Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 2717–2729
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.25999.3185

On Ostrowski type inequalities via the Taylor's formula

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(Communicated by Michael Th. Rassias)

Abstract

In the present article, our aim is to obtain new Ostrwoski type inequalities by using Taylor's formula with weighted Montgomery identity and the Green's function. The results we acquire containing the identities for the sum $\sum_{i=1}^{m} p_i f_1(\lambda_i)$ and the integral $\int_\mu^\nu p(\xi)\ f_1(g(\xi))\,d\xi$. We also estimate the difference of two weighted integral means for the result obtained by the Taylor formula with weighted Montgomery identity.

Keywords: Ostrowski inequality, Taylor's formula, Green's function, Integral means
2020 MSC: 26A51, 26D15, 26D20

1 Introduction

Integral inequalities that establish bounds on the physical quantities have great significance in the sense that these types of inequalities are not only applicable in numerical integration, approximation theory, nonlinear analysis,statistics, probability theory, stochastic analysis, integral operator theory and information theory but also we can track its applications in different areas of technology, biological sciences and physics. A renowned integral inequality involving mapping with bounded derivative known as Ostrowski's inequality was presented by Alexander Markovich Ostrowski in 1938 [16] which may be produced in various ways by using different techniques: using direct calculation, using Lagrange mean value theorem and also by using Montgomery identity etc. can be stated as:

Theorem 1.1. Let f_1 be a real-valued continuous mapping on $[\eta,\theta]$ and differentiable on (η,θ) such that f_1' is bounded by some real constant K. Then

$$\begin{aligned}\left|f_1(\lambda)-\frac{1}{\theta-\eta}\int_\eta^\theta f_1(t)dt\right| &\le \left[\frac{1}{4}+\frac{\left(\lambda-\frac{\eta+\theta}{2}\right)^2}{(\theta-\eta)^2}\right]K(\theta-\eta)\\ &= \left[\frac{(\lambda-\eta)^2+(\theta-\lambda)^2}{2(\theta-\eta)}\right]K \qquad (1.1)\end{aligned}$$

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Received: January 2022 *Accepted:* January 2023

Here the constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by any smaller constant. In recent versions K is usually replaced by $\|f_1'\|_\infty = ess \sup_{\lambda\in(\eta,\theta)} |f_1'(\lambda)| < \infty$. This outcome is also valid for functions of bounded variation since f_1' is bounded.

In recent years, a rapid advancement in generalizations and improvements of Ostrowski type inequalities has been observed. The large number of research articles books provide extensive range of literature on inequalities of Ostrowski type that can provide remarkable results for new researchers in this field. Furthermore, for detailed discussion on the topic some generalizations and related results, see the books [14], [15], [20] the papers [1], [4], [8], [9], [10], [12] and so many papers with the key word "Ostrowski inequality" can be accessed on-line.

Before we further proceed, let us denote the class of absolutely continuous functions by $AC(I)$ which defined on some real interval I.

Let us mention the Taylor's formula which has a crucial role in this work. Let I be an interval in $\mathbb{R}$ and $f_1 : I \to \mathbb{R}$ be a function such that $f_1^{(n-1)}$ is absolutely continuous on $I \subset \mathbb{R}, \eta, \theta \in I, \eta < \theta$. Then for $c, \lambda \in [\eta, \theta]$ the following formula holds

$$f_1(\lambda) = \sum_{k=0}^{n-1} \frac{f_1^{(k)}(c)}{k!}(\lambda - c)^k + \frac{1}{(n-1)!}\int_c^\lambda (\lambda - s)^{n-1} f_1^{(n)}(s)ds. \tag{1.2}$$

It is called the Taylor expansion of a function f_1 around a point c.

In the further text we use notation $(\lambda - s)_+^k$, $k \in \mathbb{N}_0$, for the following

$$(\lambda - s)_+^k = \begin{cases} (\lambda - s)^k, & \text{if } \lambda \geq s \\ 0, & \text{if } \lambda < s. \end{cases}$$

By using above defined notation Taylor's formulae can be defined over whole interval $[\eta, \theta]$ applying (1.2) for $c = \eta$ and then $c = \theta$ we get as follows which we would use in the rest of the paper.

$$f_1(\lambda) = \sum_{k=0}^{n-1} \frac{f_1^{(k)}(\eta)}{k!}(\lambda - \eta)^k + \frac{1}{(n-1)!}\int_\eta^\theta (\lambda - s)_+^{n-1} f_1^{(n)}(s)ds. \tag{1.3}$$

and

$$f_1(\lambda) = \sum_{k=0}^{n-1} (-1)^k \frac{f_1^{(k)}(\theta)}{k!}(\theta - \lambda)^k + \frac{(-1)^n}{(n-1)!}\int_\eta^\theta (s - \lambda)_+^{n-1} f_1^{(n)}(s)ds. \tag{1.4}$$

Now we recall the celebrated Montgomery identity from "Inequalities for Functions and their Integrals and Derivatives" by Mitrinović et al. [15].

Theorem 1.2. Let $f_1 : [\eta, \theta] \to \mathbb{R}$ be differentiable on $[\eta, \theta]$ and $f_1' : [\eta, \theta] \to \mathbb{R}$ integrable on $[\eta, \theta]$. Then

$$f_1(\lambda) = \frac{1}{\theta - \eta}\int_\eta^\theta f_1(t)dt + \frac{1}{\theta - \eta}\int_\eta^\theta P(\lambda, t) {f_1}'(t)dt, \tag{1.5}$$

where

$$P(\lambda, t) = \begin{cases} t - \eta, & \text{if } \quad t \in [\eta, \lambda], \\ \\ t - \theta, & \text{if } \quad t \in (\lambda, \theta]. \end{cases} \tag{1.6}$$

The weighted generalized Montgomery identity can be find in [17]:

Theorem 1.3. Let $f_1 : [\eta, \theta] \to \mathbb{R}$ be differentiable on $[\eta, \theta]$ and ${f_1}' : [\eta, \theta] \to \mathbb{R}$ integrable on $[\eta, \theta]$. Suppose that $w : [\eta, \theta] \to [0, \infty\rangle$ is a normalized weighted function, i.e., it is a positive integrable function satisfying $\int_\eta^\theta w(t)dt = 1$ and

$$W(t) = \begin{cases} 0, & t < \eta, \\ \int_\eta^t w(\lambda)d\lambda, & t \in [\eta, \theta], \\ 1, & t > \theta. \end{cases}$$

Then

$$f_1(\lambda) = \int_\eta^\theta w(t) f_1(t)\, dt + \int_\eta^\theta P_w(\lambda, t) f_1'(t)\, dt, \tag{1.7}$$

where

$$P_w(\lambda, t) = \begin{cases} W(t), & \eta \le t \le \lambda, \\ W(t) - 1, & \lambda < t \le \theta. \end{cases}$$

The following theorems obtained in [10] and [11] that contains identities for the sum $\sum_{i=1}^m p_i f_1(\lambda_i)$ and the integral $\int_\mu^\nu p(\xi) f_1(g(\xi))\, d\xi$.

Theorem 1.4. Let $f_1 : I \to \mathbb{R}$ be a function such that $I \subset \mathbb{R}, \eta, \theta \in I, \eta < \theta$ and $f_1^{(n-1)}$ is absolutely continuous. Furthermore let $n, m \in \mathbb{N}$, $\lambda_i \in [\eta, \theta]$ and $p_i \in \mathbb{R}$ for $i \in \{1, 2, \ldots, m\}$. Then

$$\sum_{i=1}^m p_i f_1(\lambda_i) = \sum_{k=0}^{n-1} \frac{f_1^{(k)}(\eta)}{k!} \sum_{i=1}^m p_i (\lambda_i - \eta)^k + \frac{1}{(n-1)!} \int_\eta^\theta \left(\sum_{i=1}^m p_i (\lambda_i - s)_+^{n-1} \right) f_1^{(n)}(s) ds. \tag{1.8}$$

and

$$\sum_{i=1}^m p_i f_1(\lambda_i) = \sum_{k=0}^{n-1} (-1)^k \frac{f_1^{(k)}(\theta)}{k!} \sum_{i=1}^m p_i (\theta - \lambda_i)^k + \frac{(-1)^n}{(n-1)!} \int_\eta^\theta \left(\sum_{i=1}^m p_i (s - \lambda_i)_+^{n-1} \right) f_1^{(n)}(s) ds. \tag{1.9}$$

Now we state the integral version as follows.

Theorem 1.5. Let $f_1 : I \to \mathbb{R}$ be a function such that $I \subset \mathbb{R}$, $\eta, \theta \in I$, $\eta < \theta$ and $f_1^{(n-1)}$ is absolutely continuous. Furthermore, let $n \in \mathbb{N}$ and $g : [\mu, \nu] \to [\eta, \theta]$ and $p : [\mu, \nu] \to \mathbb{R}$ be integrable functions. Then

$$\int_\mu^\nu p(\xi) f_1(g(\xi))\, d\xi = \sum_{k=0}^{n-1} \frac{f_1^{(k)}(\eta)}{k!} \int_\mu^\nu p(\xi)(g(\xi) - \eta)^k d\xi + \frac{1}{(n-1)!} \int_\eta^\theta f_1^{(n)}(s) \int_\mu^\nu p(\xi)\, (g(\xi) - s)_+^{n-1}\, d\xi ds,$$

$$\int_\mu^\nu p(\xi) f_1(g(\xi))\, d\xi = \sum_{k=0}^{n-1} (-1)^k \frac{f_1^{(k)}(\theta)}{k!} \int_\mu^\nu p(\xi)\, (\theta - g(\xi))^k d\xi + \frac{(-1)^n}{(n-1)!} \int_\eta^\theta f_1^{(n)}(s) \int_\mu^\nu p(\xi)\, (s - g(\xi))_+^{n-1}\, d\xi ds.$$

Definition 1.6. We say (q, r) is a pair of conjugate exponents if $1 < q, r < \infty$ and $\frac{1}{q} + \frac{1}{r} = 1$; or if $q = 1$ and $r = \infty$; or if $q = \infty$ and $r = 1$.

Let us denote by $G : [\eta, \theta] \times [\eta, \theta] \to \mathbb{R}$ the Green's function of the boundary value problem

$$z''(\lambda) = 0, \;\; z(\eta) = z(\theta) = 0.$$

The function G is given by

$$G(t, s) = \begin{cases} \frac{(t-\theta)(s-\eta)}{\theta-\eta} & \text{for } \eta \le s \le t, \\ \frac{(s-b)(t-\eta)}{\theta-\eta} & \text{for } t \le s \le \theta \end{cases} \tag{1.10}$$

and integration by parts easily yields that for any function $f_1 \in C^2[\eta, \theta]$ the following identity holds

$$f_1(\lambda) = \frac{\theta - \lambda}{\theta - \eta} f_1(\eta) + \frac{\lambda - \eta}{\theta - \eta} f_1(\theta) + \int_\eta^\theta G(\lambda, s) f_1''(s) ds. \tag{1.11}$$

The function G is continuous, symmetric and convex with respect to each variable.
As a special choice AbelGontscharoff polynomial for two-point right focal interpolating polynomials for $n = 2$ can be given as (see [19]):

$$f_1(\lambda) = f_1(\eta) + (\lambda - \eta) f_1'(\theta) + \int_\eta^\theta G_1(\lambda, t) f_1''(t) dt. \tag{1.12}$$

where $G_1(s,t)$ is Green's function for two-point right focal problem defined as

$$G_1(s,t) = \begin{cases} \eta - t & \text{for } \eta \le t \le s, \\ \eta - s & \text{for } s \le t \le \theta \end{cases} \tag{1.13}$$

