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The first and second K-Banhatti indices of some graph operations

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Abstract

Graph operations are an essential tool in graph theory, as we can develop large graphs from small graphs. The aim of this paper is to compute the values of K-Banhatti indices for some graph operations, more precisely join and corona product of two graphs. Path, cycle, and complete graphs are taken for the graph operations. The result found may be applicable in locating some buried information on large graphs.

Keywords: K-Banhatti Indices, Composed graphs, Corona product
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1 Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The number of elements present in the set $V(G)$ and $E(G)$ are known as the order and size of the graph G respectively and denoted by $|V(G)| = n$, and $|E(G)| = m$. The degree of the vertex $p \in V(G)$ is the number of edges occurring to a vertex p and denoted by, d_p. Let d_e denote the degree of an edge e in G which is defined by, $d_e = d_p + d_q - 2$ where $e = pq$. Throughout this paper, $p \sim e$ denotes for vertex p and edge e are adjacent in the graph G. A walk (W) in a graph G, is a sequence of vertices in G which are consecutive vertices in W are adjacent to G. If in a walk no vertex is repeated, then it is known as a path and it is denoted by, P_n with n vertices. A cycle is a closed path, denoted by C_n. Every two vertices are adjacent in a graph, then it is known as a Complete graph and it is denoted by, K_n.

Let G_1 and G_2 be two vertex-disjoint graphs, the join of G_1 and G_2 (denoted by, $G_1 + G_2$). The vertex and edge sets of $G_1 + G_2$ are $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{pq : p \in V(G_1), /q \in V(G_2)\}$.

Let G_1 be a connected graph with n vertices and G_2 (not necessarily connected) be a graph with at least two vertices. The corona product of G_1 and G_2 (denoted by, $G_1 \odot G_2$) is a graph which is formed by taking n copies of G_2 and connecting i^{th} vertex of G_1 to the vertices of G_2 in each copies. The general formula for vertex-degree-based topological indices is given by,

$$TI(G) = \sum_{p \sim q} F(d_p, d_q),$$

where $F(u, v)$ is a symmetric function [2]. In [3], the first and second K-Banhatti indices are defined and as follows:

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Table 1: Number of edges in each partitions of $P_n + P_m$ based on degree of end vertices of each edge.

(d_p, d_q), where $p \sim q$	Degree of edge d_e	Number of edges
$(m+1, m+1)$	$2m$	1 if $n = 2$, 0 if $n > 2$
$(m+1, m+2)$	$2m+1$	0 if $n = 2$, 2 if $n > 2$
$(m+2, m+2)$	$2m+2$	0 if $n = 2$, $n-3$ if $n > 2$
$(n+1, n+1)$	$2n$	1 if $m = 2$, 0 if $m > 2$
$(n+1, n+2)$	$2n+1$	0 if $m = 2$, 2 if $m > 2$
$(n+2, n+2)$	$2n+2$	0 if $m = 2$, $m-3$ if $m > 2$
$(m+1, n+1)$	$m+n$	4
$(m+1, n+2)$	$m+n+1$	0 if $m = 2$, $2(m-2)$ if $m > 2$
$(m+2, n+1)$	$m+n+1$	0 if $n = 2$, $2(n-2)$ if $n > 2$
$(m+2, n+2)$	$m+n+2$	0 if $n = 2$ or $m = 2$, $(n-2)(m-2)$ if $n, m > 2$

Definition 1.1. The first K-Banhatti index of a graph G is defined as

$$B_1(G) = \sum_{p \sim e} (d_p + d_e). \tag{1.1}$$

Definition 1.2. The second K-Banhatti index of a graph G is defined as

$$B_2(G) = \sum_{p \sim e} (d_p \cdot d_e). \tag{1.2}$$

Recently, mathematical properties and studies on K-Banhatti indices can be found in [4, 10, 6, 8, 7, 5, 9, 1, 11].

2 Main Results

In this section, we are computing the first and second K-Banhatti indices of P_n+P_m, C_n+C_m, K_n+K_m, $P_n \odot P_m$, $C_n \odot C_m$, $K_n \odot K_m$, and some other composite graphs.

2.1 Results on $P_n + P_m$

Consider two paths P_n and P_m with order n and m respectively. The order and size of the graph $P_n + P_m$ will be $n+m$ and $nm+n+m-2$ respectively. When both n and m are greater then one, then we have four types of vertices with degree $m+1$, $m+2$, $n+1$, and $n+2$. By considering those vertices, we have ten types of edge partitions for the graph $P_n + P_m$ which is shown in Table 1.

