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Estimate the interval of the fuzzy parameters of the inverse Weibull distribution

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Abstract

In this article, two estimation methods are used to estimate the interval of the parameters for the inverse Weibull distribution in the case of fuzzy data. These two methods are based on, the maximum likelihood method and the relative maximum likelihood method. In addition, we compare the Maximum likelihood intervals with relative maximum likelihood intervals for both real and fuzzy data. The results of the comparison showed that the fuzziness interval estimation is better than the real one. Examples of applications are given.

Keywords: Fuzzy data, Inverse Weibull distribution, Interval Estimation
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1 Introduction

Recently, many models in real life can be more accurate if we describe them in form of fuzzy sets or numbers, especially uncertainty models such as statistical variation of observed lifetimes. In addition, there is another type of uncertainty, which is in every individual observation called vagueness such that observation is not a real number but can be more or less fuzzy. For example, if we have a contestant of fitness and the weight criteria to be in this competition is (40, 45, 50, 55, or 60) kg. If we have a (with competitors, then it is natural that we will choose persons such that their weight is one of the given weights. What about persons with 56 or 49 kg.? The registration will be more accurate and clear if the weight criteria are (45-60) kg. Then, each element of the registers list will have a certain degree of fitness, and an element in the set of competitors is distinguished by a membership function that gives values between (0, 1). On the other hand, reliability is a very powerful tool to evaluate the work of systems or items. It measures the probability of units or vehicles working for a specific period without failure. In lifetime probability functions, the parameters are fragile in their classic form. Hence and due to inaccuracies, it is needful to generalize the classical statistical estimation methods of real data into fuzzy data which means the reliability model may give an exact formula of traditional functions. As a result, we can deal with an equivalent term to the original term of reliability. This equivalent model of functions is defined as the fuzzy probability of the mathematical formula of unit period and the degree of membership.

Zadeh in 1965 [15], used the term “fuzzy variable” to express inaccurate linguistic idiom and parlance. Which was the beginning of the radix of the theory of fuzzy sets theory. A fuzzy set is a group of elements with continuous degrees

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of membership. Each object in the set is distinguished by the function of membership and this degree is between zero and one [16]. In statistical inference, there are two types of parameter estimation: estimation in point, which means we have one value for the estimated parameters, and estimation in the interval, which means we estimate the interval of parameters [4]. Wu in 2004 used the Bayes approach which is fuzzy handling of fuzzy data with fuzzy distributions [14]. Pak and Saraj 2013 used the Bayes approach to estimate the reliability of Rayleigh distribution where the data is fuzzy lifetime data and estimate the parameters of distributions from that data [10]. Also, Pak, used three methods to estimate the shape parameter for lognormal distribution such that the data was in the fuzzy format [9]. In 2020 Raphael Masila Mweleli and others estimate the Weibull distribution's parameters in the interval where the data was lifetime data with censoring of type two [7]. In this article, we used fuzzy data to estimate the interval in the inverse Weibull distribution.

Definition 1.1. Fuzzy set: each element in this set has a definite degree of belonging. This degree of membership in the interval [0,1]. The element or object is allowed to belong to a partial membership. Let X be the universe set and the $\widetilde{A}$ is a fuzzy subset distinguished by a membership function $\mu_{\tilde{A}}(x)$, such that:

$$\tilde{A} = \{(x_i, \mu_{\tilde{A}}(x_i)), x_i \in X, i = 1, 2, 3 \ldots\ldots n, 0 < \mu_{\tilde{A}} < 1\} \tag{1.1}$$

where if $\mu_{\tilde{A}}(x) = 1$ then x completely belongs to $\tilde{A}$ and if $\mu_{\tilde{A}}(x) = 0$ then x does not belong to $\tilde{A}$ while if $\mu_{\widetilde{A}}(x) = 0.6$ then x the degree of belonging is 0.6 to $\tilde{A}$ [2, 17].

Fuzzy Numbers: it is a tool that is used to characterize uncertainty, which is triangular, trapezoidal, or any other formula. Or in other words, it is a partial number symbolized by $\tilde{A}$ to set of real numbers and characterized by a membership function. The fuzzy numbers form a fuzzy set with the following conditions [6]:

1. Convex and normalized.
2. The belonging function $\mu_{\widetilde{A}}$ is semi-continuous from the top.
3. The α level group must be assigned for each $\alpha \in [0, 1]$.
4. It must be defined on real set numbers.

