37]), such as, biological populations [16], reactor dynamics ([17], [20]), theory of automatic systems [23] and many other fields. In many cases, most of stochastic functional equations can't be solved analytically. So, different studies have been introduced to solve these problems, such as Block pulse approximation [1], second kind Chebyshev wavelets Galerkin method [21], Galerkin method [15], finite element method [2], operational matrix method [12], Runge-Kutta method [40], wavelets Galerkin method [11], meshless method ([8], [7]) and Petrov-Galerkin method [13].

Wavelets analysis is a relatively new area in different fields of science and engineering. It is very useful in signal analysis, image processing, harmonic analysis, sampling theory, edge extrapolation, time-frequency analysis and fast algorithms [6]. Wavelets permit to represent a function at different levels of resolution. Another advantage of wavelets scheme is after discretizing, the coefficients matrix of algebraic equations is spare [9], so the computational cost is

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low. Different types of wavelets schemes for approximating the solution of the integral equations and differential equations are known such as, Haar wavelets [34], Legendre wavelets [14], Bernoulli wavelets [31], Chebyshev wavelets [18], fractional-order Bernoulli wavelets [32] and Müntz-Legendre wavelets ([29], [33]).

In this study, we propose a new numerical approach for solving the stochastic fractional integro-differential equations. Our approach is based upon the fractional-order Bernstein wavelets. First, the fractional-order Bernstein wavelets and their properties, are given. Then, the fractional integral operator for these wavelets was formed. This operator together with the Gaussian integration formula and collocation scheme are utilized to reduce the solution of the stochastic fractional integro-differential equations to the solution of algebraic equations.

The structure of this article is as: in Section 2, we recall some necessary definitions and mathematical preliminaries of fractional calculus and Brownian motion. In Section 3, we introduce the fractional-order Bernstein wavelets and some properties of them. In Section 4, we derive the Riemann-Liouville fractional integral operator for the fractional-order Bernstein wavelets. In Section 5, the numerical method for solving the stochastic fractional integro-differential equations is presented. In Section 6, we give the error bound for our scheme. In Section 7, we present our numerical results and demonstrate the efficiency and practicability of our method by considering six numerical examples. The main results are summarized in Section 8.

2 Preliminaries and notations

In this section, we recall some basic definitions and properties that are further used in this paper.

2.1 Fractional calculus

In this section, we introduce some necessary definitions and mathematical preliminaries of the fractional calculus theory.

Definition 2.1. The Riemann-Liouville fractional integrals of order ν are defined by means of [32]

$$^{R}I^{\nu}f(t)=\begin{cases}\frac{1}{\Gamma(\nu)}\int_0^t\frac{f(s)}{(t-s)^{1-\nu}}ds, & \nu>0, t>0,\\ f(t), & \nu=0.\end{cases} \tag{2.1}$$

Definition 2.2. The Caputo fractional derivatives of order ν are defined by means of [32]

$$^{C}D^{\nu}f(t)=\frac{1}{\Gamma(n-\nu)}\int_0^t(t-s)^{n-\nu-1}f^{(n)}(s)ds, \qquad n-1<\nu\le n. \tag{2.2}$$

Proposition 2.3. The Caputo fractional derivatives and Riemann-Liouville fractional integrals satisfies the following properties [32]:

1. $^{C}D^{\nu\,R}I^{\nu}f(t)=f(t)$,
2. $^{R}I^{\nu\,C}D^{\nu}f(t)=f(t)-\sum_{i=0}^{n-1}f^{(i)}(0)\frac{t^i}{i!}$,
3. $^{C}D^{\nu}(\lambda f(t)+\theta g(t))=\lambda^{C}D^{\nu}f(t)+\theta^{C}D^{\nu}g(t)$,
4. $^{C}D^{\nu}t^{\beta}=\begin{cases}0, & \nu\in N_0,\ \beta<\nu,\\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)}t^{\beta-\nu}, & \text{otherwise},\end{cases}$
5. $^{C}D^{\nu}\lambda=0$,

where λ,θ are real constants and $n-1<\nu\le n$.

2.2 Brownian motion and its properties

Definition 2.4. A real-valued stochastic process $B(t), t\in[0,T]$ is called Brownian motion, if it satisfies the following properties ([24], [39]):

- $B(t)-B(\tau)$, for $t>\tau$, is independent of the past.
- $B(t)-B(\tau)$ is normally distributed with mean 0 and variance $t-\tau$.

- $B(t), t \geq 0$, is a continuous function of t, ($B(0) = 0$ (with probability 1)).

Property 1. Suppose $f(\tau,\omega) = f(\tau)$ is continuous and bounded variation in $[0,t]$ and f is only depends on τ. Then we have ([24], [39])

$$\int_0^t f(\tau)dB_\tau = f(t)B_t - \int_0^t B_s df_s.$$

Property 2. Let $f \in \gamma(Q,T)$, then ([24], [39])

$$E\Big[\Big(\int_Q^T f(t,\omega)dB_t(\omega)\Big)^2\Big] = E\Big[\int_Q^T f^2(t,\omega)dt\Big],$$

where $\gamma = \gamma(Q,T)$ is the class of functions $f(t,\omega) : [0,\infty) \to \Delta \times R$ such that

- The function $(t,\omega) \to f(t,\omega)$ is $B \times F-$measurable, where B denotes the Borel algebra on the interval $[0,\infty)$ and F is the $\sigma-$algebra on Δ.
- f is adapted to F_t, where F_t is the $\sigma-$algebra generated by the random variables $B(s), s \leq t$.
- $E[\int_Q^T f^2(t,\omega)dt] < \infty$.

3 Fractional-order Bernstein wavelets

In this section, first we review the definition of the Bernstein wavelets, then we introduce the fractional-order Bernstein wavelets and some their properties.

3.1 Bernstein wavelets

The Bernstein wavelets $\psi_{n,i,m}(t) = \psi(k,\hat{n},i,t)$ have four arguments: $\hat{n} = n-1; n = 1,2,\ldots,2^{k-1}$, k can assume any positive integer, i is the order of Bernstein polynomials and t is the normalized time. They are defined on the interval $[0,1)$ as follows

$$\psi_{n,i,m}(t) = \begin{cases} 2^{\frac{k-1}{2}}\beta_{i,m}B_{i,m}(2^{k-1}t-\hat{n}), & \frac{\hat{n}}{2^{k-1}} \leq t < \frac{\hat{n}+1}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

with

$$\beta_{i,m} = \frac{\sqrt{(2m+1)\begin{pmatrix} 2m \\ 2i \end{pmatrix}}}{\begin{pmatrix} m \\ i \end{pmatrix}},$$

where $i = 0,1,\ldots,m; m = M-1$, $\hat{m} = 2^{k-1}M$ and $B_{i,m}(t)$ denotes the Bernstein polynomials defined on the interval $[0,1]$, by [22]:

$$B_{i,m}(t) = \begin{pmatrix} m \\ i \end{pmatrix} t^i(1-t)^{m-i} = \sum_{j=0}^{m-i} \begin{pmatrix} m \\ i \end{pmatrix}\begin{pmatrix} m-i \\ j \end{pmatrix}(-1)^{m-i-j}t^{m-j}, \quad 0 \leq i \leq m. \tag{3.2}$$