Theorem 2.1. The first K-Banhatti index of $P_n + P_m$ is given by

$$B_1(P_n + P_m) = \begin{cases} 82 & \text{if } m = n = 2 \\ 38m + 2(m-2)(3m+11) + 8 & \text{if } n = 2 \ \&\ m > 2 \\ 38n + 2(n-2)(3n+11) + 8 & \text{if } m = 2 \ \&\ n > 2 \\ 24(m+n) + (n-3)(6m+8) + (m-3)(6n+8) + (3m+3n+5)(2m+2n-8) & \\ +(n-2)(m-2)(3m+3n+8) + 28 & \text{if } m, n > 2 \end{cases}$$

Proof . With the help of Table 1 and equation 1.1, we compute first K-Banhatti index of $P_n + P_m$ in the following cases:
Case 1 If $n = m = 2$

$$B_1(P_n + P_m) = (2(m+1) + 4m) + [2(n+1) + 4n] + 4[m+1+2m+2n+n+1] = 18m + 18n + 10$$

After substituting $m = n = 2$, we get

$$B_1(P_n + P_m) = 82.$$

Case 2 If $n = 2$, $m > 2$

$$\begin{aligned} B_1(P_n + P_m) &= 2(m+1+4m) + 2[(n+1) + 2(2n+1) + n + 2] + (m-3)[2(n+2) + 2(2n+2)] \\ &\quad + 4[m+1+n+2+2m+2n] + 2(m-2)[m+1+2m+2n+2+n+2] \end{aligned}$$

After substituting $n = 2$, we get

$$B_1(P_n + P_m) = 38m + 2(m-2)(3m+11) + 8.$$

Case 3 If $n > 2$, $m = 2$

$$\begin{aligned} B_1(P_n + P_m) &= 2[(m+1) + (m+2) + 4m + 2] + (n-3)(2m+4+4m+4) \\ &\quad + (2n+2+4n) + 4[3m+2+3n] + 2(n-2)[3m+3n+5] \end{aligned}$$

After substituting $m = 2$, we get

$$B_1(P_n + P_m) = 38n + 2(n-2)(3n+11) + 8.$$

Case 4 If $n, m > 2$

$$\begin{aligned} B_1(P_n + P_m) =& 6[6m+5] + (n-3)(6m+8) + 2[6n+5] + (m-3)(6n+8) + 4[3m+3n+2] \\ &+ 2(m-2)[3m+3n+5] + (n-2)(m-2)[3m+3n+8] + 2(m-2)[3m+3n+5] \\ =& 24(m+n) + (n-3)(6m+8) + (m-3)(6n+8) + (3m+3n+5)(2m+2n-8) \\ &+ (n-2)(m-2)(3m+3n+8) + 28. \end{aligned}$$

□

Theorem 2.2. The second K-Banhatti index of $P_n + P_m$ is given by

$$B_2(G) = \begin{cases} 114 & \text{if } m = n = 2 \\ 4m(m+1) + 72(m-3) + 4(m+2)(m+4) + 2(m-2)(m+3)(m+5) + 70 & \text{if } n = 2 \ \& \ m > 2 \\ 4n(n+1) + 72(n-3) + 4(n+2)(n+4) + 2(n-2)(n+3)(n+5) + 70 & \text{if } m = 2 \ \& \ n > 2 \\ 4(2m+3)[m+n+1] + 4(n-3)(m+1)(m+2) + 4(m-3)(n+1)(n+2) & \\ +4(m+n)(m+n+2) + 2(m+n+1)(m+n+3)(m+n-4) & \\ +(n-2)(m-2)(m+n+2)(m+n+4) & \text{if } m, n > 2. \end{cases}$$

Proof . The proof is similar to Theorem 2.1. □

2.2 Results on $C_n + C_m$

The graph $C_n + C_m$ is obtained by joining two cycle graphs C_n and C_m of order n and m respectively. Then the order and size of $C_n + C_m$ is $n + m$ and $n + m + nm$ respectively. According to vertex degree, we have two types of vertices for $C_n + C_m$. One type of vertices has degree $m + 2$ and the other has degree $n + 2$. So, there are three edge partitions shown in Table 2.

Table 2: Number of edges in each partitions of $C_n + C_m$ based on degree of end vertices of each edge.