Triangular fuzzy number: this number is the most famous type because it can be represented with three points (a_1, a_2, a_3) such that $a_1 < a_2 < a_3$ it is the triangle within the interval $[a1, a2]$. The head of the triangle at $x = a_2$ and it can be written as $\widetilde{A} = (a1/a2/a3)$. This number has a membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & a_1 \le x \le a_2 \\ \frac{a_3-x}{a_3-a_2} & a_2 \le x \le a_3 \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

Fuzzy sample space

It is fuzzy parts $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots\ldots, \tilde{x}_n)$ from $X = (X_1, X_2, X_3, \ldots\ldots X_n)$. Fuzzy sets for X with membership functions has Borel measure, with orthogonal constraint [13]:

$$\sum \widetilde{x} \in X \qquad \mu_{\widetilde{x}}(x) = 1, \text{ for each } x \in X. \tag{1.3}$$

In addition, this space is called a fuzzy information system (FIS).

Fuzzy event

If $X = (X_1, X_2, X_3, \ldots\ldots X_n)$ in space and B_x is the smallest Borel field in X. Then, the fuzzy event is a fuzzy subset $\widetilde{A}$ in which its membership has a measurable Borel field [12].

2 Inverse Weibull Distribution (IW)

The inverse Weibull distribution is one of the continuous lifetime probability distributions. Maurice Frechet (1828-1973) introduced this distribution. This distribution has many applications in the modeling and analysis of many natural events such as earthquakes, floods, rainfalls, wind speeds, life tests, and sea currents. Also, it is used to model failure rates that are commonly used in reliable biological studies as well as in infant mortality modeling [1, 5].

If x is a random variable with a Weibull distribution, then $\left(\frac{1}{y}\right)$ is the inverse of the values of the random variable x and it has a probability distribution called the Inverse Weibull distribution with the following density [3]:

$$f\left(x,\alpha,\beta\right)=\alpha\beta^{\alpha}x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right),\quad x>0,\alpha,\beta>0 \tag{2.1}$$

where α is the shape parameter and β is the scale parameter. The cumulative distribution function is given as follows:

$$F\left(x,\alpha,\beta\right)=p\left(X\leq x\right)=\int_{0}^{x}f\left(u\right)du=\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\qquad ;x>0 \tag{2.2}$$

3 Interval Estimators of the Parameters of IW Distribution

Estimation is considered a basic concept in statistical inference because estimation provides us with parameter values of the tested models of the population to the statistics that are sorted by drawn sample.

In this section, a summarized description of some estimation methods for finding interval estimators of the parameters of the IW distribution with fuzzy data.

Fuzzy Maximum Likelihood Estimation Method [7, 8]

If we have a random, sample of size n such that $x=(x_1,\ldots\ldots\ldots.,x_n)$ is based on inverse Weibull distribution with a density function:

$$f\left(x,\alpha,\beta\right)=\alpha\beta^{\alpha}x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right). \tag{3.1}$$

In addition, let $X=(X_1,\ldots\ldots,X_n)$ be a random vector representing the sample space. The likelihood function for complete data *((Crisp set))* is

$$L\left(\alpha,\beta;x\right)=\prod_{i=1}^{n}f\left(\alpha,\beta;x\right)=\alpha^{n}\beta^{n\alpha}\prod_{i=1}^{n}x^{-(\alpha+1)}\exp\left(-\sum_{i=1}^{n}\left(\frac{\beta}{x}\right)^{\alpha}\right) \tag{3.2}$$

where x is purposely visible and available with full information about the crisp vector. Now, if we have a case such that x is not purposely visible and available in a partial way in a fuzzy subset form with a membership function $\mu_{\tilde{A}}x$ having Borel measurement. The fuzzy observation $\widetilde{x}$ can express the partial observation x from the random vector X. the membership function $\mu_{\tilde{A}}$ can be considered as a probability distribution that expresses the restrictions of that partial observation $\widetilde{x}$. The fuzzy set X has two characteristics:

- X is drawn from X.
- The vector X observer after the part-time reference is encrypted in $\mu_{\tilde{A}}x$

Note that in this model only the first characteristic is a random experiment. While the second one is about collecting information about x and modeling this information as a fuzzy probability distribution. The information on x can be considered as the following probability distribution:

$$\mu_{\widetilde{x}}(x)=\mu_{\widetilde{x}_1}(x)\times\ldots\ldots\ldots\times\mu_{\widetilde{x}_n}(x). \tag{3.3}$$

So, if x is given and its function is supposed to have Boral measurement. Then we can compute the probability of it by the definition of fuzzy probability. Moreover, we can have the maximum likelihood function as follows:

$$L\left(\alpha,\beta;\widetilde{x}\right)=p\left(\widetilde{x};\alpha,\beta\right)=\int f\left(\widetilde{x};\alpha,\beta\right)xxdx \tag{3.4}$$