These polynomials satisfy the following condition [22]:

$$\int_0^1 B_{i,m}(t)B_{j,n}(t)dt = \frac{\begin{pmatrix} m \\ i \end{pmatrix}\begin{pmatrix} n \\ j \end{pmatrix}}{(m+n+1)\begin{pmatrix} m+n \\ i+j \end{pmatrix}}.$$

3.2 Fractional-order Bernstein wavelets

We introduce the fractional-order Bernstein wavelets on the interval $[0,1)$ as

$$\psi_{n,i,m}^{(\alpha)}(t)=\begin{cases} 2^{\frac{k-1}{2}}\beta_{i,m}^{(\alpha)}B_{i,m}^{(\alpha)}(2^{k-1}t-\hat{n}), & \frac{\hat{n}}{2^{k-1}}\leq t<\frac{\hat{n}+1}{2^{k-1}},\\ 0, & \text{otherwise,}\end{cases} \tag{3.3}$$

with

$$\beta_{i,m}^{(\alpha)}=\frac{\sqrt{\alpha(2m+1)\begin{pmatrix} 2m\\ 2i\end{pmatrix}}}{\begin{pmatrix} m\\ i\end{pmatrix}},$$

where $B_{i,m}^{(\alpha)}(t)$ denotes the fractional-order Bernstein functions on the interval $[0,1]$. These functions are defined as:

$$B_{i,m}^{(\alpha)}(t)=\sum_{j=0}^{m-i}\begin{pmatrix} m\\ i\end{pmatrix}\begin{pmatrix} m-i\\ j\end{pmatrix}(-1)^{m-i-j}t^{(m-j)\alpha}, \qquad 0\leq i\leq m. \tag{3.4}$$

These functions satisfy the following condition:

$$\int_0^1 B_{i,m}^{(\alpha)}(t)B_{j,n}^{(\alpha)}(t)t^{\alpha-1}dt=\frac{\begin{pmatrix} m\\ i\end{pmatrix}\begin{pmatrix} n\\ j\end{pmatrix}}{\alpha(m+n+1)\begin{pmatrix} m+n\\ i+j\end{pmatrix}}.$$

A function f belong to $L^2[0,1)$ can be expanded by the fractional-order Bernstein wavelets in the form

$$f(t)=\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}c_{n,i}\psi_{n,i,m}^{(\alpha)}(t). \tag{3.5}$$

Truncated series for f in Eq. (3.5) is

$$f(t)\simeq\sum_{n=1}^{2^{k-1}}\sum_{i=0}^{M-1}c_{n,i}\psi_{n,i,m}^{(\alpha)}(t)=\mathcal{C}^T\Psi^{(\alpha)}(t), \tag{3.6}$$

where

$$\mathcal{C}=[c_{1,0},c_{1,1},c_{1,2},\ldots,c_{2^{k-1},M-1}]^T, \tag{3.7}$$

and

$$\Psi^{(\alpha)}(t)=[\psi_{1,0,m}^{(\alpha)}(t),\psi_{1,1,m}^{(\alpha)}(t),\psi_{1,2,m}^{(\alpha)}(t),\ldots,\psi_{2^{k-1},M-1,m}^{(\alpha)}(t)]^T. \tag{3.8}$$

4 Fractional integral operator of the fractional-order Bernstein wavelets

The fractional integral operator ${}^RI^{\nu}$ for $\Psi^{(\alpha)}(t)$ in Eq. (3.8) is given by

$${}^RI^{\nu}\Psi^{(\alpha)}(t)=R(\alpha,\nu,t), \tag{4.1}$$

where

$$R(\alpha,\nu,t)=[{}^RI^{\nu}\psi_{1,0,m}^{(\alpha)}(t),{}^RI^{\nu}\psi_{1,1,m}^{(\alpha)}(t),{}^RI^{\nu}\psi_{1,2,m}^{(\alpha)}(t),\ldots,{}^RI^{\nu}\psi_{2^{k-1},M-1,m}^{(\alpha)}(t)]^T. \tag{4.2}$$

For this aim, we use the Laplace transform. From Eq. (3.3), yield

$$\psi_{n,i,m}^{(\alpha)}(t)=\mu_{\frac{\hat{n}}{2^{k-1}}}(t)\delta_{i,m,k}^{(\alpha)}B_{i,m}^{(\alpha)}(2^{k-1}t-\hat{n})-\mu_{\frac{\hat{n}+1}{2^{k-1}}}(t)\delta_{i,m,k}^{(\alpha)}B_{i,m}^{(\alpha)}(2^{k-1}t-\hat{n}), \tag{4.3}$$

where $\mu_c(t)$ is unit step function defined as

$$\mu_c(t) = \begin{cases} 1, & t \geq c, \\ 0, & t < c, \end{cases}$$

and

$$\delta^{(\alpha)}_{i,m,k} = 2^{\frac{k-1}{2}} \beta^{(\alpha)}_{i,m}.$$

Then we use the Laplace transform to both sides of Eq. (4.3):

$$\begin{aligned}
\ell[\psi^{(\alpha)}_{n,i,m}] &= \delta^{(\alpha)}_{i,m,k}\ell[\mu_{\frac{\hat{n}}{2^{k-1}}}(t)B^{(\alpha)}_{i,m}(2^{k-1}t-\hat{n})] - \delta^{(\alpha)}_{i,m,k}\ell[\mu_{\frac{\hat{n}+1}{2^{k-1}}}(t)B^{(\alpha)}_{i,m}(2^{k-1}t-\hat{n})] \\
&= \delta^{(\alpha)}_{i,m,k}e^{-\frac{\hat{n}}{2^{k-1}}s}\ell[B^{(\alpha)}_{i,m}(2^{k-1}t)] - \delta^{(\alpha)}_{i,m,k}e^{-\frac{\hat{n}+1}{2^{k-1}}s}\ell[B^{(\alpha)}_{i,m}(2^{k-1}t+1)] \\
&= \delta^{(\alpha)}_{i,m,k}e^{-\frac{\hat{n}}{2^{k-1}}s}\ell\Big[\sum_{j=0}^{m-i}\binom{m}{i}\binom{m-i}{j}(-1)^{m-i-j}2^{(k-1)(m-j)\alpha}t^{(m-j)\alpha}\Big] \\
&- \delta^{(\alpha)}_{i,m,k}e^{-\frac{\hat{n}+1}{2^{k-1}}s}\ell\Big[\sum_{j=0}^{m-i}\sum_{\tau=0}^{\lfloor(m-j)\alpha\rfloor}\binom{m}{i}\binom{m-i}{j}\binom{(m-j)\alpha}{\tau} \\
&\quad (-1)^{m-i-j}2^{(k-1)((m-j)\alpha-\tau)}t^{((m-j)\alpha-\tau)}\Big] = \delta^{(\alpha)}_{i,m,k}e^{-\frac{\hat{n}}{2^{k-1}}s} \\
&\quad \sum_{j=0}^{m-i}\binom{m}{i}\binom{m-i}{j}(-1)^{m-i-j}2^{(k-1)(m-j)\alpha}\frac{\Gamma((m-j)\alpha+1)}{s^{(m-j)\alpha+1}} \\
&- \delta^{(\alpha)}_{i,m,k}e^{-\frac{\hat{n}+1}{2^{k-1}}s}\sum_{j=0}^{m-i}\sum_{\tau=0}^{\lfloor(m-j)\alpha\rfloor}\binom{m}{i}\binom{m-i}{j}\binom{(m-j)\alpha}{\tau} \\
&\quad (-1)^{m-i-j}2^{(k-1)((m-j)\alpha-\tau)}\frac{\Gamma((m-j)\alpha-\tau+1)}{s^{(m-j)\alpha-\tau+1}}.
\end{aligned} \tag{4.4}$$