(d_p, d_q), where $p \sim q$	Degree of edge d_e	Number of edges
$(m+2, m+2)$	$2m+2$	n
$(m+2, n+2)$	$m+n+2$	nm
$(n+2, n+2)$	$2n+2$	m

Table 3: Number of edges in each partitions of $P_n + K_m$ based on degree of end vertices of each edge.

(d_p, d_q), where $p \sim q$	Degree of edge d_e	Number of edges
$(m+1, m+1)$	$2m$	1 if $n = 2$, 0 if $n > 2$
$(m+1, m+2)$	$2m+1$	0 if $n = 2$, 2 if $n > 2$
$(m+1, n+m-1)$	$2m+n-2$	$2m$
$(m+2, m+2)$	$2m+2$	0 if $n = 2$, $(n-3)$ if $n > 2$
$(m+2, n+m-1)$	$2m+n-1$	0 if $n = 2$, $m(n-2)$ if $n > 2$
$(n+m-1, n+m-1)$	$2n+2m-4$	$\binom{m}{2}$

Theorem 2.3. For positive integers $m, n(\geq 3)$, the first and second K-Banhatti indices for $C_n + C_m$ are given as follows,

$$\begin{aligned} B_1(C_n + C_m) =& 20nm + (m+n)(8+3mn), \\ B_2(C_n + C_m) =& 4n(m+2)(m+1) + mn(m+n+4)(m+n+2) + 4m(n+2)(n+1). \end{aligned}$$

Proof . By substituting the values in Table 2 in the equation 1.1, one can easily prove the results. □

Remark 2.4. The graph $K_n + K_m$ is obtained by joining two complete graphs K_n and K_m. The order and size of $K_n + K_m$ are $n + m$ and $\dfrac{(n+m)(n+m-1)}{2}$. It is obvious that, $K_n + K_m = K_{n+m}$ and the degree of each vertex in K_{n+m} is $m + n - 1$. Therefore, we have only one edge partition and that is, $(n + m - 1, n + m - 1)$, and edge degree $2m + 2n - 4$. So,

$$\begin{aligned} K_1(K_{n+m}) =& (n+m)(n+m-1)[3m+3n-5], \\ K_2(K_{n+m}) =& 2(n+m)(n+m-1)^2(m+n-2). \end{aligned}$$

2.3 Results for $P_n + K_m$

The graph $P_n + K_m$ can be easily obtained by joining the graphs P_n and K_m. It contains $n + m$ vertices and $\binom{m}{2} + mn + n - 1$ edges. There are three types of vertices, $(m + 1)$, $(m + 2)$, and $(n + m - 1)$ and have six edge partitions as shown in Table 3.

Theorem 2.5. The first K-Banhatti index of $P_n + K_m$ is given by

$$B_1(P_n + K_m) = \begin{cases} (6m+2) + 2m[6m+2] + \binom{m}{2}(6m+2) & \text{if } n = 2 \\ 2(6m+5) + 2m(6m+3n-4) + (n-3)(6m+8) & \\ +m(m-2)(6m+3n-1) + \binom{m}{2}(6n+6m-10) & \text{if } n > 2. \end{cases}$$

Table 4: Number of edges in each partitions of $C_n + K_m$ based on degree of end vertices of each edge.

(d_p, d_q), **where** $p \sim q$	**Degree of edge**	**Number of edges**
$(m+2, m+2)$	$2m+2$	n
$(m+2, n+m-1)$	$2m+n-1$	nm
$(n+m-1, n+m-1)$	$2n+2m-4$	$\binom{m}{2}$

Proof . This can be proved in following cases for n:
Case 1 Suppose $n = 2$

$$\begin{aligned}B_1(P_n + K_m) =& 1(2m+2+4m) + 2m[(m+1) + (2n+4m-4) + n + m - 1] + \binom{m}{2}(2n+2m-2+4n+4m-8)\\ =& (6m+2) + 2m[6m+2] + \binom{m}{2}(6m+2).\end{aligned}$$

Case 2 Suppose $n > 2$

$$\begin{aligned}B_1(P_n + K_m) =& 2(6m+5) + 2m[6m+3n-4] + (n-3)[2m+4+4m+4]\\ &+ m(n-2)(4m+2n-2+2m+n+1) + \binom{m}{2}(6n+6m-10)\\ =& 2(6m+5) + 2m(6m+3m-4) + (n-3)[6m+8] + m(m-2)(6m+3n-1) + \binom{m}{2}(6n+6m-10).\end{aligned}$$