Motivated by Abel-Gontscharoff identity (1.12) and related Green's function (1.13), we would like to recall some new types of Green functions $G_l : [\eta, \theta] \times [\eta, \theta] \longrightarrow \mathbb{R}$ for $l \in \{2, 3, 4\}$ defined as in [7]:

$$G_2(s,t) = \begin{cases} s - \theta & \text{for } \eta \le t \le s, \\ t - \theta & \text{for } s \le t \le \theta \end{cases} \tag{1.14}$$

$$G_3(s,t) = \begin{cases} s - \eta & \text{for } \eta \le t \le s, \\ t - \eta & \text{for } s \le t \le \theta \end{cases} \tag{1.15}$$

$$G_4(s,t) = \begin{cases} \theta - t & \text{for } \eta \le t \le s, \\ \theta - s & \text{for } s \le t \le \theta \end{cases} \tag{1.16}$$

In [7], it is also shown that all four Green functions are symmetric and continuous. These new Green functions enables us to introduce some new identities, stated in form of following

$$f_1(\lambda) = f_1(\theta) + (\theta - \lambda) f_1'(\eta) + \int_\eta^\theta G_2(\lambda, t) f_1''(t) dt. \tag{1.17}$$

$$f_1(\lambda) = f_1(\theta) - (\theta - \eta) f_1'(\theta) + (\lambda - \eta) f_1'(\eta) + \int_\eta^\theta G_3(\lambda, t) f_1''(t) dt. \tag{1.18}$$

$$f_1(\lambda) = f_1(\eta) + (\theta - \eta) f_1'(\eta) - (\theta - \lambda) f_1'(\theta) + \int_\eta^\theta G_4(\lambda, s) f_1''(t) dt. \tag{1.19}$$

The following result obtained by using green's function in [11]

Theorem 1.7. Let $f_1 : I \to \mathbb{R}$, $[\eta, \theta] \subset I$, be a function such that $f_1^{(n-1)}$ is absolutely continuous. Furthermore, let $m, n \in \mathbb{N}$, $n \ge 3$ $\lambda_i \in [\eta, \theta]$ and $p_i \in \mathbb{R}$ for $i \in \{1, 2, \ldots, m\}$ be such that

$$\sum_{i=1}^m p_i = 0, \quad \sum_{i=1}^m p_i \lambda_i = 0.$$

Then

$$\sum_{i=1}^m p_i f_1(\lambda_i) = \sum_{k=0}^{n-3} \frac{f_1^{(k+2)}(\eta)}{k!} \int_\eta^\theta \sum_{i=1}^m p_i G(\lambda_i, t)(t - \eta)^k dt + \frac{1}{(n-3)!} \int_\eta^\theta f_1^{(n)}(s) \left(\int_s^\theta \sum_{i=1}^m p_i G(\lambda_i, t)(t - s)^{n-3} dt \right) ds \tag{1.20}$$

and

$$\sum_{i=1}^m p_i f_1(\lambda_i) = \sum_{k=0}^{n-3} (-1)^k \frac{f_1^{(k+2)}(\theta)}{k!} \int_\eta^\theta \sum_{i=1}^m p_i G(\lambda_i, t)(\theta - t)^k dt - \frac{1}{(n-3)!} \int_\eta^\theta f_1^{(n)}(s) \left(\int_\eta^s \sum_{i=1}^m p_i G(\lambda_i, t)(t - s)^{n-3} dt \right) ds. \tag{1.21}$$

The integral versions of the previous theorem may also be stated.

Theorem 1.8. Let $f_1 : I \to \mathbb{R}$, $[\eta, \theta] \subset I$, be a function such that $f_1^{(n-1)}$ is absolutely continuous. Additionally, let $n \geq 3$ and $g : [\mu, \nu] \to [\eta, \theta]$, $p : [\mu, \nu] \to \mathbb{R}$ be integrable functions such that

$$\int_\mu^\nu p(\xi) d\xi = 0, \int_\mu^\nu p(\xi) g(\xi) d\xi = 0. \tag{1.22}$$

Then we get the following identities

$$\int_\mu^\nu p(\xi) f_1(g(\xi)) d\xi = \sum_{k=0}^{n-3} \frac{f_1^{(k+2)}(\eta)}{k!} \int_\eta^\theta \left(\int_\mu^\nu p(\xi) G(g(\xi), t) d\xi \right) (t-\eta)^k dt$$
$$+ \frac{1}{(n-3)!} \int_\eta^\theta f_1^{(n)}(s) \left(\int_s^\theta \left(\int_\mu^\nu p(\xi) G(g(\xi), t) d\xi \right) (t-s)^{n-3} dt \right) ds,$$

$$\int_\mu^\nu p(\xi) f_1(g(\xi)) d\xi = \sum_{k=0}^{n-3} (-1)^k \frac{f_1^{(k+2)}(\theta)}{k!} \int_\eta^\theta \left(\int_\mu^\nu p(\xi) G(g(\xi), t) d\xi \right) (\theta - t)^k dt$$
$$- \frac{1}{(n-3)!} \int_\eta^\theta f_1^{(n)}(s) \left(\int_\eta^s \left(\int_\mu^\nu p(\xi) G(g(\xi), t) d\xi \right) (t-s)^{n-3} dt \right) ds.$$

The aim of this paper to provide some Ostrowski type inequalities and estimation of difference of two weighted integral means by using the taylor formula with weighted Montgomery identity and Green functions. for this purpose we provide discrete and integral inequalities related to Taylor's formula in Section 2. In the Section 3 we establish some results which belongs to Green's function via Taylor's formula. Section 4 covers the discussion related to estimation of difference of two weighted integral means.