Since the data vector x is identically and independently distributed and the membership function is analytic then, the fuzzy maximum likelihood function of fuzzy inverse Weibull distribution is written in the following form:

$$L^* = \log(L_0(\alpha,\beta;\widetilde{x})) == n\log(\alpha) + n\alpha\log(\beta) + \sum_{i=1}^{n}\log\left(\int_0^\infty x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{x_i}(x)dx\right) \tag{3.5}$$

and estimators of ML of α and β can be obtained by maximizing L^* and partial derivation for α and β with the equivalence of zero as follows:

$$= \frac{n}{\widehat{\alpha}} + n\log(\hat{\beta}) - \left[\sum_{i=1}^{n}\frac{\int_0^\infty\left[\left(x^{-(\widehat{\alpha}+1)}\cdot\ln(x) + x^{-2\widehat{\alpha}-1}(\widehat{\beta})^{\widehat{\alpha}}\cdot\ln\left(\frac{\widehat{\beta}}{x}\right)\right)\right)\exp\left(-\left(\frac{\widehat{\beta}}{x}\right)^{\widehat{\alpha}}\right)\cdot\mu_{\widetilde{x}_i}(x)dx.\right]}{\int_0^\infty x^{-(\widehat{\alpha}+1)}\exp\left(-\left(\frac{\widehat{\beta}}{x}\right)^{\widehat{\alpha}}\right)\mu_{\widetilde{x}_i}(x)dx}\right] = 0 \tag{3.6}$$

$$\frac{\partial L^*}{\partial\beta} = \frac{n\hat{\alpha}}{\hat{\beta}} - \sum_{i=1}^{n}\frac{\int_0^\infty x^{-\alpha-1}\left(\frac{\hat{\beta}}{x}\right)^{\hat{\alpha}}\frac{\hat{\alpha}}{\hat{\beta}}\exp\left(-\left(\frac{\hat{\beta}}{x}\right)^{\hat{\alpha}}\right)\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{-(\hat{\alpha}+1)}\exp\left(-\left(\frac{\hat{\beta}}{x}\right)^{\hat{\alpha}}\right)\mu_{\tilde{x}_i}(x)dx} = 0 \tag{3.7}$$

Equations 3.6 and 3.7 have no closed formula for the solution. Hence, we will depend on a numerical technique called ***Newton–Raphson method*** to obtained ML estimators $\widehat{\alpha}_{fmle}$ and $\widehat{\beta}_{fmle}$. Let $\theta = (\alpha,\beta)^T$ be parameters vector, after (h+1) step from iterative steps, we have the parameters

$$\theta^{h+1} = \theta^h - \left[\left.\frac{\partial^2 L^*(\alpha,\beta;\tilde{x})}{\partial\theta\partial\theta^T}\right|_{\theta=\theta^h}\right]^{-1}\cdot\left[\left.\frac{\partial L^*(\alpha,\beta;\tilde{x})}{\partial\theta}\right|_{\theta=\theta^h}\right]$$

such that

$$\frac{\partial L^*(\alpha,\beta;\widetilde{x})}{\partial\theta} = \begin{pmatrix}\frac{\partial L^*(\alpha,\beta;\widetilde{x})}{\partial\alpha}\\ \frac{\partial L^*(\alpha,\beta;\widetilde{x})}{\partial\beta}\end{pmatrix}$$

$$\frac{\partial^2 L^*(\alpha,\beta;\tilde{x})}{\partial\theta\partial\theta^T} = \begin{pmatrix}\frac{\partial L^*(\alpha,\beta;\tilde{x})}{\partial\alpha^2} & \frac{\partial L^*(\alpha,\beta;\tilde{x})}{\partial\alpha\partial\beta}\\ \frac{\partial L^*(\alpha,\beta;\tilde{x})}{\partial\alpha\partial\beta} & \frac{\partial L^*(\alpha,\beta;\tilde{x})}{\partial\beta^2}\end{pmatrix}$$

and

$$S_1 = \frac{\partial L^{*2}}{\partial\alpha^2} = -\frac{n}{\alpha^2} - \sum_{i=1}^{n}\left(\frac{1}{(\int_0^\infty x^{-(\alpha+1)}\exp(-(\frac{\beta}{x})^\alpha)\mu_{\tilde{x}_i}(x)dx)}\left(\int_o^\infty \ln(x)^2\left(\frac{\beta}{x}\right)^\alpha + \right.\right.$$

$$\left.2\ln(x)\ln\left(\frac{\beta}{x}\right) - \ln\left(\frac{\beta}{x}\right)^2 + \left(\frac{\beta}{x}\right)^\alpha\ln\left(\frac{\beta}{x}\right)^2\right)\cdot\left(x^{-\alpha-1}\left(\frac{\beta}{x}\right)^\alpha\exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx\right) +$$

$$\frac{\left(\int_0^\infty\ln(x) + \left(\frac{\beta}{x}\right)^\alpha\ln\left(\frac{\beta}{x}\right)\right)\cdot\left(x^{-\alpha-1}\exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx\right)^2}{\left(\int_0^\infty x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx\right)^2}$$