□

Theorem 2.6. The second K-Banhatti index of $P_n + K_m$ is given by

$$B_2(P_n + K_m) = \begin{cases} 4m(m+1) + 4m^2 + 2(m+1) + 2m\binom{m}{2}(m+1)^2 & \text{if } n = 2\\ 2(2m+1)(2m+3) + 2m(2m+n-2)(2m+n) + 4(n-3)(m+1)(m+2) & \\ +m(n-2)(2m+n-1)(2m+n+1) + 2\binom{m}{2}(2n+2m-4)(n+m-1) & \text{if } n > 2.\end{cases}$$

Proof . The proof is similar to Theorem 2.6. □

2.4 Results on $C_n + K_m$

We know that both C_n and K_m are regular graphs.More precisely, C_n is 2-regular and K_m is $(m-1)$-regular. The graph $C_n + K_m$, contains $m + n$ vertices and $n + mn + \binom{m}{2}$ edges. Clearly, the graph $C_n + K_m$ have two types of vertices, that is $(m-1)$ and $(m+n-1)$. It has three edge partitions and is shown in Table 4.

Theorem 2.7. The first and second K-Banhatti indices for $K_n + C_m$ are given as follows

$$\begin{aligned}B_1(K_n + C_m) =& n(6m+8) + mn(6m+3n-1) + \binom{m}{2}(6m+6m-10),\\ B_2(K_n + C_m) =& 2n(2m+2)(m+2) + mn(2m+n-1)(2m+n+1) + 2\binom{m}{2}(2m+2n-4)(n+m-1).\end{aligned}$$

Proof . The proof is similar to Theorem 2.3. □

2.5 Results on the corona product of two path graphs

Consider the corona product of two paths P_n and P_m denoted by, $P_n \odot P_m$. Clearly, we can see it has a $nm + n$ order and $2mn - 1$ size. Based on vertex degree of $P_n \odot P_m$ contains four different types of vertices namely,2, 3, $(m+1)$, and $(m+2)$. There are ten different edge partitions as seen in Table 5.

Table 5: Number of edges in each partitions of $P_n \odot P_m$ based on degree of end vertices of each edge.

(d_p, d_q), **where** $p \sim q$	**Degree of edge**	**Number of edges**
$(2,2)$	2	n if $m=2$, 0 if $m>2$
$(3,3)$	4	0 if $m=2$, $n(m-3)$ if $m>2$
$(m+1,m+1)$	$2m$	1 if $n=2$, 0 if $n>2$
$(m+2,m+2)$	$2m+2$	0 if $n=2$, $(n-3)$ if $n>2$
$(2,3)$	3	0 if $m=2$, $2n$ if $m>2$
$(2,m+1)$	$m+1$	4
$(2,m+2)$	$m+2$	0 if $n=2$, $2(n-2)$ if $n>2$
$(3,m+1)$	$m+2$	0 if $m=2$, $2(m-2)$ if $m>2$
$(3,m+2)$	$m+3$	0 if $n=2$, $(n-2)(m-2)$ if $n>2$
$(m+1,m+2)$	$2m+1$	0 if $n=2$, 2 if $n>2$

Theorem 2.8. The first K-Banhatti index of $P_n \odot P_m$ is given by

$$B_1(P_n \odot P_m) = \begin{cases} 74 & \text{if } m=n=2 \\ (6m+2)+28(m-3)+44+4(3m+5)+2(m-2)(3m+11) & \text{if } n=2\ m>2 \\ 8n+20(n-3)+28(n-2)+78 & \text{if } n>2\ m=2 \\ 14n(m-3)+(n-3)(6m+8)+22n+4(3m+5)+2(3m+8)(m+n-4) & \\ +(n-2)(m-2)(3m+11)+2(6m+5) & \text{if } n,m>2. \end{cases}$$

Proof . Substituting the values in Table 5 and using equation 1.1, we can compute first K-Banhatti index of $P_n \odot P_m$ in the following cases:
Case 1 If $m=n=2$

$$B_1(P_n \odot P_m) = n(4+4)+1[2m+2+4m]+4[(m+3)+(2m+2)] = 74$$

Case 2 $n=2,\ m>2$

$$\begin{aligned} B_1(P_n \odot P_m) =& 1[2m+2+4m]+n(m-3)[7+7]+2n[5+6]+4[m+3+2m+2]+2(m-2)[m+6+2m+5] \\ =& (6m+2)+28(m-3)+44+4(3m+5)+2(m-2)(3m+11). \end{aligned}$$