2 Ostrowski type inequalities by using Taylor's formula

Theorem 2.1. Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a function such that $f_1^{(n-1)}$ is absolutely continuous on $I \subset \mathbb{R}, \eta, \theta \in I, \eta < \theta$

$$f_1(\lambda) - \int_\eta^\theta w(t) f_1(t) dt = \sum_{k=0}^{n-1} \frac{f_1^{(k)}(\eta)}{(k-1)!} \int_\eta^\theta p_w(\lambda, t)(t-\eta)^{k-1} dt + \frac{1}{(n-2)!} \int_\eta^\theta \left(\int_\eta^\theta p_w(\lambda, t)(t-s)_+^{n-2} dt \right) f_1^{(n)}(s) ds. \tag{2.1}$$

Proof . State (1.3) for the variable t instead of λ and then multiplying identity by $w(t)$ and integrating with respect to t from η to θ we obtain the following identity:

$$\int_\eta^\theta w(t) f_1(t) dt = \sum_{k=0}^{n-1} \frac{f_1^{(k)}(\eta)}{k!} \int_\eta^\theta w(t)(t-\eta)^k dt + \frac{1}{(n-1)!} \int_\eta^\theta \left(\int_\eta^\theta w(t)(t-s)_+^{n-1} dt \right) f_1^{(n)}(s) ds. \tag{2.2}$$

If we subtract (2.2) from identity (1.3), we get

$$f_1(\lambda) - \int_\eta^\theta w(t) f_1(t) dt = \sum_{k=0}^{n-1} \frac{f_1^{(k)}(\eta)}{k!} \left[(\lambda-\eta)^k - \int_\eta^\theta w(t)(t-\eta)^k dt \right]$$
$$+ \frac{1}{(n-1)!} \int_\eta^\theta \left[(\lambda - s)_+^{n-1} - \int_\eta^\theta w(t)(t-s)_+^{n-1} dt \right] f_1^{(n)}(s) ds \tag{2.3}$$

By applying the weighted Montgomery identity (1.7) for $(\lambda-\eta)^k$ and $(\lambda-s)_+^{n-1}$

$$(\lambda-\eta)^k = \int_\eta^\theta w(t)(t-\eta)^k dt + k \int_\eta^\theta p_w(\lambda, t)(t-\eta)^{k-1} dt \tag{2.4}$$

and

$$(\lambda - s)_+^{n-1} \quad = \quad \int_\eta^\theta w(t)(t-s)_+^{n-1}dt + (n-1)\int_\eta^\theta p_w(\lambda,t)(t-s)_+^{n-2}dt \tag{2.5}$$

Finally inserting (2.4) and (2.5) into (2.3), we obtain (2.1). □

Now we give the Ostrowski type inequality for the above result. Here we denote $\tau_w(s) = \int_\eta^\theta p_w(\lambda,t)(t-s)_+^{n-2}dt$.

Theorem 2.2. Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a function such that $f_1^{(n-1)}$ is absolutely continuous on $I \subset \mathbb{R}, \eta, \theta \in I, \eta < \theta$. Moreover, assume that (q,r) is a pair of conjugate exponents. Let $|f_1^{(n)}|^q : [\eta,\theta] \to \mathbb{R}$ be an R-integrable function for some $n \geq 2$. Then we have

$$\left| f_1(\lambda) - \int_\eta^\theta w(t)f_1(t)dt - \sum_{k=0}^{n-1}\frac{f_1^{(k)}(\eta)}{(k-1)!}\int_\eta^\theta p_w(\lambda,t)(t-\eta)^{k-1}dt \right| \quad \leq \quad \|\tau_w(s)\|_r \|f_1^{(n)}\|_q. \tag{2.6}$$

Proof . By applying the *Hölder* inequality to (2.1), we obtain (2.6). □

Remark 2.3. If we put $w(t) = \frac{1}{\theta-\eta}$ in 2.1 we get

$$\begin{aligned} & f_1(\lambda) - \frac{1}{\theta-\eta}\int_\eta^\theta f_1(t)dt - \sum_{k=0}^{n-1}\frac{f_1^{(k)}(\eta)}{k!}\left[(\lambda-\eta)^k - \frac{(\theta-\eta)^k}{(k+1)}\right] \\ & + \frac{1}{(\theta-\eta)(n-2)!}\int_\eta^\theta \left[\int_\eta^\lambda (t-\eta)(t-s)_+^{n-2}dt + \int_\lambda^\theta (t-\theta)(t-s)_+^{n-2}dt\right] f_1^{(n)}(s)ds. \end{aligned}$$

Remark 2.4. In the same manner we can prove the similar results by using (1.4).

Theorem 2.5. Let $n, m \in \mathbb{N}$ and $f_1 : I \to \mathbb{R}$ be a function such that $f_1^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}, \eta, \theta \in I, \eta < \theta$. Furthermore let $\lambda_i \in [\eta,\theta]$ and $p_i \in \mathbb{R}$ for $i \in \{1,2,\ldots,m\}$. Additionally assume that (q,r) is a pair of conjugate exponents. Let $|f_1^{(n)}|^q : [\eta,\theta] \to \mathbb{R}$ be an R-integrable function for some $n \geq 2$. Then

$$\left| f_1(\lambda) - \sum_{i=1}^m p_i f_1(\lambda_i) - \sum_{k=0}^{n-1}\frac{f_1^{(k)}(\eta)}{k!}\left[(\lambda-\eta)^k - \sum_{i=1}^m p_i(\lambda_i-\eta)^k\right]\right| \quad \leq \quad \left(\int_\eta^\theta |\varphi_1(s)|^r ds\right)^{\frac{1}{r}} \|f_1^{(n)}\|_q. \tag{2.7}$$

where

$$\varphi_1(s) = \frac{1}{(n-1)!}\left[(\lambda-s)_+^{n-1} - \sum_{i=1}^m p_i(\lambda_i-s)_+^{n-1}\right]$$

and

$$\begin{aligned} & \left| f_1(\lambda) - \sum_{i=1}^m p_i f_1(\lambda_i) - \sum_{k=0}^{n-1}\frac{(-1)^n f_1^{(k)}(\theta)}{k!}\left[(\theta-\lambda)^k - \sum_{i=1}^m p_i(\theta-\lambda_i)^k\right]\right| \\ & \qquad \leq \left(\int_\eta^\theta |\varphi_2(s)|^r ds\right)^{\frac{1}{r}} \|f_1^{(n)}\|_q. \end{aligned} \tag{2.8}$$

where

$$\varphi_2(s) = \frac{(-1)^n}{(n-1)!}\left[(s-\lambda)_+^{n-1} - \sum_{i=1}^m p_i(s-\lambda_i)_+^{n-1}\right].$$

The constants on the right-hand side of (2.7) and (2.8) are sharp for $1 < q \leq \infty$ and the best possible for $q = 1$.