$$S_2 = \frac{\partial L^{*2}}{\partial\beta^2} = -\frac{n\alpha}{\beta^2} - \sum_{i=1}^{n}\left(\frac{\left(\frac{\alpha^2}{\beta^2} - \frac{\alpha}{\beta^2} - \frac{\alpha^2}{\beta^2}\left(\frac{\beta}{x}\right)^\alpha\right)\cdot\left(x^{-\alpha-1}\exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\right)\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx} + \right.$$

$$\left.\frac{\int_0^\infty x^{-\alpha-1}\left(\frac{\beta}{x}\right)^\alpha\frac{\alpha}{\beta}\exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx}{\left(\int_0^\infty x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx\right)^2}\right)$$

$$S_3 = \frac{\partial L^{*2}}{\partial\alpha\partial\beta} = -\frac{n}{\beta} - \sum_{i=1}^{n}\left(\frac{-\ln(x)\frac{\alpha}{\beta} + \frac{\alpha}{\beta} + \frac{1}{\beta} - \left(\frac{\beta}{x}\right)^\alpha\ln(x)\frac{\alpha}{\beta}\right)\cdot\left(x^{-\alpha-1}\exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\right)\mu_{\tilde{x}_i}(x)dx}{\left(\int_0^\infty x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx\right)} + \right.$$

$$\left.\frac{\int_0^\infty x^{-\alpha-1}\ln(x)\exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right) + x^{-\alpha-1}\left(\frac{\beta}{x}\right)^\alpha\ln\left(\frac{\beta}{x}\right)\exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx\int_0^\infty x^{-\alpha-1}\left(\frac{\beta}{x}\right)^\alpha\frac{\alpha}{\beta}\exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx}{\left(\int_0^\infty x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx\right)^2}\right).$$

We proceed with replication until $||\theta^{h+1} - \theta^h||$ is an approach to ε where $\varepsilon > 0$ is a very small positive number. The Wald Technique in this method we need to compute the variance and covariance matrix for the parameters that we want to estimate the confidence interval for it by using the property of *(Gramer –Roa lower Bound)* which is equal to the inverse of fisher information ‘s matrix detriment as follows

$$(-\frac{1}{I(\hat{\alpha},\ \hat{\beta})}.$$

Fisher information matrix can be founded from:

$$I\left(\hat{\alpha},\ \hat{\beta}\right) = -E\begin{pmatrix} S_1 & S_3 \\ S_3 & S_2 \end{pmatrix}$$

That means the variance and covariance matrix to the parameters is

$$\Sigma = \frac{1}{E\begin{pmatrix} -S_1 & -S_3 \\ -S_3 & -S_2 \end{pmatrix}} = \begin{pmatrix} \widehat{\sigma}^2(\hat{\alpha}) & \hat{\sigma}(\hat{\alpha},\hat{\beta}) \\ \hat{\sigma}(\hat{\alpha},\hat{\beta}) & \hat{\sigma}^2(\hat{\beta}) \end{pmatrix}$$

Hence, the confidence intervals to the parameters that we want to estimate can be computed according to converge theory of MLE, then the sampling distribution

$$z = \frac{\hat{\alpha} - \alpha}{\sqrt{\hat{\alpha}^2(\hat{\alpha})}}$$

$$z = \frac{\hat{\beta} - \beta}{\sqrt{\hat{\beta}^2(\hat{\alpha})}}$$

Can be approximated by using a standard normal distribution with a confident interval of 95% and the confidence interval for each parameter is:

$$\hat{\alpha} - Z_{1-\frac{\Psi}{2}}\sqrt{\widehat{\sigma}^2(\hat{\alpha})} < \alpha < \hat{\alpha} + Z_{1-\frac{\Psi}{2}}\sqrt{\widehat{\sigma}^2(\hat{\alpha})} \tag{3.8}$$

$$\hat{\beta} - Z_{1-\frac{\psi}{2}}\sqrt{\widehat{\sigma}^2(\hat{\beta})} < \beta < \hat{\beta} + Z_{1-\frac{\psi}{2}}\sqrt{\widehat{\sigma}^2(\hat{\beta})} \tag{3.9}$$

Fuzzy Relative MLE method [7, 8]

The fuzzy relative maximum likelihood function is given by the following formula:

$$L^* = \log(L_0(\alpha,\beta;\widetilde{x})) = n\log(\alpha) + n\alpha\log(\beta) + \sum_{i=1}^{n}\log\left(\int_0^\infty x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx\right) \tag{3.10}$$

