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Well-posedness and analyticity for the viscous primitive equations of geophysics in critical Fourier-Besov-Morrey spaces with variable exponents

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Abstract

In this paper, we establish the global well-posedness result of the viscous primitive equations of geophysics in the critical Fourier-Besov-Morrey space with variable exponents $\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}(\mathbb{R}^3)$, when the initial data are small and Pandtl number $P = 1$, we also show the Gevrey class regularity of the solution.

Keywords: Fourier-Besov-Morrey spaces with variable exponents, viscous primitive equations of geophysics, global well-posedness, Gevrey class regularity
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1 Introduction

It is known that the viscous primitive equations are a fundamental mathematical model in the field of fluid geophysics. In this paper we study the initial value problem of this model in $\mathbb{R}^3$ which reads as follows:

$$\begin{cases} \partial_t u - \mu\Delta u + \Omega e_3 \times u + (u.\nabla)u + \nabla p = g\theta e_3 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t \theta - \kappa\Delta\theta + (u.\nabla)\theta = -\mathcal{N}^2 u_3 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \nabla \cdot u = 0 & \\ u\mid_{t=0}= u_0, \quad \theta\mid_{t=0}= \theta_0 & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where the unknown functions $u(t,x) = (u^1(t,x), u^2(t,x), u^3(t,x))$ denote the fluid velocity , $p = p(t,x)$ denote the pressure of the fluid, $\theta = \theta(t,x)$ is a function representing the density fluctuation in the fluid. μ, κ and g are positive constants related to viscosity, diffusivity and gravity, respectively. $u_0 = u_0(x)$ denotes the given initial velocity field satisfying the incompressible condition $\nabla \cdot u_0 = 0$. Moreover, Ω is the so-called Coriolis parameter, it is the speed of rotation around the vertical unit vector $e_3(0,0,1)$ and $\mathcal{N}$ is the stratification parameter. The ratio $P := \dfrac{\mu}{\kappa}$ is known as the Prandtl number and $B := \dfrac{\Omega}{\mathcal{N}}$ is the Burger number of geophysics.

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There is a rich literature about global-in-time well-posedness for fluid dynamics PDEs with singular data in different spaces, where the smallness conditions are taken in the weak-norms of the critical spaces. For instance, we have results in the Lebesgue space L^3, homogeneous Besov $\dot{\mathcal{B}}^{\frac{3}{p}-1}_{p,\infty}$, Fourier-Besov $\mathcal{F}B^{2-\frac{3}{p}}_{p,q}$, Fourier-Besov-Morrey $\mathcal{F}\mathcal{N}^{s}_{p,k,q}$[10, 11, 12, 13, 14], Besov-Morrey spaces $\mathcal{N}^{3-\frac{k}{p}}_{p,k,\infty}$, and BMO^{-1}, among others (see, e.g., the book [36] for a nice review). It should be noted that the variable exponent Fourier-Besov-Morrey space $\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}(\mathbb{R}^3)$ is invariant under the scaling of (1.1). In fact, if $\left(u(t,x), \frac{\sqrt{g}\theta(t,x)}{\mathcal{N}}\right)$ is the solution of (1.1) with the initial data $(u_0, \frac{\sqrt{g}\theta_0}{\mathcal{N}})$, then $\left(u_\gamma(t,x), \frac{\sqrt{g}\theta_\gamma(t,x)}{\mathcal{N}}\right)$ with $\left(u_\gamma(t,x), \frac{\sqrt{g}\theta_\gamma(t,x)}{\mathcal{N}}\right) := \left(\gamma u(\gamma^2 t, \gamma x), \gamma\frac{\sqrt{g}\theta(\gamma^2 t,\gamma x)}{\mathcal{N}}\right)$ is also a solution of the same system with the initial data

$$\left(u_{0,\gamma}(x), \frac{\sqrt{g}\theta_{0,\gamma}(x)}{\mathcal{N}}\right) := \left(\gamma u_0(\gamma x), \gamma\frac{\sqrt{g}\theta_0(\gamma x)}{\mathcal{N}}\right) \tag{1.2}$$

and

$$\left\|\left(u_{0,\gamma}(x), \frac{\sqrt{g}\theta_{0,\gamma}(x)}{\mathcal{N}}\right)\right\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}(\mathbb{R}^3)} \approx \left\|\left(u_0(x), \frac{\sqrt{g}\theta_0(x)}{\mathcal{N}}\right)\right\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}(\mathbb{R}^3)}.$$

Definition 1.1. Let A be a Banach space such that $A \in S'(\mathbb{R}^n)$, then A is a critical space for initial data of the system (1.1) if and only if whose norm is invariant under the scaling (1.2) for all $\gamma > 0$, i.e

$$\left\|\left(u_{0,\gamma}(x), \frac{\sqrt{g}\theta_{0,\gamma}(x)}{\mathcal{N}}\right)\right\|_A \approx \left\|\left(u_0(x), \frac{\sqrt{g}\theta_0(x)}{\mathcal{N}}\right)\right\|_A.$$

Under these scalings, we can show that $\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}(\mathbb{R}^3)$ is critical for (1.1). In this regard, there are numerous studies on global-in-time well-posedness for (1.1) in various functional settings. For example, Babin, Maholov and Nicolaenko [16] showed that the problem (1.1) is globally well-posed in $H^s(\mathbb{T}^3)$ with $s \geq 3/4$ for small initial data when the stratification parameter $\mathcal{N}$ is sufficiently large. Later, Charve [22] obtained the global well-posedness of (1.1) in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ for arbitrary initial data (ie. not necessarily small) when Ω and $\mathcal{N}$ are large. In 2008, Charve [23] proved the global well-posedness of (1.1) in less regular initial value spaces, also in [24] Charve and Ngo considered the global well-posedness of (1.1) with anisotropic viscosities. In Fourier-Besov spaces $\mathcal{F}\dot{B}^{2-\frac{3}{p}}_{p,q}(\mathbb{R}^3)$, Jinyi Sun and Shangbin Cui [39] proved that the Cauchy problem (1.1) with the Prandtl number $P = 1$ is locally well-posed for $1 < p \leq \infty$, $1 \leq q < \infty$ and globally well-posed in these spaces when the initial data are small. In 2019, Abbassi, Allalou and Oulha [1] considered the local well-posedness, the global well-posedness and stability for global solutions of the viscous primitive equations of geophysics in critical Fourier-Besov-Morrey spaces when Prandtl number $P = 1$ and the initial data are small. In [9] Leithold et al. established well posedness result for fractional version of system (1.1) in Fourier-Besov-Morrey. We refer to the monographs [20, 21] for the other studies of the problem (1.1).

