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Oscillation and asymptotic behavior of third-order semi-canonical difference equations with positive and negative terms

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Abstract

This paper deals with the oscillatory and asymptotic properties of third-order semi-canonical difference equations with positive and negative terms of the form

$$\Delta\left(\varphi(\ell)\Delta\left(\psi(\ell)\Delta\mu(\ell)\right)\right)+\xi(\ell)f(\mu(\sigma(\ell)))-\chi(\ell)g(\mu(\tau(\ell)))=0.$$

Using a canonical transformation technique, we offer new criteria which imply that the solutions of the studied equation are almost oscillatory. Some examples are provided to support our results.

Keywords: Third-order, semi-canonical, difference equation, oscillation, positive and negative term, nonlinear equation
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1 Introduction

Here, we consider the third-order semi-canonical difference equation with positive and negative terms of the form

$$\Delta\left(\varphi(\ell)\Delta\left(\psi(\ell)\Delta\mu(\ell)\right)\right)+\xi(\ell)f(\mu(\sigma(\ell)))-\chi(\ell)g(\mu(\tau(\ell)))=0,\quad \ell\geq\ell_0, \tag{E}$$

where

(H_1) $\{\varphi(\ell)\}$, $\{\psi(\ell)\}$, $\{\xi(\ell)\}$ and $\{\chi(\ell)\}$ are positive real sequences;

(H_2) $\{\sigma(\ell)\}$ and $\{\tau(\ell)\}$ are sequences of integers with $\sigma(\ell)\leq\ell$, $\lim_{\ell\to\infty}\sigma(\ell)=\lim_{\ell\to\infty}\tau(\ell)=\infty$;

(H_3) f, g are real valued continuous functions, $\delta f(\delta)>0$, $\delta g(\delta)>0$ for $\delta\neq0$, g is bounded and f is nondecreasing;

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(H_4) $-f(-\delta\gamma) \geq f(\delta\gamma) \geq f(\delta)f(\gamma)$ for $\delta\gamma > 0$;

(H_5) the equation (E) is in semi-canonical form, that is,

$$\sum_{\ell=\ell_0}^{\infty} \frac{1}{\varphi(\ell)} < \infty \quad \text{and} \quad \sum_{\ell=\ell_0}^{\infty} \frac{1}{\psi(\ell)} = \infty. \tag{1.1}$$

Recall that a solution of (E) is a nontrivial real-valued sequence $\{\mu(\ell)\}$ which is defined for all $\ell \geq \ell_0$, and satisfies (E). Solutions vanishing identically in some neighborhood of infinity will be excluded from our consideration. A solution $\{\mu(\ell)\}$ of equation (E) is said to be *nonoscillatory* if it is either eventually positive or eventually negative; otherwise, it is called *oscillatory*.

The investigation of oscillatory properties of third or higher order difference equations (see [1, 4, 2, 3, 9, 12, 15, 20, 18, 16, 14, 17, 5, 6, 11, 19, 22, 7, 13]) essentially makes use of some generalizations of discrete Kneser's theorem [1]. In the theorem, from the fixed sign of the highest difference, we deduce the structure of possible nonoscillatory solutions. Since the equation (E) contains both positive and negative terms, we are not able to fix the sign of the third-order quasidifferences for an eventually positive solution. So most of the authors studied the oscillatory properties of (E) in the partial case when $\xi(\ell) \equiv 0$ or $\chi(\ell) \equiv 0$.

Recently in [21, 10, 8], the authors studied the oscillatory behavior of equation (E) in the case when

$$\sum_{\ell=\ell_0}^{\infty} \frac{1}{\varphi(\ell)} = \sum_{\ell=\ell_0}^{\infty} \frac{1}{\psi(\ell)} = \infty.$$

Therefore, in this paper, we investigate the oscillatory behavior of (E) if condition (1.1) is satisfied. This is achieved by transforming (E) into canonical form, and this essentially simplifies the examination of (E). Thus the results obtained in this paper are new and complement to the results reported in the literature for third order difference equations.