Proof . By taking the difference of identities (1.3) and (1.8) we get after some arrangements

$$
\begin{aligned}
& f_1(\lambda)-\sum_{i=1}^{m} p_i f_1(\lambda_i)-\sum_{k=0}^{n-1}\frac{f_1^{(k)}(\eta)}{k!}\left[(\lambda-\eta)^k-\sum_{i=1}^{m}p_i(\lambda_i-\eta)^k\right] \\
&\quad =\frac{1}{(n-1)!}\int_\eta^\theta\left[(\lambda-s)_+^{n-1}-\sum_{i=1}^{m}p_i(\lambda_i-s)_+^{n-1}\right]f_1^{(n)}(s)ds. \qquad (2.9)
\end{aligned}
$$

Let us denote

$$
\varphi_1(s)=\frac{1}{(n-1)!}\left[(\lambda-s)_+^{n-1}-\sum_{i=1}^{m}p_i(\lambda_i-s)_+^{n-1}\right].
$$

Now in new notations we have

$$
f_1(\lambda)-\sum_{i=1}^{m} p_i f_1(\lambda_i)-\sum_{k=0}^{n-1}\frac{f_1^{(k)}(\eta)}{k!}\left[(\lambda-\eta)^k-\sum_{i=1}^{m}p_i(\lambda_i-\eta)^k\right]=\int_\eta^\theta \varphi_1(s)f_1^{(n)}(s)ds.
$$

After applying Hölder's inequality we obtain

$$
\begin{aligned}
&\left|f_1(\lambda)-\sum_{i=1}^{m} p_i f_1(\lambda_i)-\sum_{k=0}^{n-1}\frac{f_1^{(k)}(\eta)}{k!}\left[(\lambda-\eta)^k-\sum_{i=1}^{m}p_i(\lambda_i-\eta)^k\right]\right| \\
&=\left|\int_\eta^\theta \varphi_1(s)f_1^{(n)}(s)ds\right|\le\|\varphi_1(s)\|_r\left\|f_1^{(n)}\right\|_q \qquad (2.10)\\
&=\left(\int_\eta^\theta|\varphi_1(s)|^r ds\right)^{\frac{1}{r}}\|f_1^{(n)}\|_q. \qquad (2.11)
\end{aligned}
$$

For the proof of the sharpness of the constant $\left(\int_\eta^\theta|\varphi_1(s)|^r ds\right)^{\frac{1}{r}}$, let us define a function f_1 for which the equality in (2.10) is obtained.

For $1<q<\infty$ take f_1 to be such that

$$
f_1^{(n)}(s)=sgn\varphi_1(s).|\varphi_1(s)|^{\frac{1}{q-1}}.
$$

For $q=\infty$ take

$$
f_1^{(n)}(s)=sgn\varphi_1(s).
$$

For $q=1$ we shall prove that

$$
\left|\int_\eta^\theta \varphi_1(s)f_1^{(n)}(s)ds\right|\le\max_{s\in[\eta,\theta]}|\varphi_1(s)|\int_\eta^\theta\left|f_1^{(n)}(s)\right|ds \qquad (2.12)
$$

is the best possible inequality. Since $|\varphi_1(s)|$ is continuo for $n\ge 2$ we suppose that $|\varphi_1(s)|$ attains its maximum at $s_0\in[\eta,\theta]$. First we assume that $\varphi_1(s_0)>0$. For $\epsilon>0$ defined $f_{1\epsilon}(s)$ by

$$
f_{1\epsilon}(s)=\begin{cases}0, & \eta\le s\le s_0,\\[2mm] \dfrac{1}{\epsilon n!}(s-s_0)^n, & s_0<s\le s_0+\epsilon,\\[2mm] \dfrac{1}{(n-1)!}(s-s_0)^{n-1}, & s_0+\epsilon<s\le\theta.\end{cases}
$$

We have

$$\left|\int_\eta^\theta \varphi_1(s) f_{1\epsilon}^{(n)}(s)ds\right| = \left|\int_{s_0}^{s_0+\epsilon} \varphi_1(s)\frac{1}{\epsilon}ds\right| = \frac{1}{\epsilon}\int_{s_0}^{s_0+\epsilon} \varphi_1(s)ds.$$

Now from inequality (2.12) we have

$$\frac{1}{\epsilon}\int_{s_0}^{s_0+\epsilon} \varphi_1(s)ds \leq \varphi_1(s_0)\int_{s_0}^{s_0+\epsilon} ds = \varphi_1(s_0).$$

Since

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{s_0}^{s_0+\epsilon} \varphi_1(s)ds = \varphi_1(s_0),$$

the statement follows. In the case $\varphi_1(s_0) < 0$, we define $f_{1\epsilon}(s)$ by

$$f_{1\epsilon}(s) = \begin{cases} \dfrac{1}{(n-1)!}(s-s_0-\epsilon)^{n-1}, & \eta \leq s \leq s_0, \\ -\dfrac{1}{\epsilon n!}(s-s_0-\epsilon)^n, & s_0 < s \leq s_0+\epsilon, \\ 0, & s_0+\epsilon < s \leq \theta. \end{cases}$$

and the rest of the proof is the same as above. For the proof of second inequality we consider (1.4) and (1.9) then repeat the same procedure. □

Now we state the integral version of above Theorem as follows. Since the proofs of these results are same, we omit the details.