By using properties of fractional integral and Laplace transform, we have

$$\begin{aligned}
\ell[{}^RI^\nu\psi^{(\alpha)}_{n,i,m}(t)] &= \ell\Big[\frac{t^{\nu-1}}{\Gamma(\nu)}*\psi^{(\alpha)}_{n,i,m}(t)\Big] = \delta^{(\alpha)}_{i,m,k}e^{-\frac{\hat{n}}{2^{k-1}}s}\sum_{j=0}^{m-i}\binom{m}{i}\binom{m-i}{j}(-1)^{m-i-j}2^{(k-1)(m-j)\alpha} \\
&\quad \frac{\Gamma((m-j)\alpha+1)}{s^{(m-j)\alpha+\nu+1}} - \delta^{(\alpha)}_{i,m,k}e^{-\frac{\hat{n}+1}{2^{k-1}}s}\sum_{j=0}^{m-i}\sum_{\tau=0}^{\lfloor(m-j)\alpha\rfloor}\binom{m}{i} \\
&\quad \binom{m-i}{j}\binom{(m-j)\alpha}{\tau}(-1)^{m-i-j}2^{(k-1)((m-j)\alpha-\tau)} \\
&\quad \frac{\Gamma((m-j)\alpha-\tau+1)}{s^{((m-j)\alpha-\tau+\nu+1)}}.
\end{aligned} \tag{4.5}$$

By taking the inverse Laplace transform from Eq. (4.5), we have

$$\begin{aligned}
{}^RI^\nu\psi^{(\alpha)}_{n,i,m}(t) &= \delta^{(\alpha)}_{i,m,k}\sum_{j=0}^{m-i}\binom{m}{i}\binom{m-i}{j}(-1)^{m-i-j}2^{(k-1)(m-j)\alpha} \\
&\quad \frac{\Gamma((m-j)\alpha+1)}{\Gamma((m-j)\alpha+\nu+1)}(t-\frac{\hat{n}}{2^{k-1}})^{(m-j)\alpha+\nu}\mu_{\frac{\hat{n}}{2^{k-1}}}(t) \\
&- \delta^{(\alpha)}_{i,m,k}\sum_{j=0}^{m-i}\sum_{\tau=0}^{\lfloor(m-j)\alpha\rfloor}\binom{m}{i}\binom{m-i}{j}\binom{(m-j)\alpha}{\tau} \\
&\quad (-1)^{m-i-j}2^{(k-1)((m-j)\alpha-\tau)}\frac{\Gamma((m-j)\alpha-\tau+1)}{\Gamma((m-j)\alpha-\tau+\nu+1)} \\
&\quad (t-\frac{\hat{n}+1}{2^{k-1}})^{(m-j)\alpha-\tau+\nu}\mu_{\frac{\hat{n}+1}{2^{k-1}}}(t).
\end{aligned} \tag{4.6}$$

By considering, relation (4.6), we get

$$
{}^{R}I^{\nu}\psi^{(\alpha)}_{n,i,m}(t)=\begin{cases}0, & 0\leq t<\frac{\hat{n}}{2^{k-1}},\\ \delta^{(\alpha)}_{i,m,k}\sum_{j=0}^{m-i}\begin{pmatrix} m\\ i\end{pmatrix}\begin{pmatrix} m-i\\ j\end{pmatrix}(-1)^{m-i-j}2^{(k-1)(m-j)\alpha} & \\ \frac{\Gamma((m-j)\alpha+1)}{\Gamma((m-j)\alpha+\nu+1)}(t-\frac{\hat{n}}{2^{k-1}})^{(m-j)\alpha+\nu}, & \frac{\hat{n}}{2^{k-1}}\leq t<\frac{\hat{n}+1}{2^{k-1}},\\ & \\ \delta^{(\alpha)}_{i,m,k}\sum_{j=0}^{m-i}\begin{pmatrix} m\\ i\end{pmatrix}\begin{pmatrix} m-i\\ j\end{pmatrix}(-1)^{m-i-j}2^{(k-1)(m-j)\alpha} & \\ \frac{\Gamma((m-j)\alpha+1)}{\Gamma((m-j)\alpha+\nu+1)}(t-\frac{\hat{n}}{2^{k-1}})^{(m-j)\alpha+\nu}-\delta^{(\alpha)}_{i,m,k}\sum_{j=0}^{m-i}\sum_{\tau=0}^{\lfloor (m-j)\alpha\rfloor}\begin{pmatrix} m\\ i\end{pmatrix} & \\ \begin{pmatrix} m-i\\ j\end{pmatrix}\begin{pmatrix} (m-j)\alpha\\ \tau\end{pmatrix}(-1)^{m-i-j}2^{(k-1)((m-j)\alpha-\tau)} & \\ \frac{\Gamma((m-j)\alpha-\tau+1)}{\Gamma((m-j)\alpha-\tau+\nu+1)}(t-\frac{\hat{n}+1}{2^{k-1}})^{(m-j)\alpha-\tau+\nu}, & \frac{\hat{n}+1}{2^{k-1}}\leq t<1.\end{cases} \tag{4.7}
$$

5 Description of the numerical technique

In this section, we apply the fractional-order Bernstein wavelets fractional integral operator in Eq. (4.1) for solving the stochastic fractional integro-differential equation as

$$
{}^{C}D^{\nu}u(t)=\mathcal{G}(t)+\int_0^t k_1(s,t)u(s)ds+\int_0^t k_2(s,t)u(s)dB(s), \tag{5.1}
$$

with the initial conditions

$$
u^{(i)}(0)=u_i, \qquad i=0,1,\ldots,n-1; n-1<\nu\leq n. \tag{5.2}
$$

Using property 1 yields

$$
{}^{C}D^{\nu}u(t)=\mathcal{G}(t)+k_2(t,t)u(t)B(t)+\int_0^t k_1(s,t)u(s)ds-\int_0^t \mathcal{K}_2(s,t,u(s))B(s)ds, \tag{5.3}
$$

where

$$
\mathcal{K}_2(s,t,u(s))=\tfrac{\partial}{\partial s}(k_2(s,t)u(s)).
$$

To solve problem (5.3), we assume

$$
{}^{C}D^{\nu}u(t)\simeq {}^{C}D^{\nu}(P^{2^{k-1}}_{M-1}u(t))=C^T\Psi^{(\alpha)}(t). \tag{5.4}
$$