If we substitute the estimated fuzzy MLE parameters $\widehat{\alpha}_{fmle}$ and $\widehat{\beta}_{fmle}$ in equation 3.10 then we get:

$$\begin{gathered} L^*\left(\widehat{\alpha}_{fmle},\widehat{\beta}_{fmle},\tilde{x}\right) = n\log\left(\widehat{\alpha}_{fmle}\right) + n\widehat{\alpha}_{fmle}\log\left(\widehat{\beta}_{fmle}\right) + \\ \sum_{i=1}^{n}\log\left(\int_0^\infty x^{-(\widehat{\alpha}_{fmle}+1)}\exp\left(-\left(\frac{\widehat{\beta}_{fmle}}{x}\right)^{\widehat{\alpha}_{fmle}}\right)\mu_{\tilde{x}_i}(x)dx\right) \end{gathered} \tag{3.11}$$

In addition, the relative likelihood function is founded from compute the ratio of the equation 3.10 and 3.11 as follows:

$$R(\alpha,\beta;\widetilde{x}) = \frac{L^*(\alpha,\beta;\widetilde{x})}{L^*(\widehat{\alpha}_{fmle},\ \widehat{\beta}_{fmle},\ \widetilde{x})} =$$

$$\frac{\log(\alpha)+n\alpha\log(\beta)+\sum_{i=1}^{n}\log\left(\int_0^\infty x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\mu_{\tilde{x}_i}(x)dx\right)}{\log\left(\widehat{\alpha}_{fmle}\right)+n\widehat{\alpha}_{fmle}\log\left(\widehat{\beta}_{fmle}\right)+\sum_{i=1}^{n}\log\left(\int_0^\infty x^{-(\widehat{\alpha}_{fmle}+1)}\exp\left(-\left(\frac{\widehat{\beta}_{fmle}}{x}\right)^{\widehat{\alpha}_{fmle}}\right)\mu_{\tilde{x}_i}(x)dx\right)} \tag{3.12}$$

If we denote by $\hat{\alpha}(\beta)$ the MLE of β given β. Then the fuzzy relative MLE estimator of β can be founded by maximizing the value of the likelihood function of the distribution such that

$$L_p(\beta)= \ max_{\alpha}L^*(\alpha,\beta;\widetilde{x})=L^*(\hat{\alpha}(\beta),\beta;\widetilde{x})$$

$$\begin{aligned}&=\max_{\alpha}\log(\hat{\alpha}(\beta))+n\hat{\alpha}(\beta)\log(\beta)+\\&\sum_{i=1}^{n}\log\left(\int_0^\infty x^{-(\widehat{\alpha}(\beta)+1)}\exp\left(-\left(\frac{\beta}{x}\right)^{\widehat{\alpha}(\beta)}\right)\mu_{\tilde{x}_i}(x)dx\right)\end{aligned} \tag{3.13}$$

And we maximize $L_p(\beta)$ by differentiating w.r.t β and equating it to zero as follows

$$R_P(\beta)=\frac{\partial L_p(\beta)}{\partial\beta}=\frac{n\widehat{\alpha}(\widehat{\beta})}{\widehat{\beta}}+\sum_{i=1}^{n}\frac{x^{-(\widehat{\alpha}(\widehat{\beta})+1)}\widehat{\alpha}(\beta)\left(\frac{\beta}{x}\right)^{\widehat{\alpha}(\widehat{\beta})-1}\exp\left(-\left(\frac{\beta}{x}\right)^{\widehat{\alpha}(\widehat{\beta})}\right)\mu_{\tilde{x}_i}(x)}{\int_0^\infty x^{-(\widehat{\alpha}(\widehat{\beta})+1)}\exp\left(-\left(\frac{\beta}{x}\right)^{\widehat{\alpha}(\widehat{\beta})}\right)\mu_{\tilde{x}_i}(x)dx}=0 \tag{3.14}$$

In addition, we can find the relative MLE estimator of α as follows

$$L_p(\alpha)= \ max_{\alpha}L^*(\alpha,\beta;\widetilde{x})=L^*\left(\alpha,\hat{\beta}(\alpha);\widetilde{x}\right)$$

$$=\max_{\alpha}\log(\alpha)+n\alpha\log(\hat{\beta}(\alpha))+\sum_{i=1}^{n}\log\left(\int_0^\infty x^{-(\alpha+1)}\exp\left(-\left(\frac{\widehat{\beta}(\alpha)}{x}\right)^{\alpha}\right)\mu_{\tilde{x}_i}(x)dx\right) \tag{3.15}$$