Theorem 2.6. Let $g : [\mu,\nu] \to [\eta,\theta]$ and $p : [\mu,\nu] \to \mathbb{R}$ be integrable functions. Let $n \in \mathbb{N}$ and $f_1 : I \to \mathbb{R}$ be such that $f_1^{(n-1)}$ is absolutely continuous on $I \subset \mathbb{R}$, $\eta,\theta \in I$, $\eta < \theta$. Additionally assume that (q,r) is a pair of conjugate exponents. Let $|f_1^{(n)}|^q : [\eta,\theta] \to \mathbb{R}$ be an R-integrable function for some $n \geq 2$. Then

$$\left|f_1(\lambda) - \int_\mu^\nu p(\xi)f_1(g(\xi))d\xi - \sum_{k=0}^{n-1}\frac{f_1^{(k)}(\eta)}{k!}\left[(\lambda-\eta)^k - \int_\mu^\nu p(\xi)(g(\xi)-\eta)^k d\xi\right]\right| \leq \left(\int_\eta^\theta |\varphi_3(s)|^r ds\right)^{\frac{1}{r}} \|f_1^{(n)}\|_q \tag{2.13}$$

where

$$\varphi_3(s) = \frac{1}{(n-1)!}\left[(\lambda-s)_+^{n-1} - \int_\mu^\nu p(\xi)(g(\xi)-s)_+^{n-1}d\xi\right].$$

$$\left|f_1(\lambda) - \int_\mu^\nu p(\xi)f_1(g(\xi))d\xi - \sum_{k=0}^{n-1}\frac{(-1)^n f_1^{(k)}(\theta)}{k!}\left[(\theta-\lambda)^k - \int_\mu^\nu p(\xi)(\theta-g(\xi))^k d\xi\right]\right| \leq \left(\int_\eta^\theta |\varphi_4(s)|^r ds\right)^{\frac{1}{r}} \|f_1^{(n)}\|_q \tag{2.14}$$

where

$$\varphi_4(s) = \frac{(-1)^n}{(n-1)!}\left[(s-\lambda)_+^{n-1} - \int_\mu^\nu p(\xi)(s-g(\xi))_+^{n-1}d\xi\right].$$

The constants on the right-hand side of (2.13) and (2.14) are sharp for $1 < q \leq \infty$ and the best possible for $q = 1$.

3 Ostrowski type inequalities by using Taylor's formula and Green's function

Theorem 3.1. Let all the assumptions of $Theorem 1.7$ hold. Let G is as defined in (1.10). Then we have

$$\left| f_1(\lambda) - \sum_{i=1}^{m} p_i f_1(\lambda_i) - \frac{(\theta-\lambda)f_1(\eta) + (\lambda-\eta)f_1(\theta)}{\theta-\eta} - \sum_{k=0}^{n-3} \frac{f_1^{(k+2)}(\eta)}{k!} \int_\eta^\theta \left(G(\lambda,t) - \sum_{i=1}^{m} p_i G(\lambda_i,t) \right) (t-\eta)^k dt \right|$$
$$\leq \left(\int_\eta^\theta |\varphi_5(t)|^r dt \right)^{\frac{1}{r}} \|f_1^{(n)}\|_q, \tag{3.1}$$

where,

$$\varphi_5(t) = \frac{1}{(n-3)!} \int_s^\theta \left(G(\lambda,t) - \sum_{i=1}^{m} p_i G(\lambda_i,t) \right) (t-s)^{(n-3)} dt.$$

Moreover, the following inequality holds

$$\left| f_1(\lambda) - \sum_{i=1}^{m} p_i f_1(\lambda_i) - \frac{(\theta-\lambda)f_1(\eta) + (\lambda-\eta)f_1(\theta)}{\theta-\eta} - \sum_{k=0}^{n-3} (-1)^k \frac{f_1^{(k+2)}}{k!} \int_\eta^\theta \left(G(\lambda,s) - \sum_{i=1}^{m} p_i G(x_i,s) \right) (\theta-t)^k dt \right|$$
$$\leq \left(\int_\eta^\theta |\varphi_6(t)|^r dt \right)^{\frac{1}{r}} \|f_1^{(n)}\|_q, \tag{3.2}$$

where,

$$\varphi_6(t) = -\frac{1}{(n-3)!} \int_\eta^s \left(G(\lambda,t) - \sum_{i=1}^{m} p_i G(\lambda_i,t) \right) (t-s)^{(n-3)} dt.$$

Proof . Differentiating the function f_1 in (1.2) twice gives

$$f_1''(\lambda) = \sum_{k=0}^{n-3} \frac{f_1^{(k+2)}(c)}{k!} (\lambda-c)^k + \frac{1}{(n-3)!} \int_c^\lambda (\lambda-s)^{n-3} f_1^{(n)}(s) ds. \tag{3.3}$$

After putting $c=\eta$ inserting (3.3) in (1.11) we get

$$f_1(\lambda) = \frac{(\theta-\lambda)f_1(\eta) + (\lambda-\eta)f)(\theta)}{\theta-\eta} + \sum_{k=0}^{n-3} \frac{f_1^{(k+2)}(\eta)}{k!} \int_\eta^\theta G(\lambda,t)(t-\eta)^k dt$$
$$+ \frac{1}{(n-3)!} \int_\eta^\theta f_1^{(n)}(s) \left(\int_b^s G(\lambda,t)(t-s)^{n-3} dt \right) ds, \tag{3.4}$$