2 Main Results

Throughout, we assume (1.1) holds, and so we can employ the notations:

$$\Phi(\ell) = \sum_{s=\ell}^{\infty} \frac{1}{\varphi(s)}, \quad \eta(\ell) = \varphi(\ell)\Phi(\ell)\Phi(\ell+1), \quad \zeta(\ell) = \frac{\psi(\ell)}{\Phi(\ell)},$$

$$Q_1(\ell) = \Phi(\ell+1)\xi(\ell), \quad Q_2(\ell) = \Phi(\ell+1)\chi(\ell), \quad \Psi(\ell) = \sum_{s=\ell_1}^{\ell-1} \frac{1}{\eta(s)}, \quad \Theta(\ell) = \sum_{s=\ell_1}^{\ell-1} \frac{\Psi(s)}{\zeta(s)}$$

where $\ell \geq \ell_1 \geq \ell_0$ and ℓ_1 is large enough.

Theorem 2.1. Assume that

$$\sum_{\ell=\ell_0}^{\infty} \frac{1}{\zeta(\ell)} = \infty. \tag{2.1}$$

The semi-canonical equation (E) has the following canonical representation

$$\Delta\left(\eta(\ell)\Delta(\zeta(\ell)\Delta\mu(\ell))\right) + Q_1(\ell)f(\mu(\sigma(\ell))) - Q_2(\ell)g(\mu(\tau(\ell))) = 0. \tag{E_1}$$

Proof . Using the values of $\eta(\ell)$ and $\zeta(\ell)$ then applying product rule for difference calculus [1], we get

$$\begin{aligned}
\Delta\left(\eta(\ell)\Delta(\zeta(\ell)\Delta\mu(\ell))\right) &= \Delta\left(\varphi(\ell)\Phi(\ell)\Phi(\ell+1)\Delta\left(\frac{\psi(\ell)}{\Phi(\ell)}\Delta\mu(\ell)\right)\right) \\
&= \Delta\left(\varphi(\ell)\Phi(\ell)\Phi(\ell+1)\frac{\Phi(\ell)\Delta(\psi(\ell)\Delta\mu(\ell)) + \psi(\ell)\Delta\mu(\ell)}{\Phi(\ell)\Phi(\ell+1)}\right) \\
&= \Delta\left(\Phi(\ell)\varphi(\ell)\Delta(\psi(\ell)\Delta\mu(\ell)) + \psi(\ell)\Delta\mu(\ell)\right) \\
&= \Phi(\ell+1)\Delta\left(\varphi(\ell)\Delta(\psi(\ell)\Delta\mu(\ell))\right) \\
&\qquad - \Delta(\psi(\ell)\Delta\mu(\ell)) + \Delta(\psi(\ell)\Delta\mu(\ell)) \\
&= \Phi(\ell+1)\Delta\left(\varphi(\ell)\Delta(\psi(\ell)\Delta\mu(\ell))\right).
\end{aligned} \tag{2.2}$$

Now using (E) in (2.2), we get (E_1). Next, we show that (E_1) is a canonical type equation, that is,

$$\sum_{\ell=\ell_0}^{\infty} \frac{1}{\eta(\ell)} = \sum_{\ell=\ell_0}^{\infty} \frac{1}{\varphi(\ell)\Phi(\ell)\Phi(\ell+1)} = \sum_{\ell=\ell_0}^{\infty} \Delta\left(\frac{1}{\Phi(\ell)}\right) = \lim_{\ell\to\infty} \frac{1}{\Phi(\ell)} - \frac{1}{\Phi(\ell_0)} = \infty,$$

and $\sum_{\ell=\ell_0}^{\infty} \frac{1}{\zeta(\ell)} = \infty$ by (2.1). This completes the proof. □

Theorem 2.2. Assume that (2.1) holds. Then the semi-canonical equation (E) possesses a solution $\{\mu(\ell)\}$ if and only if the canonical equation (E_1) has the solution $\{\mu(\ell)\}$.

Corollary 2.3. Assume that (2.1) holds. Then the semi-canonical difference equation (E) has an eventually positive solution if and only if the canonical equation (E) has an eventually positive solution.