We remark that if $\theta \equiv 0, \mathcal{N} = 0$ and $\Omega = 0$, then we have the classical Navier-Stokes equations:

$$\begin{cases} u_t - \mu\Delta u + (u.\nabla)u + \nabla p = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\ \nabla \cdot u = 0, & \\ u(0,x) = u_0(x) & x \in \mathbb{R}^3. \end{cases}$$

The local and global well-posedness of the classical Navier-Stokes equations have been established by a lot of researches in a different function spaces, for instance [35, 37].
Also, we have the Navier-Stokes equations with Coriolis force when $\theta \equiv 0, \mathcal{N} = 0$ and $\Omega \neq 0$,

$$\begin{cases} u_t - \mu\Delta u + \Omega e_3 \times u + (u\nabla)u + \nabla p = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \nabla \cdot u = 0, & \\ u(0,x) = u_0(x) & x \in \mathbb{R}^3. \end{cases}$$

In [31] Iwabuchi introduced Fourier spaces of Besov type, Fourier-Besov spaces $\mathcal{F}\dot{B}^{s}_{p,q}$ with $p = 1$ in the context of parabolic-elliptic Keller-Segel system. Later, Iwabuchi and Takada [32] established the global existence and the uniqueness of the mild solution for the Navier-Stokes-Coriolis system in $\mathcal{F}\dot{B}^{-1}_{1,2}$. Taking in particular $\Omega = 0$, they also proved the global well-posedness result for $3D$ Navier-Stokes equations in the same spaces and they showed the ill-posedness in the space $\mathcal{F}\dot{B}^{-1}_{1,q}$ for all $\Omega \in \mathbb{R}$, if $2 < q \leq \infty$. Konieczny-Yoneda [34] obtained global well-posedness and asymptotic stability of small solutions for $3D$ Navier-Stokes-Coriolis equations (and Navier-Stokes) in critical Fourier-Besov spaces $\mathcal{F}\dot{B}^{2-\frac{3}{p}}_{p,\infty}$ with $1 < p \leq \infty$. After, Ferreira and Lima [29] introduced Fourier-Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}^{s}_{p,k,q}$ to analyze a class of active scalar equations and showed the well-posedness and asymptotic behavior results. Almeida et al. [8] obtained the uniform global well-posedness for the Navier-Stokes-Coriolis equations with small initial data in the framework of Fourier-Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}^{s}_{p,k,\infty}$ where $1 \leq p < 3$ and $1 \leq k < 3$ with $k \neq 0$ when $p = 1$. In Fourier-Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{k}{p}}_{p,k,q}$, El Baraka and Toumlilin [27] obtained the global well-posedness of the fractional Navier-Stokes equations, which the model as follows

$$\begin{cases} u_t + \mu(-\Delta)^{\alpha} u + (u\nabla)u + \nabla p = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\ \nabla \cdot u = 0 & \\ u(0,x) = u_0(x) & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

when the initial data are small.

Function spaces with variable exponents have been attracted the attention of many researches in the recent years, not only by theoretical reasons but also by the special role played in some applications, for instance resolution of some equations (fractional Navier-Stokes equations, fractional magneto-hydrodynamics equations, Generalized Porous Medium equations...).

In 2018, S. Ru and M. Z. Abidin [4] proved the global well-posedness result of (1.3) in critical variable exponent Fourier-Besov spaces $\mathcal{F}\dot{B}^{4-2\alpha-3}_{p(\cdot),q}$. Later, in 2021, M. Z. Abidin and J. Chen [3] generalized this result to the Fourier-Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}(\mathbb{R}^3)$ and they proved the global well-posedness result for (1.3) with small initial data belonging to $\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}(\mathbb{R}^3)$.

In this work, we prove that the Cauchy problem (1.1) is globally well-posed when Prandtl number $P = 1$ and the initial data are small, we also show the analyticity of the solution.

2 General notation

In this paragraph, we introduce some general notations which we will use throughout the paper.
We denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space. $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with radius $r > 0$, C will denote a positive constant such that whose value may change at different places, $x \lesssim y$ means that there exists a positive constant C such that $x \leq Cy$, we also use $(U,V) \in X$ to denote $(U,V) \in X \times X$ for a Banach space X, $\|(U,V)\|_X$ to denote $\|(U,V)\|_{X\times X}$ and we denote $\|\cdot\|_{E\cap F} = \|\cdot\|_E + \|\cdot\|_F$. The symbol $\mathcal{S}(\mathbb{R}^n)$ is the usual Schwartz space of infinitely differentiable rapidly decreasing complex-valued functions on $\mathbb{R}^n$.
By $\hat{\varphi}$ we denote the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ in the version

$$\hat{\varphi}(x) := \mathcal{F}\varphi(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(\xi) d\xi, \quad x \in \mathbb{R}^n.$$

and we define its inverse Fourier transform by

$$\check{\varphi}(\xi) = \mathcal{F}^{-1}\varphi(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \varphi(x) dx.$$

The convolution $f * g$ of two complex or extended real-valued measurable functions f, g on $\mathbb{R}^n$ is given by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y) g(y) dy, \quad for\ x \in \mathbb{R}^n.$$

$supp f$ is the support of the function f, i.e. the closure of its zero set and χ_A is the characteristic function of A.

3 Preliminaries and main results

Let us introduce some basic properties of the Littlewood-Paley theory and Fourier-Besov-Morrey spaces with variables exponent.

Let $\varphi \in S(\mathbb{R}^n)$ be a radial positive function such that $0 \leq \varphi \leq 1$, $\operatorname{supp}(\varphi) \subset \left\{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$ and

$$\sum_{j \in \mathbb{Z}} \varphi\left(2^{-j}\xi\right) = 1, \quad \text{for all } \xi \neq 0.$$

We denote

$$\varphi_j(\xi) = \varphi\left(2^{-j}\xi\right), \quad \psi_j(\xi) = \sum_{k \leq j-1} \varphi_k(\xi)$$

and

$$h(x) = \mathcal{F}^{-1}\varphi(x), \quad g(x) = \mathcal{F}^{-1}\psi(x).$$

Now, we present some frequency localization operators:

$$\Delta_j f := \mathcal{F}^{-1}\left(\varphi_j \mathcal{F}(f)\right) = 2^{nj} \int_{\mathbb{R}^n} h\left(2^j y\right) f(x-y) dy,$$

$$S_j f := \sum_{k \leq j-1} \Delta_k f = \mathcal{F}^{-1}\left(\psi_j \mathcal{F}(f)\right) = 2^{nj} \int_{\mathbb{R}^n} g\left(2^j y\right) f(x-y) dy,$$

where $\Delta_j = S_j - S_{j-1}$ is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$ and S_j is a frequency to the ball $\{|\xi| \lesssim 2^j\}$. By using the definition of Δ_j and S_j, we easily check that

$$\Delta_j \Delta_k f = 0, \quad \text{if } |j-k| \geq 2$$
$$\Delta_j\left(S_{k-1} f \Delta_k f\right) = 0, \quad \text{if } |j-k| \geq 5.$$

The following Bony para-product decomposition will be applied around the paper:

$$uv = \dot{T}_u v + \dot{T}_v u + R(u, v),$$

where $\dot{T}_u v = \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v$, $\dot{R}(u, v) = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \tilde{\Delta}_j v$ *and* $\tilde{\Delta}_j v = \sum_{|j'-j| \leq 1} \dot{\Delta}_{j'} v$. We define the Lebesgue spaces with variable exponent $L^{p(\cdot)}$.

Definition 3.1. ([6]) Let $\mathcal{P}_0$ denotes the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that

$$0 < p_- = \operatorname{ess} \inf_{x \in \mathbb{R}^n} p(x), \ \operatorname{ess} \sup_{x \in \mathbb{R}^n} p(x) = p_+ < \infty.$$

The Lebesgue space with variable exponent is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R} \text{ is measurable}, \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty \right\},$$

with Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1\right\}.$$

The space $L^{p(\cdot)}(\mathbb{R}^n)$ equipped with the norm $\|\cdot\|_{L^{p(\cdot)}}$ is a Banach space.