By taking difference of (3.4) and (1.20) after some arrangements we get

$$f_1(\lambda) - \sum_{i=1}^{m} p_i f_1(\lambda_i) - \frac{(\theta-\lambda)f_1(\eta) + (\lambda-\eta)f_1(\theta)}{\theta-\eta} - \sum_{k=0}^{n-3} \frac{f_1^{(k+2)}(\eta)}{k!} \int_\eta^\theta \left(G(\lambda,t) - \sum_{i=1}^{m} p_i G(\lambda_i,t) \right) (t-\eta)^k dt$$
$$= \frac{1}{(n-3)!} \int_\eta^\theta f_1^{(n)}(s) \left[\int_s^\theta \left(G(\lambda,t) - \sum_{i=1}^{m} p_i G(\lambda_i,t) \right) (t-s)^{n-3} dt \right] ds, \tag{3.5}$$

Let

$$\varphi_5(t) = \frac{1}{(n-3)!} \int_s^\theta \left(G(\lambda,t) - \sum_{i=1}^{m} p_i G(\lambda_i,t) \right) (t-s)^{n-3} dt.$$

After applying the Hölder's inequality we obtain

$$\begin{aligned}
&\left|f_1(\lambda)-\sum_{i=1}^{m}p_if_1(\lambda_i)-\frac{(\theta-\lambda)f_1(\eta)+(\lambda-\eta)f_1(\theta)}{\theta-\eta}-\sum_{k=0}^{n-3}\frac{f_1^{(k+2)}(\eta)}{k!}\int_\eta^\theta\left(G(\lambda,t)-\sum_{i=1}^{m}p_iG(\lambda_i,t)\right)(t-\eta)^kdt\right|\\
&=\left|\int_\eta^\theta\varphi_5(t)f_1^{(n)}(s)ds\right|\le\int_\eta^\theta\left|\varphi_5(t)f_1^{(n)}(s)\right|ds\le\|\varphi_5(t)\|_r\|f_1^{(n)}\|_q\\
&=\left(\int_\eta^\theta|\varphi_5(t)|^rdt\right)^{\frac{1}{r}}\|f_1^{(n)}\|_q,
\end{aligned}$$

which proves the inequality. The proof for sharpness and the best possibility are similar as in Theorem 2.7.

In the same manner after putting $c=\theta$ inserting (3.3) in (1.11) we get the other inequality based on Green Function. □

Theorem 3.2. Let all the assumptions of $Theorem 1.7$ hold. Let G_1 is as defined in (1.13). Then we have

$$\begin{aligned}
&\left|f_1(\lambda)-\sum_{i=1}^{m}p_if_1(\lambda_i)-f_1(\eta)-(\lambda-\eta)f_1'(\theta)-\sum_{k=0}^{n-3}\frac{f_1^{(k+2)}(\eta)}{k!}\int_\eta^\theta\left(G_1(\lambda,t)-\sum_{i=1}^{m}p_iG_1(\lambda_i,t)\right)(t-\eta)^kdt\right|\\
&\le\left(\int_\eta^\theta|\varphi_7(t)|^rdt\right)^{\frac{1}{r}}\|f_1^{(n)}\|_q,
\end{aligned}\tag{3.6}$$

where,

$$\varphi_7(t)=\frac{1}{(n-3)!}\int_s^\theta\left(G_1(\lambda,t)-\sum_{i=1}^{m}p_iG_1(\lambda_i,t)\right)(t-s)^{n-3}dt.$$

Moreover, the following inequality holds

$$\begin{aligned}
&\left|f_1(\lambda)-\sum_{i=1}^{m}p_if_1(\lambda_i)-f_1(\eta)-(\lambda-\eta)f_1'(\theta)-\sum_{k=0}^{n-3}\frac{(-1)^kf_1^{(k+2)}(\theta)}{k!}\int_\eta^\theta\left(G_1(\lambda,t)-\sum_{i=1}^{m}p_iG_1(\lambda_i,t)\right)(\theta-t)^kdt\right|\\
&\le\left(\int_\eta^\theta|\varphi_8(t)|^rdt\right)^{\frac{1}{r}}\|f_1^{(n)}\|_q,
\end{aligned}\tag{3.7}$$

where,

$$\varphi_8(t)=\frac{-1}{(n-3)!}\int_\eta^s\left(G_1(\lambda,t)-\sum_{i=1}^{m}p_iG_1(\lambda_i,t)\right)(t-s)^{n-3}dt,$$

Remark 3.3. The proof of Theorem 3.2 is similar to proof of Theorem 3.1 except use of identity (1.12) in place of (1.11), so we omitted the details. In the similar way we can state and proof results for G_2, G_3 and G_4 by using the identities, (1.17), (1.18) and (1.19) respectively.

Now we state the integral version of above result as follows. Since the proofs are of similar nature, we omit the details.

Theorem 3.4. Let all the assumptions of $Theorem 1.8$ hold. Let G_1 is as defined in (1.13). Then we have

$$\begin{aligned}
&\left|f_1(\lambda)-\int_\mu^\nu p(\xi)f_1(g(\xi))d\xi-f_1(\eta)-(\lambda-\eta)f_1'(\theta)-\sum_{k=0}^{n-3}\frac{f_1^{(k+2)}(\eta)}{k!}\int_\eta^\theta\left(G_1(\lambda,t)-\int_\mu^\nu p(\xi)G_1(g(\xi),t)d\xi\right)(t-\eta)^kdt\right|\\
&\le\left(\int_\eta^\theta|\varphi_9(t)|^rdt\right)^{\frac{1}{r}}\|f_1^{(n)}\|_q,
\end{aligned}\tag{3.8}$$

where,

$$\varphi_9(t) = \frac{1}{(n-3)!}\int_s^\theta \left(G_1(\lambda,t) - \int_\mu^\nu p(\xi)G_1(g(\xi),t)d\xi\right)(t-s)^{n-3}dt.$$