Case 3 $n>2,\ m=2$

$$\begin{aligned} B_1(P_n \odot P_m) =& n(4+4)+(n-3)[2(3m+4)]+4[m+3+2m+2]+2(n-2)[m+4+2m+4] \\ &+(n-2)(m-2)[m+6+2m+5]+2[3m+2+3m+3] \\ =& 8n+20(n-3)+28(n-2)+78. \end{aligned}$$

Case 4 $n,m>2$

$$\begin{aligned} B_1(P_n \odot P_m) =& n(m-3)[7+7]+(n-3)[3m+4+3m+4]+4[m+3+2m+2]+2n(5+6)+2(n-2)[m+4+2m+4] \\ &+2(m-2)[m+5+2m+3]+(n-2)(m-2)[m+6+2m+5]+2[3m+2+3m+3] \\ =& 14n(m-3)+(n-3)(6m+8)+22n+4(3m+5)+2(3m+8)(m+n-4) \\ &+(n-2)(m-2)(3m+11)+2(6m+5). \end{aligned}$$

□

Table 6: Number of edges in each partitions of $C_n \odot C_m$ based on degree of end vertices of each edge.

(d_p, d_q), where $p \sim q$	Degree of edge	Number of edges
$(3, 3)$	4	nm
$(3, m+2)$	$m+3$	nm
$(m+2, m+2)$	$2m+2$	n

Table 7: Number of edges in each partitions of $K_n \odot K_m$ based on degree of end vertices of each edge.

(d_p, d_q), where $p \sim q$	Degree of edge	Number of edges
(m, m)	$2m-2$	$n\binom{m}{2}$
$(m, n+m-1)$	$2m+n-3$	nm
$(n+m-1, n+m-1)$	$2m+2n-4$	$\binom{n}{2}$

Theorem 2.9. The second K-Banhatti index of $P_n \odot P_m$ is given by

$$B_1(P_n \odot P_m) = \begin{cases} 100 & \text{if } m = n = 2 \\ 48(m-3) + 4(m+1)[2m+3] + 2(m^2-4)(m+4) + 60 & \text{if } n = 2\ m > 2 \\ 104n - 110 & \text{if } n > 2\ m = 2 \\ 24n(m-3) + 4(m+1)[(n-3)(m+2) + (m+3)] + 2(m+4)[(n-2)(m+2) & \\ +(m^2-4)] + (n-2)(m-2)(m+3)(m+5) + 2(2m+1)(2m+3) + 30n & \text{if } n, m > 2 \end{cases}$$

Proof . The proof is similar to the Theorem 2.8. □

2.6 Results on corona product of two cycle graphs

The graph $C_n \odot C_m$ is a graph, which has order $mn+n$ and size $2nm+n$. There are two different types of vertices namely, 3 and $m+2$. The edge partition can be seen in Table 6.

Theorem 2.10. The first and second K-Banhatti indices of $C_n \odot C_m$ is given by

$$\begin{aligned} B_1(C_n \odot C_m) =& 14mn + nm(3m+11) + n(6m+8), \\ B_2(C_n \odot C_m) =& 24mn + mn(m+3)(m+5) + 2(2m+2)(m+2). \end{aligned}$$

Proof . The proof is similar to the Theorem 2.8. □

2.7 Results on corona product of two complete graphs

Consider the corona product of two complete graphs K_n and K_m denoted by, $K_n \odot K_m$. The order and size the graph is given by, $mn+n$ and size $\binom{n}{2} + n\binom{m}{2} + mn$. There are two different types of vertices namely, m and $n+m-1$. The edge partition can be seen in Table 7.

Theorem 2.11. The first and second K-Banhatti indices of $K_n \odot K_m$ is given by

$$\begin{aligned} B_1(K_n \odot K_m) =& n\binom{m}{2}(6m-4) + mn(6m+3n-7) + \binom{n}{2}(6n+6m-10), \\ B_2(K_n \odot K_m) =& 4nm\binom{m}{2}(m-1) + nm(n+2m-3)(3m+n-1) + 2\binom{n}{2}(n+m-1)(2m+2n-4). \end{aligned}$$

Proof . The proof is similar to the Theorem 2.8. □

Conclusion

The main focus of this work is to evaluate the K-Banhatti indices of some graph operations. Further, we also present their graphical representation in Figure 1 and 2. These graphs show how the values of K-Banhatti indices behave differently depending on the parameters. These comparisons can be helpful to know some hidden information the considering larger graph structures.