In what follows, we shall assume that

(H_5) $\sum_{\ell=\ell_0}^{\infty} \frac{1}{\zeta(\ell)} \sum_{s=\ell}^{\infty} \frac{1}{\eta(s)} \sum_{t=s}^{\infty} Q_2(t) < \infty$.

It will be shown that this condition reduces the influence of the negative term in the equation (E_1) and this permit us to study the oscillatory property of (E). By *almostoscillatory* we mean that every nonoscillatory solution of (E) tends to zero as $\ell \to \infty$.

Theorem 2.4. Assume that (2.1) holds. Let $\ell_1 \geq \ell_0$ be large enough

$$\sum_{\ell=\ell_1}^{\infty} \frac{1}{\zeta(\ell)} \sum_{s=\ell}^{\infty} \frac{1}{\eta(s)} \sum_{t=s}^{\infty} Q_1(t) = \infty \tag{2.3}$$

and

$$\sum_{\ell=\ell_1}^{\infty} Q_1(\ell) f(\Theta(\sigma(\ell))) = \infty. \tag{2.4}$$

If

$$\begin{aligned} \lim_{\ell\to\infty} \sup \Bigg\{ \frac{1}{\Psi(\sigma(\ell))} & \sum_{s=\ell_1}^{\sigma(\ell)-1} Q_1(s) f(\Theta(\sigma(s)))\Psi(s+1) + \sum_{s=\sigma(\ell)}^{\ell-1} Q_1(s) f(\Theta(\sigma(s))) \\ & + f(\Psi(\sigma(\ell))) \sum_{s=\ell}^{\infty} Q_1(s) f\left(\frac{\Theta(\sigma(s))}{\Psi(\sigma(s))}\right) \Bigg\} > \lim_{\delta\to 0} \sup \frac{\delta}{f(\delta)} \end{aligned} \tag{2.5}$$

then (E) is almost oscillatory.

Proof . Let $\{\mu(\ell)\}$ be a nonoscillatory solution of (E). Then without loss of generality, we may assume that $\mu(\ell)$ is a positive solution of (E), that is, there exits an integer ℓ_1 such that $\mu(\ell) > 0$, $\mu(\sigma(\ell)) > 0$ and $\mu(\tau(\ell)) > 0$ for all $\ell_1 \geq \ell_0$ (since the proof for the negative case is similar). By Corollary 2.3, the sequence $\{\mu(\ell)\}$ is also a positive solution of (E_1) for $\ell \geq \ell_1$. Introduce the auxiliary sequence $\{\nu(\ell)\}$ associated to $\{\mu(\ell)\}$ by

$$\nu(\ell) = \mu(\ell) + \sum_{s=\ell}^{\infty} \zeta(s) \sum_{t=s}^{\infty} \frac{1}{\eta(t)} \sum_{j=t}^{\infty} Q_2(j) g(\mu(\tau(j))). \tag{2.6}$$

It follows from (H_5) and the boundedness of $g(\delta)$, that the sequence $\{\nu(\ell)\}$ is well-defined and exists for all $\ell \geq \ell_1$. Further $\nu(\ell) \geq \mu(\ell)$, $\Delta\nu(\ell) \leq \Delta\mu(\ell)$ and

$$\Delta(\eta(\ell)\Delta(\zeta(\ell)\Delta\nu(\ell))) = -Q_1(\ell) f(\mu(\sigma(\ell))) < 0, \quad \ell \geq \ell_1. \tag{E_2}$$

Since $\sum_{\ell=\ell_0}^{\infty} \frac{1}{\eta(\ell)} = \sum_{\ell=\ell_0}^{\infty} \frac{1}{\zeta(\ell)} = \infty$, we have by discrete Kneser's theorem [1], we have

$$\nu(\ell) \in S_0 \Leftrightarrow \zeta(\ell)\Delta\nu(\ell) < 0, \quad \eta(\ell)\Delta(\zeta(\ell)\Delta\nu(\ell)) > 0,$$

or

$$\nu(\ell) \in S_2 \Leftrightarrow \zeta(\ell)\Delta\nu(\ell) > 0, \quad \eta(\ell)\Delta(\zeta(\ell)\Delta\nu(\ell)) > 0$$

eventually for all $\ell \geq \ell_1$.