Similarly, we maximize $L_p(\alpha)$ by differentiating w.r.t α and equating it to zero as follows

$$\begin{aligned}&R_P(\alpha)=\frac{\partial L_p(\alpha)}{\partial\alpha}=\frac{1}{\hat{\alpha}}+n\log(\hat{\beta}(\alpha))+\\&\sum_{i=1}^{n}\frac{x^{-(\alpha+1)}\exp\left(-\left(\frac{\widehat{\beta}(\alpha)}{x}\right)^{\alpha}\left(\frac{\widehat{\beta}(\alpha)}{x}\right)^{\alpha}\right)\log\left(\frac{\widehat{\beta}(\alpha)}{x}\right)\mu_{\tilde{x}_i}(x)-(\alpha+2)x^{-(\alpha+2)}\exp\left(-\left(\frac{\widehat{\beta}(\alpha)}{x}\right)^{\alpha}\right)\mu_{\tilde{x}_i}(x)}{\int_0^\infty x^{-(\alpha+1)}\exp\left(-\left(\frac{\widehat{\beta}(\alpha)}{x}\right)^{\alpha}\right)\mu_{\tilde{x}_i}(x)dx}=0.\end{aligned} \tag{3.16}$$

For the parameter β, the 100 ψ % relative-likelihood confidence interval will be the set of all values for which

$$R_P(\beta)\geq\psi. \tag{3.17}$$

After solving the following equations:

$$r_P(\beta)-Log(\psi)=0 \tag{3.18}$$

$$r_P(\beta)-Log(0.147)=0 \tag{3.19}$$

The relative likelihood confidence interval is obtained which can be solved numerically. Similarly, for the parameter α, the 100 ψ % relative-likelihood confidence interval will be the set of all values for which

$$R_P(\alpha)\geq\psi \tag{3.20}$$

and, the relative likelihood confidence interval is obtained by solving the following equations numerically:

$$r_P(\alpha)-Log(\psi)=0 \tag{3.21}$$

$$r_P(\alpha)-Log(0.147)=0 \tag{3.22}$$

Applications (with fuzzy data)

Example 3.1. In this example the real data (from the article " The Long Term Fréchet distribution: Estimation, Properties and its Application") [11] are transformed to fuzzy data using the triangular function (2) with $\alpha - Cut = 0.1$ and given in the following table:

Table 1: The real and fuzzy data

real fuzzy	membership	real data	fuzzy data	membership	real data
1.989	0.166482223	0.8575	0.5918	0	0.0301
1.989	0.176965331	0.9096	0.6	0.001670054	0.0384
2.5068	0.18799171	0.9644	0.6438	0.006619852	0.063
2.6466	0.196804765	1.0082	0.6849	0.011026379	0.0849
3.0384	0.251936659	1.2822	0.7397	0.01158977	0.0877
3.1726	0.26461297	1.3452	0.8575	0.013239703	0.0959
3.4411	0.275639349	1.4	0.9096	0.022052758	0.1397
4.4219	0.300991972	1.526	0.9644	0.026459285	0.1616
4.4356	0.340127568	1.7205	1.0082	0.028129339	0.1699
4.5863	0.3941528	1.989	1.2822	0.036942393	0.2137
4.6904	0.445421437	2.2438	1.3452	0.036942393	0.2137
4.7808	0.498340007	2.5068	1.4	0.037485664	0.2164
4.9863	0.526469345	2.6466	1.526	0.041912312	0.2384
5	0.60530393	3.0384	1.7205	0.048512042	0.2712
	0.632306485	3.1726		0.049075434	0.274
	0.686331717	3.4411		0.071671462	0.3863
	0.883679752	4.4219		0.082154571	0.4384
	0.886436347	4.4356		0.085454436	0.4548
	0.916758889	4.5863		0.113020383	0.5918
	0.937704984	4.6904		0.114670315	0.6
	0.955894485	4.7808		0.12348337	0.6438
	0.997243405	4.9863		0.131753154	0.6849
	1	5		0.142779533	0.7397

Table 2: Point estimation of the parameters α and β for the fuzzy and real data with error parentheses

Parameters	Par. of MLE	Par. of Relative MLE	K-s
(fuzzy) α=4	4.4501 (.4501)	4.4753 (.4753)	1.5682
(real)	0.6459 (3.3541)	1.8549 (2.1451)	0.0945
(fuzzy) $\beta = 0.9$	0.9072 (.0072)	0.9079 (0.0079)	
(real)	0.1728 (0.7272)	0.1300 (0.7700)	