From Eqs. (4.1), (5.2) and (5.4), we obtain:

$$
u(t)\simeq P^{2^{k-1}}_{M-1}u(t)=C^TR(\alpha,\nu,t)+\sum_{i=0}^{n-1}u_i\frac{t^i}{i!}. \tag{5.5}
$$

By substituting Eqs. (5.4) and (5.5) in Eq. (5.3), we have

$$
\begin{aligned}
C^T\Psi^{(\alpha)}(t) &= \mathcal{G}(t)+k_2(t,t)(C^TR(\alpha,\nu,t)+\sum_{i=0}^{n-1}u_i\frac{t^i}{i!})B(t)\\
&+ \int_0^t k_1(s,t)(C^TR(\alpha,\nu,s)+\sum_{i=0}^{n-1}u_i\frac{s^i}{i!})ds\\
&- \int_0^t \mathcal{K}_2(s,t,C^TR(\alpha,\nu,s)+\sum_{i=0}^{n-1}u_i\frac{s^i}{i!})B(s)ds+Res(t),
\end{aligned} \tag{5.6}
$$

where $Res(t), t \in [0,1)$ is residual error.
For using the Gauss-Legendre numerical integration, we transmit the interval $[0,t]$ to $[-1,1]$ as

$$\begin{aligned}
C^T\Psi^{(\alpha)}(t) &= \mathcal{G}(t) + k_2(t,t)(C^T R(\alpha,\nu,t) + \sum_{i=0}^{n-1} u_i \frac{t^i}{i!})B(t) \\
&+ \frac{t}{2}\int_{-1}^{1} k_1(\frac{t}{2}+\frac{t}{2}x,t)(C^T R(\alpha,\nu,\frac{t}{2}+\frac{t}{2}x) + \sum_{i=0}^{n-1} u_i \frac{(\frac{t}{2}+\frac{t}{2}x)^i}{i!})dx \\
&- \frac{t}{2}\int_{-1}^{1} \mathcal{K}_2(\frac{t}{2}+\frac{t}{2}x,t,C^T R(\alpha,\nu,\frac{t}{2}+\frac{t}{2}x) + \sum_{i=0}^{n-1} u_i \frac{(\frac{t}{2}+\frac{t}{2}x)^i}{i!}) \\
&\quad B(\frac{t}{2}+\frac{t}{2}x)dx + Res(t). \qquad (5.7)
\end{aligned}$$

By applying the Gauss-Legendre numerical integration [38] in Eq. (5.7), we deduce

$$\begin{aligned}
C^T\Psi^{(\alpha)}(t) &= \mathcal{G}(t) + k_2(t,t)(C^T R(\alpha,\nu,t) + \sum_{i=0}^{n-1} u_i \frac{t^i}{i!})B(t) \\
&+ \frac{t}{2}\sum_{j=1}^{\tilde{n}} \omega_j k_1\Big[(\frac{t}{2}+\frac{t}{2}\tau_j,t)(C^T R(\alpha,\nu,\frac{t}{2}+\frac{t}{2}\tau_j) \\
&+ \sum_{i=0}^{n-1} u_i \frac{(\frac{t}{2}+\frac{t}{2}\tau_j)^i}{i!})\Big] - \frac{t}{2}\sum_{j=1}^{\tilde{n}} \omega_j \Big[\mathcal{K}_2(\frac{t}{2}+\frac{t}{2}\tau_j,t,C^T R(\alpha,\nu,\frac{t}{2}+\frac{t}{2}\tau_j) \\
&+ \sum_{i=0}^{n-1} u_i \frac{(\frac{t}{2}+\frac{t}{2}\tau_j)^i}{i!})B(\frac{t}{2}+\frac{t}{2}\tau_j)\Big] + Res(t), \qquad (5.8)
\end{aligned}$$

where $\tau_j, j = 1,2,\ldots\tilde{n}$ are zeros of the Legendre polynomial $P_{\tilde{n}}$ and $\omega_j = \frac{-2}{(\tilde{n}+1)P'_{\tilde{n}}(\tau_j)P_{\tilde{n}+1}(\tau_j)}$. By collocating [36] Eq. (5.8) at the zeros of the shifted Legendre polynomials $L_{2^{k-1}M}; (t_i; i = 1,2,\ldots,2^{k-1}M)$ we have:

$$\begin{aligned}
C^T\Psi^{(\alpha)}(t_i) &= \mathcal{G}(t_i) + k_2(t_i,t_i)(C^T R(\alpha,\nu,t_i) + \sum_{s=0}^{n-1} u_s \frac{t_i^s}{s!})B(t_i) \\
&+ \frac{t_i}{2}\sum_{j=1}^{\tilde{n}} \omega_j k_1\Big[(\frac{t_i}{2}+\frac{t_i}{2}\tau_j,t_i)(C^T R(\alpha,\nu,\frac{t_i}{2}+\frac{t_i}{2}\tau_j) \\
&+ \sum_{s=0}^{n-1} u_s \frac{(\frac{t_i}{2}+\frac{t_i}{2}\tau_j)^s}{s!})\Big] - \frac{t_i}{2}\sum_{j=1}^{\tilde{n}} \omega_j \Big[\mathcal{K}_2(\frac{t_i}{2}+\frac{t_i}{2}\tau_j,t_i,C^T R(\alpha,\nu,\frac{t_i}{2}+\frac{t_i}{2}\tau_j) \\
&+ \sum_{s=0}^{n-1} u_s \frac{(\frac{t_i}{2}+\frac{t_i}{2}\tau_j)^s}{s!})B(\frac{t_i}{2}+\frac{t_i}{2}\tau_j)\Big] + Res(t_i). \qquad (5.9)
\end{aligned}$$

Now, we can use the Newton's iterative scheme for finding the unknown coefficients $c_i; i = 1,2,\ldots,2^{k-1}M$.
For finding the values of B at points $\frac{t_i}{2}+\frac{t_i}{2}\tau_j; i = 1,2,\ldots,2^{k-1}M; j = 1,2,\ldots,\tilde{n}$, we use the Definition 3 in which the Brownian motion has normal distribution as

$$B(t) - B(\tau) \sim \sqrt{t-\tau}\mathcal{N}(0,1),$$

where $t > \tau$. Also, we let $\Delta\zeta = \frac{1}{T}$, $\zeta_j = j\Delta\zeta$ and $B_0 = 0$, from Definition 3, we have

$$B_j = B_{j-1} + dB_j, \qquad j = 0,1,\ldots,T,$$

where dB_j is an independent random variable of the form $\sqrt{\Delta\zeta}\mathcal{N}(0,1)$. For finding approximate function for $B(\zeta)$ we use linear spline interpolation at points $(\zeta_j, B_j); j = 0,1,\ldots,T$.