First assume that $\nu(\ell) \in S_2$. Using $\eta(\ell)\Delta(\zeta(\ell)\Delta\nu(\ell))$ is decreasing, we have

$$\zeta(\ell)\Delta\nu(\ell) \geq \sum_{s=\ell_1}^{\ell-1} \eta(s)\frac{\Delta(\zeta(s)\Delta\nu(s))}{\eta(s)} \geq \Psi(\ell)\eta(\ell)\Delta(\zeta(\ell)\Delta\nu(\ell)). \tag{2.7}$$

Hence

$$\Delta\left(\frac{\zeta(\ell)\Delta\nu(\ell)}{\Psi(\ell)}\right) = \frac{\Psi(\ell)\eta(\ell)\Delta(\zeta(\ell)\Delta\nu(\ell)) - \zeta(\ell)\Delta\nu(\ell)}{\eta(\ell)\Psi(\ell)\Psi(\ell+1)} \leq 0,$$

and consequently

$$\frac{\zeta(\ell)\Delta\nu(\ell)}{\Psi(\ell)} \text{ is decreasing.} \tag{2.8}$$

Then

$$\mu(\ell) \geq \sum_{s=\ell_1}^{\ell-1} \Delta\mu(s) \geq \sum_{s=\ell_1}^{\ell-1} \frac{\zeta(s)\Delta\nu(s)\Psi(s)}{\Psi(s)\zeta(s)} \geq \frac{\zeta(\ell)\Delta\nu(\ell)}{\Psi(\ell)}\Theta(\ell).$$

Letting the last estimate into (E_2), we see that

$$\omega(\ell) = \zeta(\ell)\Delta\nu(\ell) \tag{2.9}$$

is a positive solution of the difference inequality

$$\Delta(\eta(\ell)\Delta\omega(\ell)) + Q_1(\ell)f\left(\frac{\Theta(\sigma(\ell))}{\Psi(\sigma(\ell))}\omega(\sigma(\ell))\right) \leq 0. \tag{2.10}$$

It is easy to see that $\eta(\ell)\Delta\omega(\ell) > 0$ and by combining (2.8) with (2.9), the sequence $\left\{\frac{\omega(\ell)}{\Psi(\ell)}\right\}$ is decreasing.

On the other hand, a summation of (2.10) from ℓ to ∞ and then from ℓ_1 to $\ell - 1$ yields

$$\begin{aligned}
\omega(\ell) &\geq \sum_{s=\ell_1}^{\ell-1} \frac{1}{\eta(s)} \sum_{t=s}^{\infty} Q_1(t)f\left(\frac{\Theta(\sigma(t))}{\Psi(\sigma(t))}\omega(\sigma(t))\right) \\
&= \sum_{s=\ell_1}^{\ell-1} \frac{1}{\eta(s)} \sum_{t=s}^{\ell-1} Q_1(t)f\left(\frac{\Theta(\sigma(t))}{\Psi(\sigma(t))}\omega(\sigma(t))\right) + \sum_{s=\ell_1}^{\ell-1} \frac{1}{\eta(s)} \sum_{s=\ell}^{\infty} Q_1(s)f\left(\frac{\Theta(\sigma(s))}{\Psi(\sigma(s))}\omega(\sigma(s))\right) \\
&= \sum_{s=\ell_1}^{\ell-1} Q_1(s)f\left(\frac{\Theta(\sigma(s))}{\Psi(\sigma(s))}\omega(\sigma(s))\right)\Psi(s+1) + \Psi(t)\sum_{s=\ell}^{\infty} Q_1(s)f\left(\frac{\Theta(\sigma(s))}{\Psi(\sigma(s))}\omega(\sigma(s))\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\omega(\sigma(\ell)) \geq & \sum_{s=\ell_1}^{\sigma(\ell)-1} Q_1(s)f\left(\frac{\omega(\sigma(s))}{\Psi(\sigma(s))}\omega(\sigma(s))\right)\Psi(s+1) \\
& + \Psi(\sigma(\ell)) \sum_{s=\sigma(\ell)}^{\ell-1} Q_1(s)f\left(\frac{\Theta(\sigma(s))}{\Psi(\sigma(s))}\omega(\sigma(s))\right) \\
& + \Psi(\sigma(\ell)) \sum_{s=\ell}^{\infty} Q_1(s)f\left(\frac{\Theta(\sigma(s))}{\Psi(\sigma(s))}\omega(\sigma(s))\right).
\end{aligned}$$