Definition 3.2. ([6]) Let $p : \mathbb{R}^n \to \mathbb{R}$.

i) We say that p is locally log-Hölder continuous, $p \in C_{loc}^{\log}(\mathbb{R}^n)$, if there exists a constant $c_{\log} > 0$ with

$$|p(x) - p(y)| \leq \frac{c_{\log}}{\log\left(e + \frac{1}{|x-y|}\right)} \quad \text{for all} \quad x, y \in \mathbb{R}^n \text{and } x \neq y.$$

ii) We say that p is globally log-Hölder continuous, $p \in C^{\log}(\mathbb{R}^n)$, if $p \in C_{loc}^{\log}(\mathbb{R}^n)$ and there exists a $p_\infty \in \mathbb{R}$ and a constant $c_\infty > 0$ with

$$|p(x) - p_\infty| \leq \frac{c_\infty}{\log(e + |x|)} \quad \text{for all} \quad x \in \mathbb{R}^n.$$

iii) We write $p \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ if $0 < p^- \leq p(x) \leq p^+ \leq \infty$ with $1/p \in C^{\log}(\mathbb{R}^n)$.

Definition 3.3. ([5]) Let $p(\cdot),\ k(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $0 < p_- \leq p(x) \leq k(x) \leq \infty$, the Morrey space with variable exponent $\mathcal{M}^{k(\cdot)}_{p(\cdot)} := \mathcal{M}^{k(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is defined as the set of all measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{\mathcal{M}^{k(\cdot)}_{p(\cdot)}} := \sup_{x_0 \in \mathbb{R}^n, r>0} \|r^{\frac{n}{k(x)} - \frac{n}{p(x)}} f \chi_{B(x_0,r)}\|_{L^{p(\cdot)}} < \infty.$$

According to the definition of the $L^{p(\cdot)}$-norm, $\|f\|_{\mathcal{M}^{k(\cdot)}_{p(\cdot)}}$ also has the following form

$$\|f\|_{\mathcal{M}^{k(\cdot)}_{p(\cdot)}} := \sup_{x_0 \in \mathbb{R}^n, r>0} \inf \left\{ \lambda > 0 : \rho_{p(\cdot)}(r^{\frac{n}{k(x)} - \frac{n}{p(x)}} \frac{f}{\lambda} \chi_{B(x_0,r)}) \leq 1 \right\}.$$

We now give some important lemmas.

Lemma 3.4. ([5]) Let $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. For any measurable function f

$$\sup_{x \in \mathbb{R}^n, r>0} \rho_{p(\cdot)}(f \chi_{B(x,r)}) = \rho_{p(\cdot)}(f) \ .$$

Lemma 3.5. ([5]) If $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, then $\|f\|_{\mathcal{M}^{p(\cdot)}_{p(\cdot)}} = \|f\|_{L^{p(\cdot)}}$.

Lemma 3.6. ([18]) Let X be a Banach space with norm $\|\cdot\|$ and $B : X \longrightarrow X$ a bilinear operator, such that for any $x_1, x_2 \in X$, $\|B(x_1, x_2)\| \leq \eta \|x_1\| \|x_2\|$, then for any $y \in X$ such that $\|y\| < \dfrac{1}{4\eta}$ the equation $x = y + B(x, x)$ has a solution $x \in X$. In particular, the solution is such that $\|x\| \leq 2\|y\|$ and it is the only one such that $\|x\| < \dfrac{1}{2\eta}$.

Definition 3.7. ([5]) Let $p(\cdot),\ q(\cdot),\ k(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $p(\cdot) \leq k(\cdot)$, the mixed Morrey-sequence space $l^{q(\cdot)}(\mathcal{M}^{k(\cdot)}_{p(\cdot)})$ includes all sequences $\{f_j\}_{j\in \mathrm{Z}}$ of measurable functions in $\mathbb{R}^n$ such that
$\rho_{l^{q(\cdot)}(\mathcal{M}^{k(\cdot)}_{p(\cdot)})}(\lambda \{f_j\}_{j\in\mathrm{Z}}) < \infty$ for some $\lambda > 0$. For $\{f_j\}_j \in l^{q(\cdot)}(\mathcal{M}^{k(\cdot)}_{p(\cdot)})$ we define

$$\|\{f_j\}_{j\in\mathrm{Z}}\|_{l^{q(\cdot)}(\mathcal{M}^{k(\cdot)}_{p(\cdot)})} := \inf \left\{ \lambda > 0,\ \rho_{l^{q(\cdot)}(\mathcal{M}^{k(\cdot)}_{p(\cdot)})} \left(\{\frac{f_j}{\lambda}\}_{j\in\mathrm{Z}} \right) \leq 1 \right\} < \infty,$$

where

$$\rho_{l^{q(\cdot)}(\mathcal{M}^{k(\cdot)}_{p(\cdot)})}(\{f_j\}_{j\in\mathrm{Z}}) := \sum_{j\in Z} \inf \left\{ \gamma > 0,\ \int_{\mathbb{R}^n} \left(\frac{|r^{\frac{n}{k(x)} - \frac{n}{p(x)}} f_j \chi_{B(x_0,r)}|}{\gamma^{\frac{1}{q(x)}}} \right)^{p(x)} dx \leq 1 \right\}.$$

Notice that if $q_+ < \infty$ or $q_+ < \infty$ and $p(x) \geq q(x)$, then

$$\rho_{l^{q(\cdot)}(\mathcal{M}^{k(\cdot)}_{p(\cdot)})}(\{f_i\}_{i\in\mathbb{N}_0}) = \sum_{i\in\mathbb{N}_0} \sup_{x_0\in\mathbb{R}^n, r>0} \|(|r^{\frac{n}{k(x)} - \frac{n}{p(x)}} f_i| \chi_{B(x_0,r)})^{q(\cdot)}\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

Definition 3.8. ([4]) Let $s(\cdot) \in C^{log}(\mathbb{R}^n)$ and $p(\cdot),\ q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$ with $0 < p_- \leq p(\cdot) \leq \infty$. The homogeneous Fourier-Besov space with variable exponent $\mathcal{F}\dot{B}^{s(\cdot)}_{p(\cdot),q(\cdot)}$ is defined by the set of all $f \in \mathcal{Z}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{F}\dot{B}^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)} \varphi_j \hat{f}\}_{-\infty}^{\infty}\|_{l^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

The space $\mathcal{Z}'(\mathbb{R}^n)$ is the dual space of

$$\mathcal{Z}(\mathbb{R}^n) = \{f \in S(\mathbb{R}^n) : (D^\alpha f)(0) = 0, \forall \alpha \text{ multi-index}\}.$$

Definition 3.9. ([5]) Let $s(\cdot) \in C^{log}(\mathbb{R}^n)$ and $p(\cdot),\ q(\cdot),\ k(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$ with $0 < p_- \leq p(x) \leq k(x) \leq \infty$. The homogeneous Besov-Morrey space with variable exponent $\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),k(\cdot),q(\cdot)}$ is defined by the set of all $f \in \mathcal{Z}'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),k(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)} \Delta_j f\}_{j\in\mathbb{Z}}\|_{l^{q(\cdot)}(\mathcal{M}^{k(\cdot)}_{p(\cdot)})} < \infty.$$