The data in a table 1 was tested by the goodness of fit tests (Kolmogorov-Smirnov) to ensure the data is IW distributed or not according to the hypothesis:

$$H_0 : t \quad IW\ disterbution$$

$$H_1 : t \quad \nsim IW\ disterbution$$

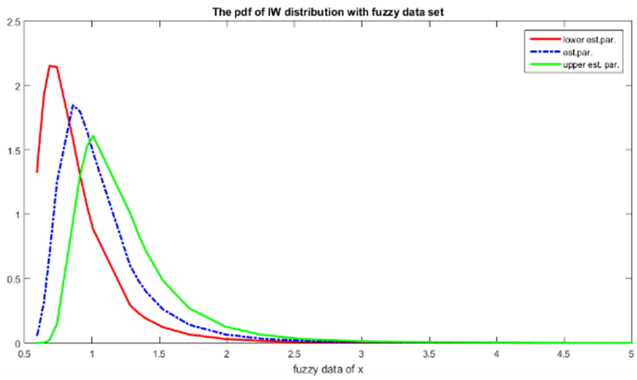
As shown in table 2 the test is applied for both real and fuzzy data and the test's result (1.5682) for fuzzy values while (0.0945) is the result of real data so, despite both real and fuzzy data are IW distributed but the fuzzy data is appropriate to IW distribution more than the real one with a significant value greater than 0.05. In addition, note that the value of estimated parameters in the fuzzy case is with less error than real data as given in table 2.

Also, results in table 2 showed that the MLE method exhibits results of point estimated with error values better than the relative MLE method such that at parameter $\alpha = 4$ the estimator of MLE is (4.4501) with relative MLE estimator ((4.4753). Same analysis to the second parameter $\beta = 0.9$, the MLE method gives a closer result in estimating according to the relative MLE.

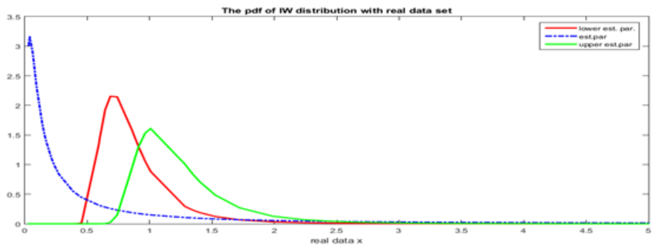
Table 3: Interval estimations of the parameters α and β for the fuzzy data

Fuzzy Data	MLE	CL for mle	Relative Mle	CL for r-mle
Integral Length Of αh	(4.3119, 4.5883)	0.2764	(4.3370, 4.6136)	0.2766
Of βh	(0.7446, 1.0699)	0.3253	(0.7443, 1.0715)	0.3272

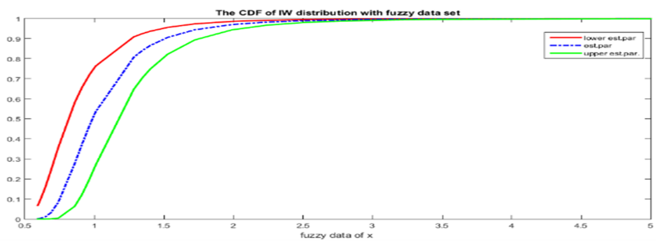
Interval estimation values to the parameters of fuzzy data are listed in tables (3) as the MLE method, the 95% approximate confidence intervals for α and β are obtained as [4.5883,4.3119] and [0.7446, 1.0699] respectively. Further, using the relative MLE method, the 95% approximate confidence intervals for α and β are obtained as [4.3370, 4.6136] and [0.7443, 1.0715], respectively. So, it can have noticed that the interval estimates obtained by the MLE method are narrower as compared to those obtained by the relative MLE method.



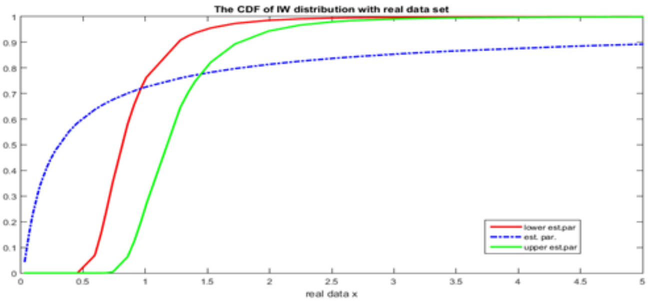
(a)



(b)



(c)



(d)

Figure 1: The pdf and CDF curves of fuzzy and real data at estimated intervals.

It is evident from Figure 1. (a, b, c and d) that the estimated parameters of IW distribution with fuzzy data are located within interval estimates while in the case of real data, the estimated parameters given in table 1 are located out of the estimated intervals.