In view of (H_4) and the fact that $\omega(\ell)$ is increasing and $\frac{\omega(\ell)}{\Psi(\ell)}$ is decreasing, we have

$$\begin{aligned}\omega(\sigma(\ell)) \quad \geq \quad & f\left(\frac{\omega(\sigma(\ell))}{\Psi(\sigma(\ell))}\right)\sum_{s=\ell_1}^{\ell-1} Q_1(s) f\left(\Theta(\sigma(s))\right)\Psi(s+1) \\ & + \Psi(\sigma(\ell)) f\left(\frac{\omega(\sigma(\ell))}{\Psi(\sigma(\ell))}\right)\sum_{s=\sigma(\ell)}^{\ell-1} Q_1(s) f(\Theta(\sigma(s))) \\ & + \Psi(\sigma(\ell)) f\left(\omega(\sigma(\ell))\right)\sum_{s=\ell}^{\infty} Q_1(s) f\left(\frac{\Theta(\sigma(s))}{\Psi(\sigma(s))}\right). \end{aligned} \tag{2.11}$$

Therefore, by letting $\delta = \frac{\omega(\sigma(\ell))}{\Psi(\sigma(\ell))}$, we get

$$\begin{aligned}\frac{\delta}{f(\delta)} \quad \geq \quad & \frac{1}{\Psi(\sigma(\ell))}\sum_{s=\ell_1}^{\sigma(\ell)-1} Q_1(s) f(\Theta(\sigma(s)))\Psi(s+1) + \sum_{s=\sigma(\ell)}^{\ell-1} Q_1(s) f(\Theta(\sigma(s))) \\ & + f(\Psi(\sigma(\ell)))\sum_{s=\ell}^{\infty} Q_1(s) f\left(\frac{\Theta(\sigma(s))}{\Psi(\sigma(s))}\right). \end{aligned} \tag{2.12}$$

Condition (2.4) implies that $\frac{\omega(\ell)}{\Psi(\ell)} \to 0$ as $\ell \to \infty$. Indeed, if $\frac{\omega(\ell)}{\Psi(\ell)} \to m > 0$, then $\frac{\omega(\ell)}{\Psi(\ell)} \geq m$ and using the last inequality into (2.10), we obtain

$$0 \geq \Delta(\eta(\ell)\Delta\omega(\ell)) + Q_1(\ell) f(m\Theta(\sigma(\ell))).$$

Summing up from ℓ_1 to ∞ yields

$$\eta(\ell_1)\Delta\omega(\ell_1) \geq f(m)\sum_{\ell=\ell_1}^{\infty} Q_1(\ell) f(\Theta(\sigma(\ell)))$$

which contradicts with (2.4). Now, taking lim sup on the both sides of (2.12), one obtains a contradiction with (2.5).

Next, we assume that $\nu(\ell) \in S_0$. Since $\nu(\ell)$ is positive and decreasing, there exists $\lim_{\ell\to\infty} \nu(\ell) = 2m > 0$. It follows from (2.6) that $\lim_{\ell\to\infty} \mu(\ell) = 2m$. If we assume that $m > 0$, then $\mu(\sigma(\ell)) \geq m > 0$ eventually. Summation of (E_2) yields

$$\eta(\ell)\Delta(\zeta(\ell)\Delta\nu(\ell)) \geq \sum_{s=\ell}^{\infty} Q_1(s) f(\mu(\sigma(s))) \geq f(m)\sum_{s=\ell}^{\infty} Q_1(s).$$

Summing up from ℓ to ∞ and then from ℓ_1 to ∞ one gets

$$\nu(\ell_1) \geq f(m)\sum_{\ell=\ell_1}^{\infty}\frac{1}{\zeta(\ell)}\sum_{s=\ell}^{\infty}\frac{1}{\eta(s)}\sum_{t=s}^{\infty} Q_1(t)$$

which contradicts with (2.3) and the proof is complete. □

Let $f(\delta) = \delta$ in (E), then we have the following verifiable criterion.