Definition 3.10. ([3]) Let $s(\cdot) \in C^{log}(\mathbb{R}^n)$ and $p(\cdot),\ q(\cdot),\ k(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$ with $0 < p_- \le p(\cdot) \le k(\cdot) \le \infty$. The homogeneous Fourier-Besov-Morrey space with variable exponent $\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),k(\cdot),q(\cdot)}$ is defined by the set of all $f \in \mathcal{Z}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),k(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\varphi_j \hat{f}\}_{-\infty}^{\infty}\|_{l^{q(\cdot)}(\mathcal{M}^{k(\cdot)}_{p(\cdot)})} < \infty.$$

Definition 3.11. ([3]) Let $s(\cdot) \in C^{log}(\mathbb{R}^n)$, $p(\cdot),\ q(\cdot),\ k(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$, $T \in [0,\infty)$ and $1 \le q,\ \rho \le \infty$. We define the Chemin-Lerner type homogeneous Fourier-Besov-Morrey space with variable exponents $\mathcal{L}^\rho([0,T);\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),k(\cdot),q})$ by

$$\mathcal{L}^\rho\ ([0,T);\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),k(\cdot),q}) = \left\{ f \in \mathcal{Z}'(\mathbb{R}^n);\ \|f\|_{\mathcal{L}^\rho([0,T);\cdot\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),k(\cdot),q})} < \infty \right\},$$

with the norm

$$\|f\|_{\mathcal{L}^\rho([0,T);\cdot\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),k(\cdot),q})} = \left(\sum_{j\in\mathbb{Z}} \|2^{js(\cdot)}\varphi_j \hat{f}\|^q_{L^\rho([0,T);\mathcal{M}^{k(\cdot)}_{p(\cdot)})} \right)^{\frac{1}{q}}.$$

Proposition 3.12. ([3]) For Morrey spaces with variable exponents, the following inclusions are established.
(1) (Hölder inequality)([3]) Let $p(\cdot),\ p_1(\cdot),\ p_2(\cdot),\ k(\cdot),\ k_1(\cdot),\ k_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, such that $p(x) \le k(x)$, $p_1(x) \le k_1(x)$, $p_2(x) \le k_2(x)$, $\dfrac{1}{p(x)} = \dfrac{1}{p_1(x)} + \dfrac{1}{p_2(x)}$ and $\dfrac{1}{k(x)} = \dfrac{1}{k_1(x)} + \dfrac{1}{k_2(x)}$. Then there exists a constant C depending only on p_- and p_+ such that

$$\|fg\|_{\mathcal{M}^{k(\cdot)}_{p(\cdot)}} \le C\|f\|_{\mathcal{M}^{k_1(\cdot)}_{p_1(\cdot)}}\|g\|_{\mathcal{M}^{k_2(\cdot)}_{p_2(\cdot)}},$$

holds for every $f \in \mathcal{M}^{k_1(\cdot)}_{p_1(\cdot)}$ and $g \in \mathcal{M}^{k_2(\cdot)}_{p_2(\cdot)}$.
(2) ([3]) Let $p_0(\cdot),\ p_1(\cdot),\ k_0(\cdot),\ k_1(\cdot),\ q(\cdot) \in \mathcal{P}_0$, and $s_0(\cdot),\ s_1(\cdot) \in L^\infty \cap C^{log}(\mathbb{R}^n)$ with $s_0(\cdot) \ge s_1(\cdot)$. If $\dfrac{1}{q}$ and $s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}$ are locally log-Hölder continuous, then

$$\mathcal{N}^{s_0(\cdot)}_{p_0(\cdot),k_0(\cdot),q} \hookrightarrow \mathcal{N}^{s_1(\cdot)}_{p_1(\cdot),k_1(\cdot),q}.$$

(3) ([5]) For $p(\cdot) \in C^{log}(\mathbb{R}^n)$ and $\psi \in L^1(\mathbb{R}^n)$, assume $\Psi(x) = \sup_{y \notin B(0,|x|)} |\psi(y)|$ is integrable. Then

$$\|f * \psi_\epsilon\|_{\mathcal{M}^{k(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}^{k(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}\|\Psi\|_{L^1(\mathbb{R}^n)},$$

for all $f \in \mathcal{M}^{k(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, where $\psi_\epsilon = \dfrac{1}{\epsilon^n}\psi(\dfrac{1}{\epsilon})$ and C depends only on n.

The following result will be used to prove the main theorems.

Proposition 3.13. ([3]) Let $I = (0,T]$, $s > 0$, $1 \le \gamma,\ \rho,\ \rho_1,\ \rho_2,\ q \le \infty$, $p(\cdot),\ k(\cdot),\ r(\cdot) \in C^{\log} \cap \mathcal{P}_0(\mathbb{R}^n)$, $\frac{1}{k(\cdot)} = \frac{1}{k_1(\cdot)} + \frac{1}{k_2(\cdot)}$, $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$ and $\frac{1}{\gamma} = \frac{1}{r(\cdot)} + \frac{1}{p(\cdot)}$.Then, we have

$$\begin{aligned}\|ab\|_{\mathcal{L}^\rho(I,\dot{\mathcal{N}}^s_{\gamma,k(\cdot),q})} \lesssim & \|a\|_{\mathcal{L}^{\rho_1}(I,\mathcal{M}^{k_1(\cdot)}_{r(\cdot)})}\|b\|_{\mathcal{L}^{\rho_2}(I,\dot{\mathcal{N}}^s_{p(\cdot),k_2(\cdot),q})} \\ & + \|b\|_{\mathcal{L}^{\rho_1}(I,\mathcal{M}^{k_1(\cdot)}_{r(\cdot)})}\|a\|_{\mathcal{L}^{\rho_2}(I,\dot{\mathcal{N}}^s_{p(\cdot),k_2(\cdot),q})}.\end{aligned}$$

Below, we shall present our first main result that establishes the global existence.

Theorem 3.14. Let Prandtl number $P = 1$, i.e. $\mu = \kappa$, $\Omega \in \mathbb{R}$, $p(\cdot), k(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ such that $p(\cdot) \le k(\cdot) < \infty$, $2 \le p(\cdot) \le 6$, $1 \le \rho < \infty$, $1 \le q < 3$. Then there exists a small σ such that for any $v_0 = (u_0, \frac{\sqrt{g}\theta_0}{\mathcal{N}}) \in \mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}$ satisfying $\nabla \cdot u_0 = 0$ with $\|(u_0, \frac{\sqrt{g}\theta_0}{\mathcal{N}})\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}} < \sigma$, the problem (1.1) admits a unique global solution $v = (u, \frac{\sqrt{g}\theta}{\mathcal{N}})$ in

$$\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}) \cap \mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q}) \cap \mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q}).$$

Moreover, let $p_1(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$, $s_1(\cdot) = \frac{2}{\rho} - \frac{3}{p_1(\cdot)} + 2$ and $s_1(\cdot) \in C^{\log}(\mathbb{R}^n)$, if there exists $c > 0$ such that $2 \le p_1(\cdot) \le c \le p(\cdot)$, then we obtain that $v = (u, \frac{\sqrt{g}\theta}{\mathcal{N}}) \in \mathcal{L}^\rho([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),k(\cdot),q})$.

The next theorem assures the Gevrey class regularity of the solution.