Example 3.2. In this example the real data (from the article "Comparison of some interval estimation methods for the parameters of the gamma distribution") [8] are transformed to fuzzy data using the triangular function (2) with $\alpha - Cut = 0.1$ and given in the following table:

Table 4: The real and fuzzy data

membership	real data	fuzzy data	membership	real data
0.120718435	0.5425	0.5096	0	0.0001
0.142663195	0.6411	0.5425	2.23E-05	0.0002
0.204246511	0.9178	0.6411	0.001802764	0.0082
0.226191271	1.0164	0.9178	0.003026863	0.0137
0.278649485	1.2521	1.0164	0.005475062	0.0247
0.296921947	1.3342	1.2521	0.014622421	0.0658
0.298146046	1.3397	1.3342	0.026195722	0.1178
0.365827602	1.6438	1.3397	0.032894883	0.1479
0.373149941	1.6767	1.6438	0.044490441	0.2
0.437782377	1.9671	1.6767	0.08655494	0.389
0.470721773	2.1151	1.9671	0.089003138	0.4
0.821949211	3.6932	2.1151	0.090227237	0.4055
1	4.4932	3.6932	0.091451336	0.411
		4.4932	0.113396096	0.5096

Table 5: The results of point estimations of the parameters α and β for the fuzzy and real data

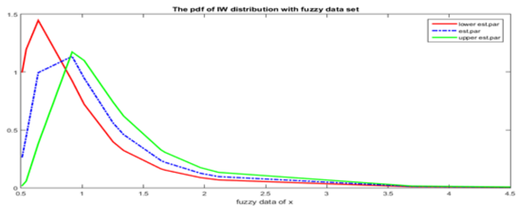
Parameters	Par. of MLE	Par. of Relative MLE	ks
(fuzzy) α=0.3	0.4744 (0.1744)	0.4070 (0.1070)	0.9856
(real)	0.5499 (0.2499)	0.5840 (0.2840)	0.0895
(fuzzy) $\beta = .08$	0.0851 (0.0051)	0.0778 (0.0022)	
(real)	0.0227 (0.05730)	0.0608 (0.0102)	

The data in table 4 was tested to indicate if the data fitting IW distribution and the results of the KS test are (0.9856,0.0895) for fuzzy and real data, respectively. The values of KS indicate that both cases are WI distributions, but the fuzzy data fitted the IW distribution more than the real one ($0.9856 > 0.0895 > 0.05$). In addition, the error values in parentheses listed in table 5 indicate that the fuzzy data are more appropriate to estimate parameters than real data. In addition, relative MLE gives estimators with errors less than MLE's estimators. Hence, with data in table 4 the relative MLE method is better than the MLE method.

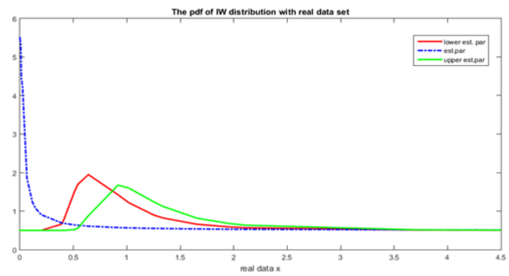
Table 6

Fuzzy Data	MLE	CL of mle	Relative Mle	CL of r-mle
Interval Length Of αh	(0.4580, 0.4909)	0.0329	(0.3920, 0.4222)	0.0302
Of βh	(0.0669, 0.1033)	0.0364	(0.0623, 0.0933)	0.0310

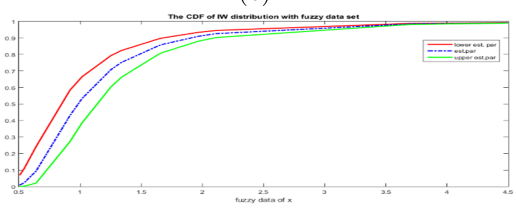
As a result of the table 5 the interval estimation is calculated for the fuzzy case and listed in the table 6, we note that the intervals estimation of α and β are (0.3920, 0.4222) and (0.0623,0.0933) respectively by relative MLE is narrower than interval estimation by MLE such that the CL of relative MLE is 0.0302 which is less than 0.0329, Cl of MLE.



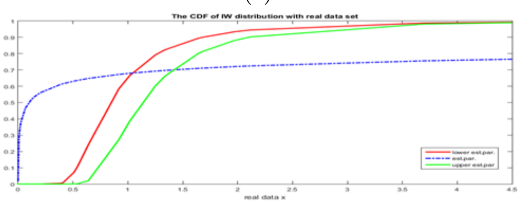
(a)



(b)



(c)



(d)

Figure 3: The pdf and cdf curves of estimated intervals for fuzzy and real data

The curves in Figure 3 of pdf and cdf of IW distribution, declare that fuzzy data is more accurate in point and interval estimation for the parameters of WI distribution.

4 Conclusion

Based on the results of estimations in fuzzy and real cases, it can be concluded that fuzzy data gives more accurate estimators in points than in real cases. In addition, the interval estimation in the case of fuzzy data exhibits narrower intervals in both estimated methods than the real data. Both methods, MLE and relative MLE applied in this paper can be extended to construct approximate confidence intervals of IW distribution with fuzzy data.

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