Corollary 2.5. Assume that (2.1) holds. Let (2.3) hold and for all ℓ_1 large enough

$$\sum_{\ell=\ell_1}^{\infty} Q_1(\ell)\Theta(\sigma(\ell)) = \infty. \tag{2.13}$$

Assume that

$$\begin{aligned}\limsup_{\ell\to\infty}\Bigg\{\frac{1}{\Psi(\sigma(\ell))} \quad & \sum_{s=\ell_1}^{\sigma(\ell)-1} Q_1(s)\Theta(\sigma(s))\Psi(s+1) + \sum_{s=\sigma(\ell)}^{\ell-1} Q_1(s)\Theta(\sigma(s)) \\ & + \Psi(\sigma(\ell))\sum_{s=\ell}^{\infty} Q_1(s)\frac{\Theta(\sigma(s))}{\Psi(\sigma(s))}\Bigg\} > 1, \end{aligned} \tag{2.14}$$

then (E) is almost oscillatory.

Theorem 2.6. Assume that (2.1) holds. Let (2.3) hold and for all ℓ_1 large enough

$$\sum_{\ell=\ell_1}^{\infty}\frac{1}{\eta(\ell)}\sum_{s=\ell}^{\infty}Q_1(s)f\left(\frac{\Theta(\sigma(s))}{\Psi(\sigma(s))}\right)=\infty. \tag{2.15}$$

If

$$\begin{aligned}\limsup_{\ell\to\infty}\quad & \Bigg\{f\left(\frac{1}{\Psi(\sigma(\ell))}\right)\sum_{s=\ell_1}^{\sigma(\ell)-1}Q_1(s)f(\Theta(\sigma(s)))\Psi(s+1)\\ & +\Psi(\sigma(s))f\left(\frac{1}{\Psi(\sigma(\ell))}\right)\sum_{s=\sigma(\ell)}^{\ell-1}Q_1(s)f(\Theta(\sigma(s)))\\ & +f(\Psi(\sigma(\ell)))\sum_{s=\ell}^{\infty}Q_1(s)f\left(\frac{\Theta(\sigma(s))}{\Psi(\sigma(s))}\right)\Bigg\}>\lim_{\gamma\to 0}\frac{\gamma}{f(\gamma)},\end{aligned} \tag{2.16}$$

then (E) is almost oscillatory.

Proof . Let $\{\mu(\ell)\}$ be a nonoscillatory solution of (E). Then without loss of generality, we may assume that $\mu(\ell)$ is a positive solution of (E), that is, there exits an integer ℓ_1 such that $\mu(\ell)>0$, $\mu(\sigma(\ell))>0$ and $\mu(\tau(\ell))>0$ for all $\ell_1\geq\ell_0$ (since the proof for the negative case is similar). By Corollary 2.3, the sequence $\{\mu(\ell)\}$ is also a positive solution of (E_1) for all $\ell\geq\ell_1$. Proceeding exactly as in the proof of Theorem 2.4, we verify that the associated sequence $\{\nu(\ell)\}$ belongs to the class S_0 or S_2.