Theorem 3.15. Let $p(\cdot), k(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ such that $p(\cdot) \le k(\cdot) < \infty$, $2 \le p(\cdot) \le 6$, $1 \le \rho < \infty$, $1 \le q < 3$. Then there exists a small σ_0 such that for any $v_0 = (u_0, \frac{\sqrt{g}\theta_0}{\mathcal{N}}) \in \mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}$ satisfying $\nabla \cdot u_0 = 0$ with $\|(u_0, \frac{\sqrt{g}\theta_0}{\mathcal{N}})\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}} < \sigma_0$, the solution $v = (u, \frac{\sqrt{g}\theta}{\mathcal{N}})$ obtained in Theorem 3.14 is analytic, in the sense that

$$\|e^{\mu\sqrt{t}|D|}v\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}) \cap \mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q}) \cap \mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})} \le \|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q})}.$$

Moreover, let $p_1(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$, $s_1(\cdot) = \frac{2}{\rho} - \frac{3}{p_1(\cdot)} + 2$ and $s_1(\cdot) \in C^{\log}(\mathbb{R}^n)$, if there exists $c > 0$ such that $2 \le p_1(\cdot) \le c \le p(\cdot)$, then we obtain that $e^{\mu\sqrt{t}|D|}v(t) \in \mathcal{L}^\rho([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),k(\cdot),q})$.
where $e^{\mu\sqrt{t}|D|}$ is a Fourier multiplier and $e^{\mu\sqrt{t}|\xi|}$ defines its symbol.

4 Global well-posedness

To ensure the existence of global solution with small initial data, we first transform (1.1) to an equivalent Cauchy problem.
By setting $N := \mathcal{N}\sqrt{g}$, $v := (v^1, v^2, v^3, v^4) = \left(u^1, u^2, u^3, \frac{\sqrt{g}\theta}{\mathcal{N}}\right)$, $v_0 := (v_0^1, v_0^2, v_0^3, v_0^4) = \left(u_0^1, u_0^2, u_0^3, \frac{\sqrt{g}\theta_0}{\mathcal{N}}\right)$ and $\bar{\nabla} := (\partial_1, \partial_2, \partial_3, 0)$, we can convert the system (1.1) to

$$\begin{cases} v_t + \mathcal{A}v + \mathcal{B}v + \tilde{\nabla}p = -(v \cdot \tilde{\nabla})v & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \tilde{\nabla} \cdot v = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ v(0,x) = v_0(x) & x \in \mathbb{R}^3, \end{cases} \tag{4.1}$$

where

$$\mathcal{A} = \begin{pmatrix} -\mu\Delta & 0 & 0 & 0 \\ 0 & -\mu\Delta & 0 & 0 \\ 0 & 0 & -\mu\Delta & 0 \\ 0 & 0 & 0 & -\kappa\Delta \end{pmatrix} \text{ and } \mathcal{B} = \begin{pmatrix} 0 & -\Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -N \\ 0 & 0 & N & 0 \end{pmatrix}.$$

In order to solve the problem (4.1), we consider the following integral equation:

$$\begin{aligned} v(t) &= T_{\Omega,N}(t)v_0 - \int_0^t T_{\Omega,N}(t-\tau)\tilde{\mathbb{P}}\tilde{\nabla} \cdot (v \otimes v)d\tau \\ &= T_{\Omega,N}(t)v_0 + B(v,v) \end{aligned} \tag{4.2}$$

where, $B(v,v) = -\int_0^t T_{\Omega,N}(t-\tau)\tilde{\mathbb{P}}\tilde{\nabla} \cdot (v \otimes v)d\tau$ and $\tilde{\mathbb{P}} = \left(\tilde{\mathbb{P}}ij\right)_{4\times 4}$ is the Helmholtz projection onto the divergence-free vector fields defined by

$$\tilde{\mathbb{P}}_{ij} = \begin{cases} \delta_{ij} + R_iR_j & 1 \le i,j \le 3 \\ \delta_{ij} & otherwise, \end{cases}$$

where δ_{ij} denotes the Kronecker symbol and R_j $(j = 1,2,3)$ are the Riesz transforms on $\mathbb{R}^3$. $T_{\Omega,N}(\cdot)$ denotes the Stokes-Coriolis Stratification semigroup corresponding to the linear problem of (4.1), which is given by

$$T_{\Omega,N}(t)f = \mathcal{F}^{-1}\left[\cos\left(\frac{|\xi|'}{|\xi|}t\right)M_1 + \sin\left(\frac{|\xi|'}{|\xi|}t\right)M_2 + M_3\right] * \mathcal{F}^{-1}\left(e^{-\mu|\xi|^2 t}\mathcal{F}(f)\right),$$

where

$$|\xi| := \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}, \quad |\xi|' := |\xi|'_{\Omega,N} := \sqrt{N^2\xi_1^2 + N^2\xi_2^2 + \Omega^2\xi_3^2}$$

for $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$,

$$M_1=\begin{pmatrix} \frac{\Omega^2\xi_3^2}{|\xi|'^2} & 0 & -\frac{N^2\xi_1\xi_3}{|\xi|'^2} & \frac{\Omega N\xi_2\xi_3}{|\xi|'^2} \\ 0 & \frac{\Omega^2\xi_3^2}{|\xi|'^2} & -\frac{N^2\xi_2\xi_3}{|\xi|'^2} & -\frac{\Omega N\xi_1\xi_3}{|\xi|'^2} \\ -\frac{\Omega^2\xi_1\xi_3}{|\xi|'^2} & -\frac{\Omega^2\xi_2\xi_3}{|\xi|'^2} & \frac{N^2\left(\xi_1^2+\xi_2^2\right)}{|\xi|'^2} & 0 \\ \frac{\Omega N\xi_2\xi_3}{|\xi|'^2} & -\frac{\Omega N\xi_1\xi_3}{|\xi|'^2} & 0 & \frac{N^2\left(\xi_1^2+\xi_2^2\right)}{|\xi|'^2} \end{pmatrix},$$

$$M_2=\begin{pmatrix} 0 & -\frac{\Omega\xi_3^2}{|\xi||\xi|'} & -\frac{\Omega\xi_2\xi_3}{|\xi||\xi|'} & \frac{N\xi_1\xi_3}{|\xi||\xi|'} \\ \frac{\Omega\xi_3^2}{|\xi||\xi|'} & 0 & -\frac{\Omega\xi_1\xi_3}{|\xi||\xi|'} & \frac{N\xi_2\xi_3}{|\xi||\xi|'} \\ -\frac{\Omega\xi_2\xi_3}{|\xi||\xi|'} & \frac{\Omega\xi_1\xi_3}{|\xi||\xi|'} & 0 & -\frac{N\left(\xi_1^2+\xi_3^2\right)}{|\xi||\xi|'} \\ -\frac{N\xi_1\xi_3}{|\xi||\xi|'} & -\frac{N\xi_2\xi_3}{|\xi||\xi|'} & \frac{N\left(\xi_1^2+\xi_3^2\right)}{|\xi||\xi|'} & 0 \end{pmatrix}$$

and

$$M_3=\begin{pmatrix} \frac{N^2\xi_2^2}{|\xi|'^2} & -\frac{N^2\xi_1\xi_2}{|\xi|'^2} & 0 & -\frac{\Omega N\xi_2\xi_3}{|\xi|'^2} \\ -\frac{N^2\xi_1\xi_2}{|\xi|'^2} & \frac{N^2\xi_1^2}{|\xi|'^2} & 0 & \frac{\Omega N\xi_1\xi_3}{|\xi|'^2} \\ 0 & 0 & 0 & 0 \\ -\frac{\Omega N\xi_2\xi_3}{|\xi|'^2} & \frac{\Omega N\xi_1\xi_3}{|\xi|'^2} & 0 & \frac{\Omega^2\xi_3^2}{|\xi|'^2} \end{pmatrix}.$$

We now prove our first main theorem.