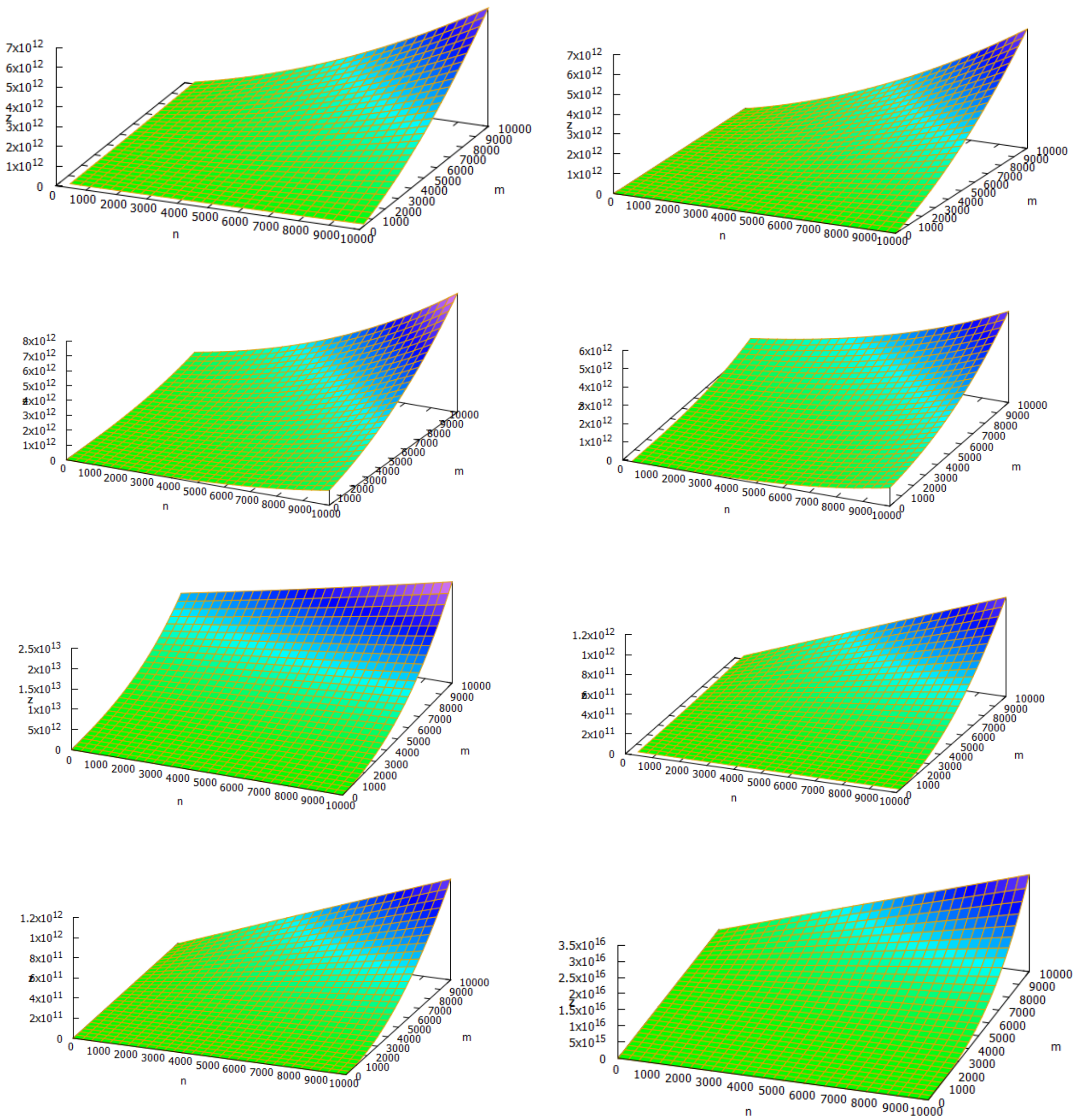

Figure 1: The graphical representation first K-Banhatti of $P_n + P_m$, $C_n + C_m$, $K_n + K_m$, $P_n + K_m$,$C_n + K_m$, $P_n \odot P_m$, $C_n \odot C_m$, and $K_n \odot K_m$ respectively.