If $\nu(\ell)\in S_2$, then $\omega(\ell)=\zeta(\ell)\Delta\nu(\ell)$ satisfies (2.11). We claim that (2.15) implies that $\omega(\ell)\to\infty$ as $\ell\to\infty$. If not, then $\omega(\ell)\to K$ as $\ell\to\infty$. Summation of (2.10) yields

$$\eta(\ell)\Delta\omega(\ell)\geq\sum_{s=\ell}^{\infty}Q_1(s)f\left(\frac{\Theta(\sigma(s))}{\Psi(\sigma(s))}\omega(\sigma(s))\right).$$

Summing once more, we get

$$\begin{aligned}K &\geq \sum_{\ell=\ell_1}^{\infty}\frac{1}{\eta(\ell)}\sum_{s=\ell}^{\infty}Q_1(s)f\left(\frac{\Theta(\sigma(s))}{\Psi(\sigma(s))}\omega(\sigma(s))\right)\\ &\geq \omega(\sigma(\ell_1))\sum_{\ell=\ell_1}^{\infty}\frac{1}{\eta(\ell)}\sum_{s=\ell}^{\infty}Q_1(s)f\left(\frac{\Theta(\sigma(s))}{\Psi(\sigma(s))}\right)\end{aligned}$$

which contradicts with (2.15) and therefore $\omega(\ell)\to\infty$ as $\ell\to\infty$. Now, setting $\gamma=\omega(\sigma(\ell))$, we get

$$\begin{aligned}\frac{\gamma}{f(\gamma)} \quad\geq\quad & \frac{1}{f(\Psi(\sigma(\ell)))}\sum_{s=\ell_1}^{\sigma(\ell)-1}Q_1(s)f(\Theta(\sigma(s)))\Psi(s+1)+\Psi(\sigma(s))\frac{1}{f(\Psi(\sigma(\ell)))}\sum_{s=\sigma(\ell)}^{\ell-1}Q_1(s)f(\Theta(\sigma(s)))\\ & +\Psi(\sigma(\ell))\sum_{s=\ell}^{\infty}Q_1(s)f\left(\frac{\Theta(\sigma(s))}{\Psi(\sigma(s))}\right).\end{aligned}$$

Taking lim sup on both sides, we obtain a contradiction with (2.16) If $\nu(\ell)\in S_0$, then as in the proof of Theorem 2.4, we see that $\mu(\ell)\to 0$ as $\ell\to\infty$. The proof is completed. □

Remark 2.1. Theorems 2.4 and 2.6 are applicable for the case

$$f(\delta)=\delta^{\beta}$$

where β is a ration of odd positive integers with $0<\beta\leq 1$ and $\beta>1$, respectively.

Remark 2.2. The summation criteria (2.5) and (2.16) of Theorem 2.4 and 2.6 contain three terms and naturally they provide the better results than one term summation criteria that are generally used.

3 Examples

In this section, we present some examples to illustrate the importance of the main results.

Example 3.1. Consider the third-order semi-canonical difference equation

$$\Delta\left(\ell(\ell+1)\Delta\left(\frac{1}{\ell}\Delta\mu(\ell)\right)\right)+\frac{\xi}{\ell(\ell+2)}\mu(\ell-2)-\frac{\chi}{\ell^3}tan^{-1}(\mu(\tau(\ell)))=0,\quad \ell\geq 1, \tag{3.1}$$

where $\xi>0,\ \chi>0$. A simple calculation shows that $\Phi(\ell)=\frac{1}{\ell},\ \eta(\ell)=\zeta(\ell)=1,\ Q_1(\ell)=\frac{\xi}{\ell(\ell+1)(\ell+2)},\ Q_2(\ell)=\frac{\chi}{\ell^3(\ell+1)},\ f(\delta)=\delta,\ \Psi(\ell)=\ell$ and $\Theta(\ell)\approx\frac{\ell^2}{2}$. Since $g(\delta)=tan^{-1}(\delta)$ is bounded, and condition (H_5) is satisfied. The condition (2.3) becomes

$$\sum_{\ell=1}^{\infty}\sum_{s=\ell}^{\infty}\sum_{t=s}^{\infty}\frac{\xi}{t(t+1)(t+2)}=\sum_{\ell=1}^{\infty}\frac{\xi}{\ell}=\infty,$$

that is, (2.3) holds. The condition (2.13) becomes

$$\sum_{\ell=1}^{\infty}\frac{\xi}{\ell(\ell+1)(\ell+2)}\frac{(\ell-2)^2}{2}\approx\sum_{\ell=1}^{\infty}\frac{\xi}{2\ell}=\infty,$$