6 Error bounds

In this section, error bound of our scheme in the sense of Sobolev norms is given. The Sobolev norm of integer order $\mu \geq 0$ defined on the interval (a,b) by [30]

$$\|u\|_{H^{\mu}(a,b)} = \left(\sum_{j=0}^{\mu}\int_a^b |u^{(j)}(t)|dt\right)^{\frac{1}{2}} = \left(\sum_{j=0}^{\mu}\|u^{(j)}(t)\|^2_{L^2(a,b)}\right)^{\frac{1}{2}}, \tag{6.1}$$

where $u^{(j)}$ denotes the distributional derivative of u of order j. The symbol $|u|_{H^{\mu;M}(0,1)}$ was defined by [30], as

$$|u|_{H^{\mu;M}(0,1)} = \left(\sum_{j=min(\mu,M+1)}^{\mu}\|u^{(j)}(t)\|^2_{L^2(0,1)}\right)^{\frac{1}{2}}. \tag{6.2}$$

The seminorm is [30]

$$|u|_{H^{s,\mu;M;N}(0,1)} = \left(\sum_{j=min(\mu,M+1)}^{\mu} N^{2s-2j}\|y^{(j)}(t)\|^2_{L^2(0,1)}\right)^{\frac{1}{2}}. \tag{6.3}$$

Theorem 6.1. Suppose $u \in H^{\mu}(0,1)$ with $\mu \geq 0$, $M \geq \mu$ and $P^{2^{k-1}}_{M-1}u$ is the best approximation of u, then we obtain

$$\|u - P^{2^{k-1}}_{M-1}u\|_{L^2(0,1)} \leq c(M-1)^{-\mu}2^{-(k-1)\mu}\|u^{(\mu)}\|_{L^2(0,1)}, \tag{6.4}$$

and for $s \geq 1$ we have

$$\|u - P^{2^{k-1}}_{M-1}u\|_{H^s(0,1)} \leq c(M-1)^{2s-\frac{1}{2}-\mu}2^{(k-1)(s-\mu)}\|u^{(\mu)}\|_{L^2(0,1)}. \tag{6.5}$$

Proof . Similar to [30]. □

Theorem 6.2. Suppose $u \in H^{\mu}(0,1)$ with $\mu \geq 0, M \geq \mu$ and $n-1 < \nu \leq n$, then

$$\begin{aligned} & \|{}^C D^{\nu}u - {}^C D^{\nu}(P^{2^{k-1}}_{M-1}u)\|_{L^2(0,1)} \\ \leq\ & \frac{c}{\Gamma(n-\nu+1)}(M-1)^{2s-\frac{1}{2}-\mu}2^{(s-\mu)(k-1)}\|u^{(\mu)}\|_{L^2(0,1)}, \end{aligned} \tag{6.6}$$

where $1 \leq s < \mu$.

Proof . Similar to [30]. □

Theorem 6.3. Suppose $u \in H^{\mu}(0,1)$ with $\mu \geq 0, M \geq \mu$ and $n-1 < \nu \leq n$, then the error bound of the suggested approach is obtained by

$$\begin{aligned} \|E^{M-1}_{2^{k-1}}\|_{L^2(0,1)} \leq\ & \frac{c}{\Gamma(n-\nu+1)}(M-1)^{2s-\frac{1}{2}-\mu}2^{(s-\mu)(k-1)}\|u^{(\mu)}\|_{L^2(0,1)} \\ +\ & c\mathcal{M}_1(M-1)^{-\mu}2^{-(k-1)\mu}\|u^{(\mu)}\|_{L^2(0,1)} \\ +\ & c\vartheta^*\mathcal{M}_2\mathcal{M}_3(M-1)^{-\mu}2^{-(k-1)\mu}\|u^{(\mu)}\|_{L^2(0,1)}, \end{aligned} \tag{6.7}$$

where

$$\|k_1\|_2 \leq \mathcal{M}_1, \quad sup_{0\leq s,t\leq 1}|k_2(s,t)| = \mathcal{M}_2, \quad sup_{0\leq t\leq 1}|B(t)| = \mathcal{M}_3.$$

Proof . We know

$$\begin{aligned} \|E^{M-1}_{2^{k-1}}\|_{L^2(0,1)} =\ & \|\mathcal{G}(t) - {}^C D^{\nu}P^{2^{k-1}}_{M-1}u(t) + \int_0^t k_1(s,t)P^{2^{k-1}}_{M-1}u(s)ds \\ +\ & \int_0^t k_2(s,t)P^{2^{k-1}}_{M-1}u(s)dB(s)\|_{L^2(0,1)}. \end{aligned} \tag{6.8}$$

By using Eq. (5.1), we get

$$
\begin{aligned}
\|E_{2^{k-1}}^{M-1}\|_{L^2(0,1)} &= \Big\| {}^C D^\nu u(t) - \int_0^t k_1(s,t)u(s)ds - \int_0^t k_2(s,t)u(s)dB(s) \\
&- {}^C D^\nu P_{M-1}^{2^{k-1}} u(t) + \int_0^t k_1(s,t)P_{M-1}^{2^{k-1}} u(s)ds \\
&+ \int_0^t k_2(s,t)P_{M-1}^{2^{k-1}} u(s)dB(s) \Big\|_{L^2(0,1)}.
\end{aligned} \tag{6.9}
$$

Now, we can write

$$
\begin{aligned}
\|E_{2^{k-1}}^{M-1}\|_{L^2(0,1)} &\leq \|{}^C D^\nu u(t) - {}^C D^\nu P_{M-1}^{2^{k-1}} u(t)\|_{L^2(0,1)} \\
&+ \Big\| \int_0^t k_1(s,t)(u(s) - P_{M-1}^{2^{k-1}} u(s))ds \Big\|_{L^2(0,1)} \\
&+ \Big\| \int_0^t k_2(s,t)(u(s) - P_{M-1}^{2^{k-1}} u(s))dB(s) \Big\|_{L^2(0,1)}.
\end{aligned} \tag{6.10}
$$

From Eq. (6.4), we have

$$
\begin{aligned}
\Big\| \int_0^t k_1(s,t)(u(s) - P_{M-1}^{2^{k-1}} u(s))ds \Big\|_{L^2(0,1)} &\leq \|k_1\|_2 \|u - P_{M-1}^{2^{k-1}} u\|_{L^2(0,1)} \\
&\leq c\mathcal{M}_1 (M-1)^{-\mu} 2^{-(k-1)\mu} \|u^{(\mu)}\|_{L^2(0,1)}.
\end{aligned} \tag{6.11}
$$

Also, we achieve

$$
sup\Big| \int_0^1 k_2(s,t)(u(s) - P_{M-1}^{2^{k-1}} u(s))dB(s) \Big| \leq \mathcal{M}_2 \mathcal{M}_3 sup|u(t) - P_{M-1}^{2^{k-1}} u(t)|. \tag{6.12}
$$

Using Eq. (6.12), we get

$$
\Big\| \int_0^1 k_2(s,t)(u(s) - P_{M-1}^{2^{k-1}} u(s))dB(s) \Big\|_\infty \leq \mathcal{M}_2 \mathcal{M}_3 \|u - P_{M-1}^{2^{k-1}} u\|_\infty. \tag{6.13}
$$