4.1 Proof of Theorem 3.14

We consider $X=\left\{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q})\cap\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})\cap\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})\right\}$, to solve the problem (4.1), we define the mapping

$$\Phi: T_{\Omega,N}(t)v_0+B(v,v).$$

Thus, it suffices to show that Φ admits a fixed point.
We start with linear estimate, according to Proposition 3.12 we obtain

$$\begin{aligned} \|T_{\Omega,N}(t)v_0\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})} &\le \left\|\left\|2^{j\left(\frac{2}{\rho}+\frac{1}{2}\right)}\varphi_j e^{-t\mu|\xi|^2}\widehat{v_0}\right\|_{L^\rho\left([0,\infty),\mathcal{M}_2^{k(\cdot)}\right)}\right\|_{\ell^q} \\ &\le \left\|\sum_{k=0;\pm1}\|2^{j(2-\frac{3}{p(\cdot)})}\varphi_j\hat{v_0}\|_{\mathcal{M}^{k(\cdot)}_{p(\cdot)}}\|r^{\frac{-3(p(\cdot)-2)}{2p(\cdot)}}2^{j(\frac{2}{\rho}-\frac{3}{2}+\frac{3}{p(\cdot)})}\varphi_{j+k}e^{-\mu|\xi|^2}\|_{L^\rho\left([0,\infty),L^{\frac{2p(\cdot)}{p(\cdot)-2}}\right)}\right\|_{\ell^q} \\ &\lesssim \left\|\|2^{j(2-\frac{3}{p(\cdot)})}\varphi_j\hat{v_0}\|_{\mathcal{M}^{k(\cdot)}_{p(\cdot)}}\right\|_{\ell^q} \\ &\lesssim \|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}}, \end{aligned}$$

where we used the following estimates

$$\begin{aligned}
\left\|r^{\frac{-3(p(\cdot)-2)}{2p(\cdot)}}2^{j(\frac{2}{\rho}-\frac{3}{2}+\frac{3}{p(\cdot)})}\varphi_{j+k}e^{-t\mu|\xi|^2}\right\|_{L^\rho\left([0,\infty),L^{\frac{2p(\cdot)}{p(\cdot)-2}}\right)} &\lesssim \left\|r^{\frac{-3(p(\cdot)-2)}{2p(\cdot)}}2^{j(\frac{3}{p(\cdot)}-\frac{3}{2})}2^{j\frac{2}{\rho}}\varphi_{j+k}e^{-t\mu 2^{2(j+k)}}\right\|_{L^\rho\left([0,\infty),L^{\frac{2p(\cdot)}{p(\cdot)-2}}\right)}\\
&\lesssim \left\|r^{\frac{-3(p(\cdot)-2)}{2p(\cdot)}}2^{j\frac{2}{\rho}}e^{-t\mu 2^{2(j+k)}}\right\|_{L^\rho([0,\infty))}\left\|\varphi_{j+k}2^{j(\frac{3}{p(\cdot)}-\frac{3}{2})}\right\|_{L^{\frac{2p(\cdot)}{p(\cdot)-2}}}\\
&\lesssim \left\|\varphi_{j+k}2^{j(\frac{3}{p(\cdot)}-\frac{3}{2})}\right\|_{L^{\frac{2p(\cdot)}{p(\cdot)-2}}}\\
&\lesssim \inf\left\{\lambda>0:\int_{\mathbb{R}^3}|\frac{\varphi_{j+k}2^{j(\frac{3}{p(\cdot)}-\frac{3}{2})}}{\lambda}|^{\frac{2p(x)}{p(x)-2}}dx\le 1\right\}\\
&\lesssim \inf\left\{\lambda>0:\int_{\mathbb{R}^3}|\frac{\varphi_{j+k}}{\lambda}|^{\frac{2p(x)}{p(x)-2}}2^{-3j}dx\le 1\right\}\\
&\lesssim \inf\left\{\lambda>0:\int_{\mathbb{R}^3}|\frac{\varphi_{k}}{\lambda}|^{\frac{2p(2^jx)}{p(2^jx)-2}}dx\le 1\right\}\\
&\le C.
\end{aligned}$$

Consequently, one obtains

$$\|T_{\Omega,N}(t)v_0\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})}\lesssim\|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}}.$$

Analogously, for $p_1(\cdot)\le c\le p(\cdot)$, we have

$$\begin{aligned}
\|T_{\Omega,N}(t)v_0\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),k(\cdot),q})} &\le \left\|\left\|2^{js_1(\cdot)}\varphi_j e^{-t\mu|\xi|^2}\hat{v}_0\right\|_{L^\rho\left([0,\infty),\mathcal{M}^{k(\cdot)}_{p_1(\cdot)}\right)}\right\|_{\ell^q}\\
&\lesssim \left\|\sum_{k=0,\pm1}\|2^{j(2-\frac{3}{c})}\varphi_j\hat{v}_0\|_{\mathcal{M}^{k(\cdot)}_c}\|r^{\frac{-3(c-p_1(\cdot))}{cp_1(\cdot)}}2^{j(\frac{2}{\rho}+\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+k}e^{-t\mu 2^{2(j+k)}}\|_{L^\rho\left([0,\infty),L^{\frac{cp_1(\cdot)}{c-p_1(\cdot)}}\right)}\right\|_{\ell^q}\\
&\lesssim \left\|\sum_{k=0,\pm1}\|2^{j(2-\frac{3}{p(\cdot)})}\varphi_j\hat{v}_0\|_{\mathcal{M}^{k(\cdot)}_{p(\cdot)}}\right\|_{\ell^q}\\
&\lesssim \|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}},
\end{aligned}$$

where

$$\begin{aligned}
&\left\|r^{\frac{-3(c-p_1(\cdot))}{cp_1(\cdot)}}2^{j(\frac{2}{\rho}+\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+k}e^{-t\mu 2^{2(j+k)}}\right\|_{L^\rho\left([0,\infty),L^{\frac{cp_1(\cdot)}{c-p_1(\cdot)}}\right)}\\
&=\left\|r^{\frac{-3(c-p_1(\cdot))}{cp_1(\cdot)}}2^{j\frac{2}{\rho}}e^{-t\mu 2^{2(j+k)}}\right\|_{L^\rho([0,\infty))}\inf\left\{\lambda>0:\int_{\mathbb{R}^3}|\frac{\varphi_{j+k}2^{j(\frac{3}{c}-\frac{3}{p_1(x)})}}{\lambda}|^{\frac{cp_1(x)}{c-p_1(x)}}dx\le 1\right\}\\
&\le \inf\left\{\lambda>0:\int_{\mathbb{R}^3}|\frac{\varphi_{j+k}}{\lambda}|^{\frac{cp_1(x)}{c-p_1(x)}}2^{-3j}dx\le 1\right\}\\
&\le C.
\end{aligned}$$