that is (2.13) holds. The condition (2.14) becomes

$$\begin{aligned}\limsup_{\ell\to\infty}\quad&\left\{\frac{1}{\ell-2}\sum_{s=1}^{\ell-3}\frac{\xi}{s(s+1)(s+2)}\frac{(s+1)(s-2)^2}{2}+\sum_{s=\ell-2}^{\ell-1}\frac{\xi}{s(s+1)(s+2)}\frac{(s-2)^2}{2}\right.\\&\left.+(\ell-2)\sum_{s=\ell}^{\infty}\frac{\xi}{s(s+1)(s+2)}\frac{(s-2)}{2}\right\}=\xi,\end{aligned}$$

that is (2.14) is satisfied if $\xi>1$. Therefore by Corollary 2.5, equation (3.1) is almost oscillatory if $\xi>1$.

Example 3.2. Consider the third-order semi-canonical difference equation

$$\Delta\left(\ell(\ell+1)\Delta\left(\frac{1}{\ell}\Delta\mu(\ell)\right)\right)+\frac{\xi}{\ell^2}\mu^3(\ell-2)-\frac{\chi}{\ell^3}tan^{-1}(\mu(\tau(\ell)))=0, \tag{3.2}$$

where $\xi>0,\ \chi>0$. A simple calculation shows that $\Phi(\ell)=\frac{1}{\ell},\ \eta(\ell)=\zeta(\ell)=1,\ Q_1(\ell)=\frac{\xi}{(\ell+1)\ell^2},\ Q_2(\ell)=\frac{\chi}{(\ell+1)\ell^3},\ f(\delta)=\delta^3,\ \Psi(\ell)\approx\ell$ and $\Theta(\ell)\approx\frac{\ell^2}{2}$. Since $g(\delta)=tan^{-1}(\delta)$ is bounded and condition (H_5) holds. Further $\lim_{\gamma\to\infty}\frac{\gamma}{f(\gamma)}=0$. The condition (2.3) becomes

$$\sum_{\ell=1}^{\infty}\sum_{s=\ell}^{\infty}\sum_{t=s}^{\infty}\frac{\xi}{(t+1)t^2}=\sum_{\ell=1}^{\infty}\frac{\xi}{\ell}=\infty,$$

therefore condition (2.3) is satisfied. The condition (2.15) becomes

$$\sum_{\ell=1}^{\infty}\sum_{s=\ell}\frac{\xi}{(\ell+1)\ell^2}\frac{(\ell-2)^3}{8}\approx\sum_{\ell=1}^{\infty}\frac{\xi}{8}=\infty,$$

therefore condition (2.15) is satisfied. The condition (2.16) becomes

$$\begin{aligned}\limsup_{\ell\to\infty}\quad&\left\{\frac{1}{(\ell-2)^2}\sum_{s=1}^{\ell-3}\frac{\xi}{(s+1)s^2}(s-2)^3(s+1)+\frac{1}{(\ell-2)^2}\sum_{s=\ell-2}^{\ell-1}\frac{\xi}{(s+1)s^2}\frac{(s-2)^3}{8}\right.\\&\left.+(\ell-2)^3\sum_{s=\ell}^{\infty}\frac{\xi}{(s+1)s^2}\frac{(s-2)^3}{8}\right\}=\infty,\end{aligned}$$

therefore condition (2.16) is satisfied. Hence by Theorem 2.6, equation (3.2) is almost oscillatory.

4 Conclusion

In this paper we provide a new technique for studying the oscillatory and asymptotic behavior of third order semi-canonical difference equations with positive and negative terms. Further the results reported in the literature cannot be applied to equations (3.1) and (3.2) since the these equations are semi-canonical and containing both positive and negative terms. Thus the results established in this paper are new and complement to the existing results obtained for third order difference equations. It is also interesting to study equation (E) if the condition

$$\sum_{\ell=\ell_0}^{\infty} \frac{1}{\varphi(\ell)} = \infty \quad \text{and} \quad \sum_{\ell=\ell_0}^{\infty} \frac{1}{\psi(\ell)} < \infty$$

is satisfied. This left as our future research.

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