Moreover, the following inequality holds

$$\left|f_1(\lambda) - \int_\mu^\nu p(\xi)f_1(g(\xi))d\xi - f_1(\eta) - (\lambda-\eta)f_1'(\theta) - \sum_{k=0}^{n-3}\frac{(-1)^k f_1^{(k+2)}(\theta)}{k!}\int_\eta^\theta \left(G_1(\lambda,t) - \int_\mu^\nu p(\xi)G_1(g(\xi),t)d\xi\right)(\theta-t)^k dt\right|$$
$$\leq \left(\int_\eta^\theta |\varphi_{10}(t)|^r dt\right)^{\frac{1}{r}} \|f_1^{(n)}\|_q, \tag{3.9}$$

where,

$$\varphi_{10}(t) = \frac{-1}{(n-3)!}\int_\eta^s \left(G_1(\lambda,t) - \int_\mu^\nu p(\xi)G_1(g(\xi),t)d\xi\right)(t-s)^{n-3}dt,$$

4 Estimation of the difference of two weighted integral means

In this section we generalize the results from [5], [13]. We denote

$$T_{w,n}[\eta,\theta](\lambda) = \sum_{k=0}^{n-1}\frac{f_1^{(k)}(\eta)}{(k-1)!}\int_\eta^\theta p_w(\lambda,t)(t-\eta)^{k-1}dt$$

and

$$T_{w,n}[\alpha,\beta](\lambda) = \sum_{k=0}^{n-1}\frac{f_1^{(k)}(\alpha)}{(k-1)!}\int_\alpha^\beta p_w(\lambda,t)(t-\alpha)^{k-1}dt.$$

We have four possible cases for the two intervals $[\eta,\theta]$ and $[\alpha,\beta]$ if $[\eta,\theta]\cup[\alpha,\beta]\neq\emptyset$. The first case is $[\alpha,\beta]\subset[\eta,\theta]$ and the second $[\eta,\theta]\cap[\alpha,\beta]=[\alpha,\theta]$. Other two possible cases we simply get by interchange $\eta\leftrightarrow\alpha, \theta\leftrightarrow\beta$.

Theorem 4.1. Let $f:[\eta,\theta]\cup[\alpha,\beta]\to\mathbb{R}$ be a function such that $f_1^{(n-1)}\in AC(I)$ function for some $n\geq 2$. Then if $[\eta,\theta]\cup[\alpha,\beta]\neq\emptyset$ and $\lambda\in[\eta,\theta]\cup[\alpha,\beta]$, we have

$$\int_\eta^\theta w(t)f_1(t)dt - \int_\alpha^\beta u(t)f_1(t)dt + T_{w,n}[\eta,\theta](\lambda) - T_{w,n}[\alpha,\beta](\lambda) = \int_{\min\{\eta,\alpha\}}^{\max\{\theta,\beta\}} K_n(\lambda,s)f_1^n(s)ds. \tag{4.1}$$

where incase $[\alpha,\beta]\subset[\eta,\theta]$

$$K_n(\lambda,s) = \begin{cases} -\dfrac{1}{(n-2)!}\displaystyle\int_\eta^\theta p_w(\lambda,t)(t-s)_+^{n-2}dt, & s\in[\eta,\alpha\rangle \\ \dfrac{1}{(n-2)!}\left[\displaystyle\int_\alpha^\beta p_u(\lambda,t)(t-s)_+^{n-2}dt - \int_\eta^\theta p_w(\lambda,t)(t-s)_+^{n-2}dt\right], & s\in\langle\alpha,\beta], \\ -\dfrac{1}{(n-2)!}\displaystyle\int_\eta^\theta p_w(\lambda,t)(t-s)_+^{n-2}dt. & s\in\langle\beta,\theta]. \end{cases}$$

and incase $[\eta,\theta]\cap[\alpha,\beta]=[\alpha,\theta]$

$$K_n(x,s) = \begin{cases} -\dfrac{1}{(n-2)!}\displaystyle\int_\eta^\theta p_w(\lambda,t)(t-s)_+^{n-2}dt, & s\in[\eta,\alpha\rangle \\ \dfrac{1}{(n-2)!}\left[\displaystyle\int_\alpha^\beta p_u(\lambda,t)(t-s)_+^{n-2}dt - \int_\eta^\theta p_w(\lambda,t)(t-s)_+^{n-2}dt\right], & s\in\langle\alpha,\theta], \\ \dfrac{1}{(n-2)!}\displaystyle\int_\alpha^\beta p_u(\lambda,t)(t-s)_+^{n-2}dt. & s\in\langle\theta,\beta]. \end{cases}$$

Proof . By taking the difference of the identities (2.1) for interval $[\eta,\theta]$ and $[\alpha,\beta]$ we get the formula (4.1). □

Theorem 4.2. Assume that (q,r) is a pair of conjugate exponents, that is, $1\le q,\ r\le\infty$, $\frac{1}{q}+\frac{1}{r}=1$. Let $|f_1^{(n)}|^q:[\eta,\theta]\to\mathbb{R}$ be an R-integrable function for some $n\ge 2$. Then we have

$$\left|\int_\eta^\theta w(t)f_1(t)dt-\int_\alpha^\beta u(t)f_1(t)dt+T_{w,n}[\eta,\theta](\lambda)-T_{w,n}[\alpha,\beta](\lambda)\right|\le\|\int_{\min\{\eta,\alpha\}}^{\max\{\theta,\beta\}}K_n(\lambda,s)\|_r\|f_1^{(n)}\|_q. \tag{4.2}$$

for every $\lambda\in[\eta,\theta]\cap[\alpha,\beta]$. The constant $\left(\int_{\min\{\eta,\alpha\}}^{\max\{\theta,\beta\}}|K_n(\lambda,s)|^r\,ds\right)^{\frac{1}{r}}$ is sharp for $1<q\le\infty$ and the best possible for $q=1$.

Proof . Use identity (4.1) and apply the Hölder inequality to obtain (4.2). □

Remark 4.3. In the same manner we can prove the similar results by using (1.4).

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