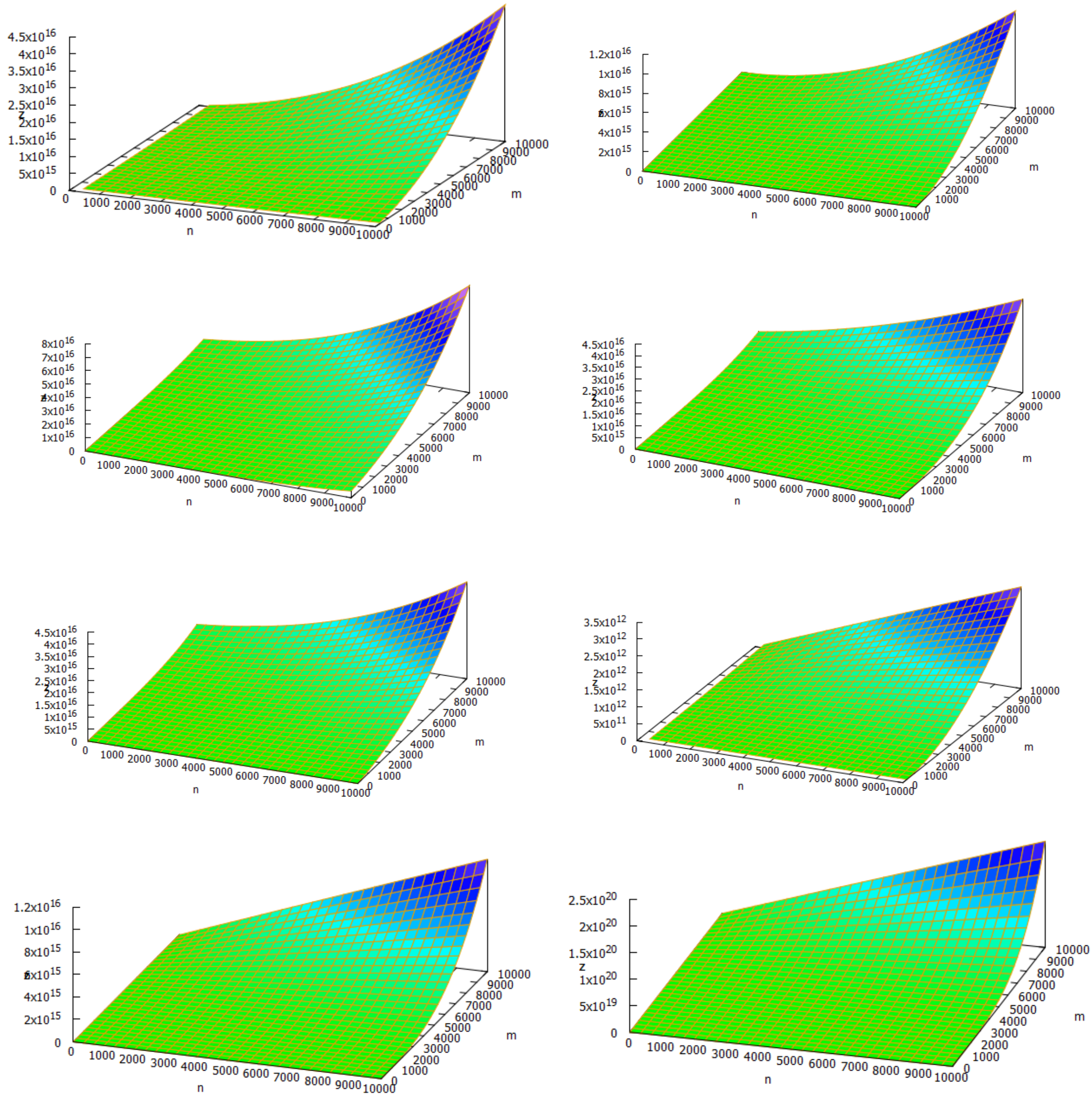

Figure 2: The graphical representation second K-Banhatti of $P_n + P_m$, $C_n + C_m$, $K_n + K_m$, $P_n + K_m$,$C_n + K_m$, $P_n \odot P_m$, $C_n \odot C_m$, and $K_n \odot K_m$ respectively.

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