From Eqs. (6.4) and (6.13) we yield

$$
\begin{aligned}
\Big\| \int_0^t k_2(s,t)(u(s) - P_{M-1}^{2^{k-1}} u(s))dB(s) \Big\|_{L^2(0,1)} &\leq \Big\| \int_0^1 k_2(s,t)(u(s) - P_{M-1}^{2^{k-1}} u(s))dB(s) \Big\|_{L^2(0,1)} \\
&\leq \vartheta \Big\| \int_0^1 k_2(s,t)(u(s) - P_{M-1}^{2^{k-1}} u(s))dB(s) \Big\|_\infty \\
&\leq \vartheta \mathcal{M}_2 \mathcal{M}_3 \|u - P_{M-1}^{2^{k-1}} u\|_\infty \\
&\leq \vartheta^* \mathcal{M}_2 \mathcal{M}_3 \|u - P_{M-1}^{2^{k-1}} u\|_{L^2(0,1)} \\
&\leq c\vartheta^* \mathcal{M}_2 \mathcal{M}_3 (M-1)^{-\mu} 2^{-(k-1)\mu} \|u^{(\mu)}\|_{L^2(0,1)}.
\end{aligned} \tag{6.14}
$$

By using the Eqs. (6.6), (6.10), (6.11) and (6.14), we obtain

$$
\begin{aligned}
\|E_{2^{k-1}}^{M-1}\|_{L^2(0,1)} &\leq \frac{c}{\Gamma(n-\nu+1)} (M-1)^{2s-\frac{1}{2}-\mu} 2^{(s-\mu)(k-1)} \|u^{(\mu)}\|_{L^2(0,1)} \\
&+ c\mathcal{M}_1 (M-1)^{-\mu} 2^{-(k-1)\mu} \|u^{(\mu)}\|_{L^2(0,1)} \\
&+ c\vartheta^* \mathcal{M}_2 \mathcal{M}_3 (M-1)^{-\mu} 2^{-(k-1)\mu} \|u^{(\mu)}\|_{L^2(0,1)},
\end{aligned} \tag{6.15}
$$

this complete the proof. □

7 Illustrative test problems

In this section, six examples are given to show the efficiency and the practicability of our scheme. The computations associated with the examples were performed using Mathematica 10.

Example 7.1. Consider the stochastic fractional integro-differential equation given by

$$^{C}D^{\nu}u(t)=\frac{\Gamma(2)t^{1-\nu}}{\Gamma(2-\nu)}-\frac{t^3}{3}+\int_0^t su(s)ds+\int_0^t u(s)dB(s), \tag{7.1}$$

with the initial condition

$$u(0)=0. \tag{7.2}$$

The residual error for different values of t,ν and $k=1, M=11$ by using our scheme with $\alpha=\frac{1}{2}$ are compared with results of the shifted Legendre polynomials scheme [39] in Table 1. The values of residual error for $k=2, M=10$, $\nu=\frac{1}{2}$ with various values of α are reported in Table 2. Also, Fig. 1 shows the numerical results of our scheme for $k=2, M=11$ and $\alpha=0.5$ with $\nu=0.15, 0.25, 0.35, 0.50$.

Table 1: The comparison of residual error for $k=1, M=11$ and $\alpha=\frac{1}{2}$ with Ref. [39] for Example 7.1.

t	$Ref.$[39]			$Present\ method$		
	$\nu=0.25$	$\nu=0.5$	$\nu=0.75$	$\nu=0.25$	$\nu=0.5$	$\nu=0.75$
0.2	6.17×10^{-3}	7.93×10^{-3}	1.03×10^{-2}	2.66×10^{-7}	3.48×10^{-9}	1.27×10^{-6}
0.4	1.06×10^{-2}	1.12×10^{-2}	1.30×10^{-2}	3.83×10^{-8}	4.50×10^{-10}	1.18×10^{-7}
0.6	1.25×10^{-2}	1.28×10^{-2}	1.32×10^{-2}	1.31×10^{-8}	1.47×10^{-10}	3.15×10^{-8}
0.8	1.31×10^{-3}	1.64×10^{-3}	1.99×10^{-3}	8.00×10^{-9}	8.66×10^{-11}	1.61×10^{-8}

Table 2: The comparison of residual error with $k=2, M=10$ and $\nu=\frac{1}{2}$ for various values of α for Example 7.1.

t	$\alpha=\frac{1}{4}$	$\alpha=\frac{1}{3}$	$\alpha=\frac{1}{2}$	$\alpha=\frac{3}{4}$	$\alpha=1$
0.1	2.35×10^{-3}	7.29×10^{-4}	1.10×10^{-6}	1.43×10^{-4}	1.47×10^{-3}
0.3	2.59×10^{-5}	1.19×10^{-5}	3.85×10^{-8}	7.29×10^{-6}	1.86×10^{-4}
0.5	1.09×10^{-5}	6.10×10^{-6}	2.91×10^{-8}	6.70×10^{-6}	2.95×10^{-4}
0.7	1.23×10^{-6}	7.82×10^{-7}	4.88×10^{-9}	1.27×10^{-6}	8.37×10^{-5}
0.9	1.44×10^{-6}	1.02×10^{-6}	7.83×10^{-9}	2.21×10^{-6}	2.01×10^{-4}

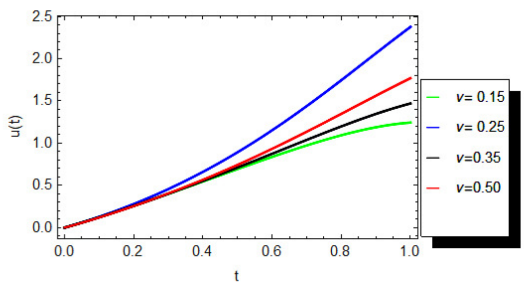

Figure 1: Approximate solutions of our scheme with $k=2, M=11$ and $\alpha=0.5$ for Example 7.1.

Example 7.2. Consider the stochastic fractional integro-differential equation given by

$$^{C}D^{\frac{1}{2}}u(t)=t^2+2\frac{t^{1.5}}{\Gamma(2.5)}-u(t)+\int_0^t sdB(s), \tag{7.3}$$

with the initial condition

$$u(0) = 0. \tag{7.4}$$

The residual error of our scheme for $k = 1, \alpha = \frac{1}{2}$ with $M = 8, 10, 12$ at some different points t are listed in Table 3. Also, the numerical results for $k = 2, M = 12$ and $\alpha = \frac{1}{2}$ is shown in Figure 2.