Therefore,

$$\|T_{\Omega,N}(t)v_0\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),k(\cdot),q})}\lesssim\|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}}.$$

On the other hand, if $\rho=\infty$ and $p_1(\cdot)=p(\cdot)$, then

$$\|T_{\Omega,N}(t)v_0\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q})}\lesssim\|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}},$$

similarly, we get

$$\|T_{\Omega,N}(t)v_0\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}\lesssim\|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}}.$$

Consequently,

$$\|T_{\Omega,N}(t)v_0\|_X \le C_1 \|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}}. \tag{4.3}$$

For bilinear estimate, according to Hölder's inequality, Hausdorf-Young's inequality, and Proposition 3.13 , we obtain

$$\begin{aligned}
&\|B(v,w)\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),k(\cdot),q})}\\
&= \left\|\int_0^t T_{\Omega,N}(t-\tau)\tilde{\mathbb{P}}\tilde{\nabla}\cdot(v\otimes w)d\tau\right\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),k(\cdot),q})}\\
&\lesssim \left\|\left\|\int_0^t 2^{js_1(\cdot)}\varphi_j e^{-\mu(t-\tau)|\xi|^2}\mathcal{F}(\tilde{\nabla}\cdot(v\otimes w))d\tau\right\|_{L^\rho([0,\infty),\mathcal{M}^{k(\cdot)}_{p_1(\cdot)})}\right\|_{\ell_q}\\
&\lesssim \left\|\left\|\int_0^t 2^{j(s_1(\cdot)+1)}\varphi_j e^{-\mu(t-\tau)|\xi|^2}\mathcal{F}(v\otimes w)d\tau\right\|_{L^\rho([0,\infty),\mathcal{M}^{k(\cdot)}_{p_1(\cdot)})}\right\|_{\ell_q}\\
&\lesssim \left\|\left\|\int_0^t \left\|2^{j(\frac{2}{\rho}-\frac{3}{p(\cdot)}+3)}\varphi_j e^{-\mu(t-\tau)|\xi|^2}r^{\frac{-3(6-p_1(\cdot))}{6p_1(\cdot)}}\right\|_{\frac{6p_1(\cdot)}{6-p_1(\cdot)}}\|\mathcal{F}(v\otimes w)\|_{\mathcal{M}^{k(\cdot)}_{6/5}}d\tau\right\|_{L^\rho([0,\infty))}\right\|_{\ell_q}\\
&\lesssim \left\|\left\|\int_0^t 2^{j(\frac{2}{\rho}+\frac{3}{2})}\left\|\varphi_j e^{-\mu(t-\tau)|\xi|^2}r^{\frac{-3(6-p_1(\cdot))}{6p_1(\cdot)}}2^{-3j\frac{6-p_1(\cdot)}{6p_1(\cdot)}}\right\|_{\frac{6p_1(\cdot)}{6-p_1(\cdot)}}\|v\otimes w\|_{\mathcal{M}^{k(\cdot)}_{6/5}}d\tau\right\|_{L^\rho([0,\infty))}\right\|_{\ell_q}\\
&\lesssim \left\|\left\|\int_0^t 2^{j(\frac{2}{\rho}+\frac{5}{2})}e^{-\mu(t-\tau)|\xi|^2}\left\|\varphi_j r^{\frac{-3(6-p_1(\cdot))}{6p_1(\cdot)}}2^{-3j\frac{6-p_1(\cdot)}{6p_1(\cdot)}}\right\|_{\frac{6p_1(\cdot)}{6-p_1(\cdot)}}\|v\otimes w\|_{\mathcal{M}^{k(\cdot)}_{6/5}}d\tau\right\|_{L^\rho([0,\infty))}\right\|_{\ell_q}\\
&\lesssim \left\|\left\|2^{j(\frac{2}{\rho}+\frac{1}{2})}\|v\otimes w\|_{\mathcal{M}^{k(\cdot)}_{6/5}}\right\|_{L^\rho([0,\infty))}\left\|2^{2j}e^{-\mu t2^{2j}}\right\|_{L^1([0,\infty))}\right\|_{\ell_q}\\
&\lesssim \|v\|_{\mathcal{L}^\rho([0,\infty),\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})}\|w\|_{\mathcal{L}^\infty([0,\infty),L^3)}+\|w\|_{\mathcal{L}^\rho([0,\infty),\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})}\|v\|_{\mathcal{L}^\infty([0,\infty),L^3)}\\
&\lesssim \|v\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})}\|w\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{B}^{1/2}_{2,q})}+\|w\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})}\|v\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{B}^{1/2}_{2,q})}\\
&\lesssim \|v\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})}\|w\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}+\|w\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})}\|v\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}.
\end{aligned}$$

As a result, it follows

$$\|B(v\otimes w)\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),k(\cdot),q})} \lesssim \|v\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})}\|w\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}+\|w\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})}\|v\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}.$$

Hence, if $\rho=\infty$ *and* $p(\cdot)=p_1(\cdot)$, we obtain

$$\|B(v\otimes w)\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q})} \lesssim \|v\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})}\|w\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}+\|w\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})}\|v\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}.$$

Similarly, we get

$$\|B(v\otimes w)\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})} \lesssim \|v\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})}\|w\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}+\|w\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})}\|v\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}$$

and

$$\|B(v\otimes w)\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})} \lesssim \|v\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})}\|w\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}+\|w\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})}\|v\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}.$$

Consequently,

$$\|B(v \otimes w)\|_X \lesssim \|v\|_X \|w\|_X .$$

Finally,

$$\|B(v \otimes v)\|_X \leq C_2 \|v\|_X \|v\|_X \quad \text{for all} \quad v \in X. \tag{4.4}$$

Then, by (4.3) and (4.4), one concludes

$$\begin{aligned} \|\Phi(v)\|_X &\leq \|T_{\Omega,N}(t)v_0\|_X + \left\| \int_0^t T_{\Omega,N}(t-\tau)\tilde{\mathbb{P}}\tilde{\nabla} \cdot (v \otimes v) d\tau \right\|_X \\ &\leq \|T_{\Omega,N}(t)v_0\|_X + \|B(v \otimes v)\|_X \\ &\leq C_1 \|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}} + C_2 \varepsilon^2. \end{aligned}$$

Put $\varepsilon < \dfrac{1}{2max(C_1, C_2)}$ for any $v_0 \in \mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}$ with

$$\|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}} < \frac{\varepsilon}{2max(C_1, C_2)},$$

we get

$$\begin{aligned} \|\Phi(v)\|_X &< C_1 \frac{\varepsilon}{2max(C_1, C_2)} + C_2 \frac{\varepsilon}{2max(C_1, C_2)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

Then, according to Lemma 3.6 we obtain a unique global solution when the initial data are small.
Moreover, let us present the space
$Y = \left\{ \mathcal{L}^\rho([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),k(\cdot),q}) \cap \mathcal{L}^\infty([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}) \cap \mathcal{L}^\rho([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q}) \cap \mathcal{L}^\infty([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q}) \right\}.$
By following a similar argument to the one presented above, we obtain

$$\|\Phi(v)\|_Y \leq \|T_{\Omega,N}(t)v_0\|_Y + \|B(v,v)\|_Y .$$

Thus, in an analogous way to the case of the space X we can show that the problem (4.1) has a unique global solution, when

$$\|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}} < \sigma.$$

5 Gevrey class regularity

The analyticity of the solution is also an important subject developed by several researchers, particularly with regard to the Navier-Stokes equations, see [17] and the references therein. In this section, we will prove the Gevrey class regularity for (1.1) in the Fourier-Besov-Morrey spaces and the following lemma is so helpful.