Table 3: The comparison of residual error with $k = 1, \alpha = \frac{1}{2}$ for various values of M for Example 7.2.

t	$M = 8$	$M = 10$	$M = 12$
0.2	1.40×10^{-8}	8.26×10^{-10}	5.31×10^{-13}
0.4	6.78×10^{-10}	8.06×10^{-11}	7.22×10^{-14}
0.6	2.78×10^{-10}	2.64×10^{-11}	1.08×10^{-13}
0.8	7.59×10^{-10}	2.13×10^{-11}	1.36×10^{-13}

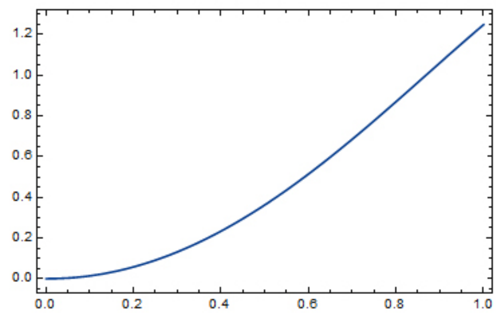

Figure 2: Approximate solution of our scheme with $k = 2; M = 12$ and $\alpha = \frac{1}{2}$ for Example 7.2.

Example 7.3. Consider the stochastic fractional integro-differential equation given by

$${}^{C}D^{\nu}u(t) = -\frac{t^5 e^t}{5} + \frac{6t^{2.25}}{\Gamma(3.25)} + \int_0^t e^t s u(s) ds + \sigma \int_0^t e^t s u(s) dB(s), \tag{7.5}$$

with the initial condition

$$u(0) = 0. \tag{7.6}$$

In Table 4, we compare L_2 errors obtained from our scheme for $k = 2, \nu = 0.75$ and $\sigma = 0$ with error achieved from the block pulse method [1]. Also, Table 5 shows the residual error for $k = 2, M = 10$ and $\nu = 0.75$ and various values of α. Numerical results for $k = 2; M = 8$ and $\nu = 0.75, \alpha = 1$ for different values of σ are shown in Figure 3.

Table 4: The comparison of L_2 errors for $k = 2, \nu = 0.75$ and $\sigma = 0$ with Ref. [1] for Example 7.3.

Ref.[1]	*Absolute errors*
$N = 2^5$	1.3×10^{-2}
$N = 2^6$	7.0×10^{-3}
$N = 2^7$	3.6×10^{-4}
$N = 2^8$	6.3×10^{-5}
Present method	
$M = 8, \alpha = \frac{1}{2}$	7.9×10^{-7}
$M = 8, \alpha = 1$	1.25×10^{-6}
$M = 10, \alpha = \frac{1}{2}$	8.66×10^{-8}
$M = 10, \alpha = 1$	2.75×10^{-7}

Example 7.4. Consider the stochastic fractional integro-differential equation given by

Table 5: The comparison of residual error with $k=2, M=10$ and $\nu=0.75$ for various values of α for Example 7.3.

t	$\alpha=\frac{1}{4}$	$\alpha=\frac{1}{2}$	$\alpha=\frac{2}{3}$	$\alpha=1$
0.2	2.08×10^{-3}	8.00×10^{-5}	2.07×10^{-4}	1.90×10^{-8}
0.4	1.18×10^{-4}	9.68×10^{-6}	4.37×10^{-5}	1.26×10^{-6}
0.6	3.01×10^{-5}	3.88×10^{-6}	2.53×10^{-5}	2.10×10^{-6}
0.8	2.19×10^{-5}	3.85×10^{-6}	3.34×10^{-5}	5.48×10^{-6}

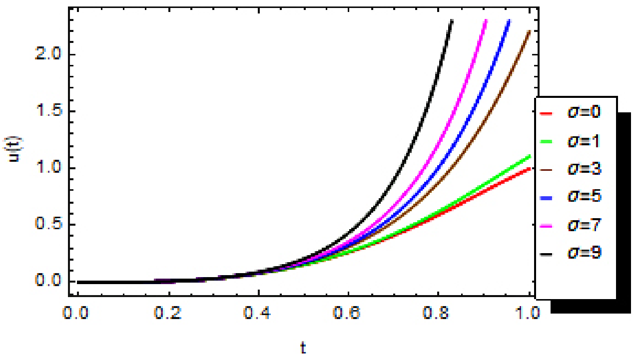

Figure 3: Approximate solution of the suggested method with $k=2; M=8$ and $\nu=0.75, \alpha=1$ for different values of σ for Example 7.3.

$$^{C}D^{\nu}u(t)=\frac{7}{12}t^4-\frac{5}{6}t^3+2\frac{t^{2-\nu}}{\Gamma(3-\nu)}+\frac{t^{1-\nu}}{\Gamma(2-\nu)}+\int_0^t(s+t)u(s)ds+\int_0^t su(s)dB(s), \tag{7.7}$$

with the initial condition

$$u(0)=0. \tag{7.8}$$

The residual error of approximate solution for $k=1, M=12$ and $\alpha=\frac{1}{2}$ at different points with the results of the technique in [39] are shown in Table 6. Also, Fig. 4 demonstrates graphs of the approximate solutions and the residual error of our scheme with $k=1, M=10$ and $\alpha=1$ for different values of ν.

Table 6: The comparison of residual error for $k=1, M=12$ and $\alpha=\frac{1}{2}$ with Ref. [39] for Example 7.4.

t	*Ref.*[39]			*Present method*		
	$\nu=0.25$	$\nu=0.5$	$\nu=0.75$	$\nu=0.25$	$\nu=0.5$	$\nu=0.75$
0.1	4.21×10^{-4}	5.28×10^{-4}	6.28×10^{-4}	1.52×10^{-5}	8.14×10^{-6}	6.92×10^{-6}
0.3	1.82×10^{-3}	1.88×10^{-3}	1.94×10^{-3}	6.48×10^{-7}	3.44×10^{-7}	2.35×10^{-7}
0.5	1.01×10^{-2}	9.84×10^{-3}	9.65×10^{-3}	3.52×10^{-7}	1.86×10^{-7}	1.15×10^{-7}
0.7	4.01×10^{-3}	3.66×10^{-3}	3.46×10^{-3}	5.71×10^{-8}	3.04×10^{-8}	1.74×10^{-8}
0.9	1.57×10^{-3}	1.99×10^{-3}	1.26×10^{-3}	4.42×10^{-8}	2.38×10^{-8}	1.29×10^{-8}

Example 7.5. Consider the stochastic fractional integro-differential equation given by

$$^{C}D^{\nu}u(t)=\frac{\Gamma(3)t^{2-\nu}}{\Gamma(3-\nu)}-\frac{t^4\sin(t)}{5}+\int_0^t\sin(t)s^2u(s)ds+\int_0^t se^tu(s)dB(s), \tag{7.9}$$

with the initial condition

$$u(0)=0. \tag{7.10}$$

In Table 7, the residual error obtained from the present scheme for $k=1, M=12$ and $\alpha=\frac{2}{3}$ with error achieved from the shifted Legendre polynomials method [39] are compared. Also, a comparison between the residual error given by our technique at $k=1, M=12$, $\nu=\frac{1}{2}$ for some values of α is described in Table 8. The graphs of the approximate solutions and the residual error of the present technique with $k=1, M=12$ and $\alpha=1$ are shown in Figure 5.

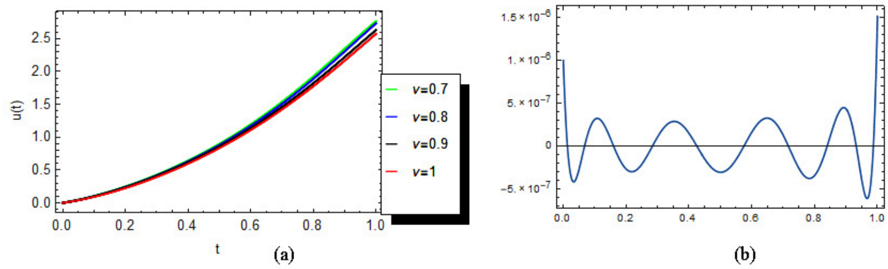

Figure 4: (a): approximate solutions, (b) residual error of the present method with $k = 1, M = 10$ and $\alpha = 1$ for Example 7.4.

Table 7: The comparison of residual error for $k = 1, M = 12$ and $\alpha = \frac{2}{3}$ with Ref. [39] for Example 7.5.

t	$Ref.$[39]			$Present\ method$		
	$\nu = 0.25$	$\nu = 0.5$	$\nu = 0.75$	$\nu = 0.25$	$\nu = 0.5$	$\nu = 0.75$
0.1	1.85×10^{-5}	1.76×10^{-4}	4.94×10^{-4}	4.71×10^{-6}	6.70×10^{-7}	5.98×10^{-7}
0.3	5.17×10^{-4}	3.94×10^{-4}	2.26×10^{-4}	5.22×10^{-7}	6.15×10^{-8}	8.11×10^{-8}
0.5	5.24×10^{-3}	5.40×10^{-3}	5.53×10^{-3}	4.87×10^{-7}	5.13×10^{-8}	7.84×10^{-8}
0.7	2.81×10^{-3}	2.92×10^{-3}	3.02×10^{-3}	1.19×10^{-7}	1.14×10^{-8}	1.92×10^{-8}
0.9	1.34×10^{-3}	1.34×10^{-3}	1.31×10^{-3}	1.32×10^{-7}	1.13×10^{-8}	2.09×10^{-8}

Table 8: The comparison of residual error with $k = 1, M = 12$ and $\nu = \frac{1}{2}$ for various values of α for Example 7.5.

t	$\alpha = \frac{1}{4}$	$\alpha = \frac{1}{3}$	$\alpha = \frac{1}{2}$	$\alpha = \frac{2}{3}$	$\alpha = 1$
0.2	1.52×10^{-5}	1.90×10^{-4}	1.57×10^{-8}	1.99×10^{-7}	2.58×10^{-6}
0.4	3.48×10^{-6}	6.25×10^{-5}	6.24×10^{-9}	2.21×10^{-7}	5.24×10^{-6}
0.6	6.05×10^{-7}	1.39×10^{-5}	5.48×10^{-10}	1.04×10^{-7}	3.50×10^{-6}
0.8	4.53×10^{-8}	1.26×10^{-6}	1.30×10^{-10}	1.62×10^{-8}	6.69×10^{-7}

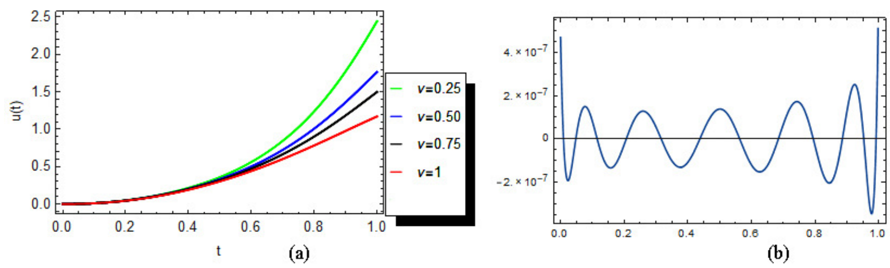

Figure 5: (a): approximate solutions, (b) residual error of our scheme with $k = 1, M = 12$ and $\alpha = 1$ for Example 7.5.

Example 7.6. Consider the stochastic fractional integro-differential equation given by

$$ {}^{C}D^{\nu}u(t) = t + \frac{\Gamma(2)}{\Gamma(2-\nu)}t^{1-\nu} - u(t) + \int_0^t dB(s), \tag{7.11} $$

with the initial condition

$$ u(0) = 0. \tag{7.12} $$

In Table 9, a comparison between the residual error of our scheme at some various values of t for some values of α is given. Figs. 6(a) and 6(b) demonstrate the plot of the numerical results for $k = 2, M = 12$ and $\alpha = 1$ and the residual error with $\nu = 1$ respectively.

Table 9: The comparison of residual error with $k = 2, M = 12, \nu = \frac{1}{2}$ for various values of α for Example 7.6.

t	$\alpha = \frac{1}{4}$	$\alpha = \frac{1}{3}$	$\alpha = \frac{1}{2}$	$\alpha = \frac{2}{3}$	$\alpha = 1$
0.2	3.83×10^{-12}	1.06×10^{-10}	7.19×10^{-11}	4.16×10^{-7}	1.03×10^{-4}
0.4	1.01×10^{-12}	2.35×10^{-11}	3.50×10^{-11}	2.82×10^{-7}	2.18×10^{-4}
0.6	1.75×10^{-13}	3.75×10^{-12}	9.09×10^{-12}	9.33×10^{-8}	1.55×10^{-4}
0.8	2.46×10^{-14}	4.61×10^{-13}	1.01×10^{-12}	1.08×10^{-8}	3.26×10^{-5}

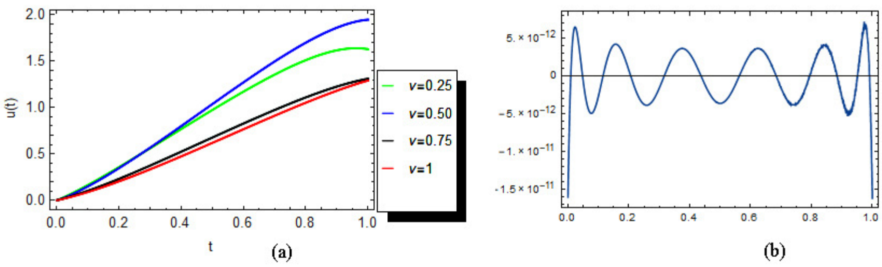

Figure 6: (a): approximate solutions, (b) residual error of our scheme with $k = 2, M = 12$ and $\alpha = 1$ for Example 7.6.

8 Conclusion

In this article, we presented a new numerical scheme based on the fractional-order Bernstein wavelets for solving the stochastic fractional integro-differential equations. Riemann-Liouville fractional integral operator of the fractional-order Bernstein wavelets, the Gaussian integration formula and collocation scheme have been applied for converting this problem to a system of algebraic equations. Then, we solved this system by using the Newton's method with zero vector as initial guess. Finally, six numerical examples are presented to demonstrate the performance and effectiveness of the proposed scheme.

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