Lemma 5.1. [40] Let $0 < s \leq t < +\infty$ and $0 \leq \alpha \leq 1$. Then, the following inequality holds

$$t \mid x \mid^\alpha - \frac{1}{2}(t^2 - s^2) \mid x \mid^{2\alpha} - s \mid x - y \mid^\alpha - s \mid y \mid^\alpha \leq \frac{1}{2}$$

for any $x, y \in \mathbb{R}^3$.

5.1 Proof of Theorem 3.15

We note $V(t) = e^{\mu\sqrt{t}|D|}v(t)$. Using (4.2), we get

$$\begin{aligned} V(t) &= e^{\mu\sqrt{t}|D|}T_{\Omega,N}(t)v_0 + e^{\mu\sqrt{t}|D|}B(v,v) \\ &= Lv_0 + \tilde{B}(v,v). \end{aligned}$$

For linear estimate, it is easy to show that

$$\begin{aligned} \left\| 2^{js_1(\cdot)}\varphi_j \hat{Lv_0} \right\|_{L^\rho([0,\infty),\mathcal{M}^{k(\cdot)}_{p_1(\cdot)})} &\lesssim \left\| 2^{js_1(\cdot)}\varphi_j e^{-\frac{\mu}{2}t|\xi|^2} e^{\mu\sqrt{t}|\xi|-\frac{\mu}{2}t|\xi|^2}\hat{v}_0 \right\|_{L^\rho([0,\infty),\mathcal{M}^{k(\cdot)}_{p_1(\cdot)})} \\ &\lesssim \left\| 2^{js_1(\cdot)}\varphi_j e^{-\frac{\mu}{2}t|\xi|^2}\hat{v}_0 \right\|_{L^\rho([0,\infty),\mathcal{M}^{k(\cdot)}_{p_1(\cdot)})}, \end{aligned}$$

where we used the following inequality $e^{\mu\sqrt{t}|\xi|-\frac{\mu}{2}t|\xi|^2} = e^{-\frac{\mu}{2}(\sqrt{t}|\xi|-1)^2+\frac{\mu}{2}} \leq e^{\frac{\mu}{2}}$. Taking ℓ_q-norm, we obtain

$$\|Lv_0\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),k(\cdot),q})} \lesssim \left\| 2^{js_1(\cdot)}\varphi_j e^{-\frac{\mu}{2}t|\xi|^2}\hat{v}_0 \right\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),k(\cdot),q})}.$$

Hence, by following a similar process as in the proof of Theorem 3.14 , we obtain

$$\|Lv_0\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),k(\cdot),q})} \lesssim \|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}}. \tag{5.1}$$

Notice that the estimate (5.1) also hold for $\rho = \infty$ and $p_1 = p$. i.e,

$$\|Lv_0\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q})} \lesssim \|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}},$$

also, it is easy to see that

$$\|Lv_0\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})} \lesssim \|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}}$$

and

$$\|Lv_0\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})} \lesssim \|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}}.$$

Then

$$\|Lv_0\|_X \lesssim \|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q}}.$$

For bilinear estimate, we have

$$\tilde{B}(v,w) = -\int_0^t e^{\mu\sqrt{t}|D|}T_{\Omega,N}(t-\tau)\tilde{\mathbb{P}}\tilde{\nabla}\cdot(v\otimes w)d\tau. \tag{5.2}$$

Then, we can rewrite (5.2) as follows:

$$\tilde{B}(V,W) = -\int_0^t e^{\mu\sqrt{t}|D|}T_{\Omega,N}(t-\tau)\tilde{\mathbb{P}}\tilde{\nabla}\cdot(e^{-\mu\sqrt{\tau}|D|}V\otimes e^{-\mu\sqrt{\tau}|D|}W)d\tau$$

with $W = e^{\mu\sqrt{\tau}|D|}w$. Taking the Fourier transform, multiplying with $2^{js_1(\cdot)}\varphi_j$, taking the $L^\rho([0,\infty),\mathcal{M}^{k(\cdot)}_{p_1(\cdot)})$-norm and using Lemma 5.1, one obtains

$$\begin{aligned}
&\left\|2^{js_1(\cdot)}\varphi_j\widehat{\tilde{B}(V,W)}\right\|_{L^\rho([0,\infty),\mathcal{M}^{k(\cdot)}_{p_1(\cdot)})}\\
&\lesssim\left\|2^{j(s_1(\cdot)+1)}\varphi_j\int_0^t e^{\mu\sqrt{t}|\xi|}e^{-\mu(t-\tau)|\xi|^2}(e^{-\mu\sqrt{\tau}|D|}\widehat{V\otimes e^{-\mu\sqrt{\tau}|D|}}W)d\tau\right\|_{L^\rho([0,\infty),\mathcal{M}^{k(\cdot)}_{p_1(\cdot)})}\\
&\lesssim\left\|2^{j(s_1(\cdot)+1)}\varphi_j\int_0^t e^{\frac{-\mu}{2}(t-\tau)|\xi|^2}\int_{\mathbb{R}^3}e^{\frac{-\mu}{2}(t-\tau)|\xi|^2+\mu\sqrt{t}|\xi|-\mu\sqrt{\tau}(|\xi-z|+|z|)}(\hat{V}(\xi-z,\tau)\otimes\hat{W}(z,\tau))dzd\tau\right\|_{L^\rho([0,\infty),\mathcal{M}^{k(\cdot)}_{p_1(\cdot)})}\\
&\lesssim\left\|2^{j(s_1(\cdot)+1)}\varphi_j\int_0^t e^{\frac{-\mu}{2}(t-\tau)|\xi|^2}\int_{\mathbb{R}^3}\hat{V}(\xi-z,\tau)\otimes\hat{W}(z,\tau)dzd\tau\right\|_{L^\rho([0,\infty),\mathcal{M}^{k(\cdot)}_{p_1(\cdot)})}\\
&\lesssim\left\|2^{j(s_1(\cdot)+1)}\varphi_j\int_0^t e^{\frac{-\mu}{2}(t-\tau)|\xi|^2}(\widehat{V\otimes W})d\tau\right\|_{L^\rho([0,\infty),\mathcal{M}^{k(\cdot)}_{p_1(\cdot)})}.
\end{aligned}$$

By using an analogous argument as in the proof of Theorem 3.14 , one reaches

$$\left\|\tilde{B}(V,W)\right\|_X\lesssim\|V\|_X\,\|W\|_X$$

and

$$e^{\mu\sqrt{t}|D|}v(t)\in X=\left\{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q})\cap\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})\cap\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})\right\}.$$

From the above and in a similar manner to the case of the space X, one checks that

$$e^{\mu\sqrt{t}|D|}v(t)\in Y=\left\{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),k(\cdot),q})\cap\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),k(\cdot),q})\cap\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,k(\cdot),q})\cap\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})\right\}.$$

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