Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 3025–3035
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.23999.2649

On subclasses of analytic functions defined by using Tremblay fractional derivative operator

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(Communicated by Mugur Alexandru Acu)

Abstract

By making use of the Tremblay operator, we introduce new subclasses of analytic functions, for which we obtain some sufficient coefficients estimates, and the consequences are some subordination properties and partial sums inequalities.

Keywords: analytic function, starlike function, convex function, subordination, factor sequence
2020 MSC: 30C45, 30C50

1 Introduction

Let $\mathcal{A}$ denote the class of univalent and analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, and normalized by the conditions $f(0) = f'(0) - 1 = 0$. It is clear that every function belonging to the class $\mathcal{A}$ can be given by the following Maclaurin's expansion:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

A function f belonging to $\mathcal{A}$ is said to be starlike function of order $\alpha\,(0 \le \alpha < 1)$, if

$$\text{Re}\left\{\frac{zf^{'}(z)}{f(z)}\right\} > \alpha \ (z \in U), \tag{1.2}$$

the suclass of all starlike functions of order α is denoted usually as $\mathcal{S}^*(\alpha)$. Further, a function f belonging to $\mathcal{A}$ is said to be convex function of order $\alpha\,(0 \le \alpha < 1)$, if

$$\text{Re}\left\{1 + \frac{zf^{''}(z)}{f'(z)}\right\} > \alpha \ (z \in U), \tag{1.3}$$

we denote by $\mathcal{K}(\alpha)$ the subclass of all convex functions of order α. We note that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$.

From (1.2) and (1.3), it is clear that

$$f \in \mathcal{K}(\alpha) \Longleftrightarrow g \in \mathcal{S}^*(\alpha); g(z) = zf^{'}(z).$$

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Received: July 2021 *Accepted:* September 2021

For details about the subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, see MacGregor [10], Schild [13], and Pinchuk [12].

The modified Hadamard product (or Convolution) of function $f \in \mathcal{A}$ given by (1.1), and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.4}$$

is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.5}$$

Also, the Subordination Principle between functions $f \in \mathcal{A}$ will be used here and is recalled as following (see [4] and [11]):

If f and g are analytic in the open unit disc U, we say that f is subordinate to g, written as $f \prec g$ in U or $f(z) \prec g(z)(z \in U)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $(z \in U\ )$ such that $f(z) = g(w(z))$ $(z \in U)$. Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence holds:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Moreover, the definitions of Riemannian fractional integral and fractional derivative are important for our results. (For details, see [15, 16, 21, 20], also references cited therein).

Definition 1.1. *The fractional integral of order δ is defined, for a function $f(z)$, by*

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\delta}} d\zeta; \quad (\delta > 0), \tag{1.6}$$

where the function f is analytic in a simply-connected domain of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{\delta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 1.2. *The fractional derivative of order δ is defined, for a function f, by*

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\delta}} d\zeta; \quad (0 \le \delta < 1), \tag{1.7}$$

where the function f is constrained, and the multiplicity of $(z-\zeta)^{-\delta}$ is removed as in Definition 1.1 above.

Under the hypothesis of Definition 1.2, the fractional derivative of order $n+\delta$ $(0 \le \delta < 1$ and $n \in \mathbb{N}_0 = \{0, 1, 2, ...\})$ is defined for a function f by

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n}\{D_z^{\delta} f(z)\} \quad (0 \le \delta < 1; n \in \mathbb{N}_0). \tag{1.8}$$

Also, by using Definitions 1.1, 1.2, then

$$D_z^{\delta}\{z^k\} = \frac{\Gamma(k+1)}{\Gamma(k-\delta+1)} z^{k-\delta} \qquad (0 \le \delta < 1; k \in \mathbb{N}), \tag{1.9}$$

and

$$D_z^{-\delta}\{z^k\} = \frac{\Gamma(k+1)}{\Gamma(k+\delta+1)} z^{k+\delta} \qquad (\delta > 0; k \in \mathbb{N}), \tag{1.10}$$

in terms of Gamma function.

In his thesis, Tremblay [22] investigated a fractional calculus operator defined in terms of the Riemann-Liouville fractional differential operator. Recently, Ibrahim and Jahangiri [9] extended the Tremblay operator in the complex plane.

Definition 1.3 [9]. *If $f \in \mathcal{A}$, then the Tremblay fractional derivative operator $T_z^{\mu,\gamma}$ of a function f is defined, for all $z \in U$, by*

$$\mathcal{T}_z^{\mu,\gamma} f(z) = \frac{\Gamma(\gamma)}{\Gamma(\mu)} z^{1-\gamma} D_z^{\mu-\gamma} z^{\mu-1} f(z), \tag{1.11}$$

$$(0 < \gamma \le 1, 0 < \mu \le 1, 0 \le \mu-\gamma < 1, \mu > \gamma),$$

It is clear that for $\mu=\gamma=1$, we obtain

$$\mathcal{T}_z^{1,1}f(z)=f(z).$$

In [5], Esa et al. defined modified of Tremblay operator of analytic functions in complex domain as follows:

Definition 1.4. *If $f\in\mathcal{A}$. Then the modified Tremblay operator denoted by $T^{\mu,\gamma}:\mathcal{A}\longrightarrow\mathcal{A}$ and defined as:*

$$\begin{aligned}\mathfrak{T}_z^{\mu,\gamma}f(z)&=\frac{\gamma}{\mu}\mathcal{T}_z^{\mu,\gamma}f(z)=\frac{\Gamma(\gamma+1)}{\Gamma(\mu+1)}z^{1-\gamma}D_z^{\mu-\gamma}z^{\mu-1}f(z)\\&=z+\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+1)\Gamma(\mu+k)}{\Gamma(\mu+1)\Gamma(\gamma+k)}a_kz^k,\\&(0<\gamma\le1,0<\mu\le1,0\le\mu-\gamma<1,\mu>\gamma)\end{aligned}\tag{1.12}$$

where $\mathcal{T}^{\mu,\gamma}$ is denoting the Tremblay fractional derivative operator defined by (1.11). For more information about Tremblay Operator see [19].

Definition 1.5. *For $-1\le B<A\le1$, $0\le\alpha<1$, $0<\gamma\le1$, $0<\mu\le1$, $0\le\mu-\gamma<1$ and $\mu>\gamma$, let $S^{\mu,\gamma}(\alpha;A,B)$ be the subclass of functions $f\in\mathcal{A}$ which satisfy*

$$\frac{z(\mathfrak{T}_z^{\mu,\gamma}f(z))'}{\mathfrak{T}_z^{\mu,\gamma}f(z)}\prec(1-\alpha)\frac{1+Az}{1+Bz}+\alpha.\tag{1.13}$$

That is, the subclass $S^{\mu,\gamma}(\alpha;A,B)$ can be described as following:

$$\mathcal{S}^{\mu,\gamma}(\alpha;A,B)=\left\{f\in\mathcal{A}:\left|\frac{\frac{z(\mathfrak{T}_z^{\mu,\gamma}f(z))'}{\mathfrak{T}_z^{\mu,\gamma}f(z)}-1}{B\frac{z(\mathfrak{T}_z^{\mu,\gamma}f(z))'}{\mathfrak{T}_z^{\mu,\gamma}f(z)}-((A-B)(1-\alpha)+B)}\right|<1,z\in U\right\}\tag{1.14}$$

Also, let $K^{\mu,\gamma}(\alpha;A,B)$ be the subclass of functions $f\in\mathcal{A}$ which satisfy

$$1+\frac{z\left(\mathfrak{T}_z^{\mu,\gamma}f(z)\right)''}{\left(\mathfrak{T}_z^{\mu,\gamma}f(z)\right)'}\prec(1-\alpha)\frac{1+Az}{1+Bz}+\alpha.\tag{1.15}$$

From (1.13) and (1.15), it is clear that

$$f\in\mathcal{K}^{\mu,\gamma}(\alpha;A,B)\Longleftrightarrow g\in\mathcal{S}^{\mu,\gamma}(\alpha;A,B);g=zf'(z).\tag{1.16}$$

Remark 1.1. *Specializing the parameters α, μ, γ, A and B in (1.13) and (1.15), we obtain the following subclasses:*

(i) $S^{\mu,\gamma}(\alpha;\beta,-\beta)=S^{\mu,\gamma}(\alpha,\beta)=\left\{f\in\mathcal{A}:\left|\dfrac{\frac{z(\mathfrak{T}_z^{\mu,\gamma}f(z))'}{\mathfrak{T}_z^{\mu,\gamma}f(z)}-1}{\frac{z(\mathfrak{T}_z^{\mu,\gamma}f(z))'}{\mathfrak{T}_z^{\mu,\gamma}f(z)}+(1-2\alpha)}\right|<\beta\right\},$

$(0<\gamma\le1,0<\mu\le1,0\le\mu-\gamma<1,\mu>\gamma,0\le\alpha<1,0<\beta\le1,z\in U)$;

$K^{\mu,\gamma}(\alpha;\beta,-\beta)=K^{\mu,\gamma}(\alpha,\beta)=\left\{f\in\mathcal{A}:\left|\dfrac{\frac{z(\mathfrak{T}_z^{\mu,\gamma}f(z))''}{(\mathfrak{T}_z^{\mu,\gamma}f(z))'}}{\frac{z(\mathfrak{T}_z^{\mu,\gamma}f(z))''}{(\mathfrak{T}_z^{\mu,\gamma}f(z))'}+2(1-\alpha)}\right|<\beta\right\},$

$(0<\gamma\le1,0<\mu\le1,0\le\mu-\gamma<1,\mu>\gamma,0\le\alpha<1,0<\beta\le1,z\in U)$;

(ii)

$$S^{\mu,\mu}(\alpha;\beta,-\beta)=S(\alpha,\beta)=\left\{f\in\mathcal{A}:\left|\frac{\frac{zf'(z)}{f(z)}-1}{\frac{zf'(z)}{f(z)}+(1-2\alpha)}\right|<\beta;0\le\alpha<1,0<\beta\le1,z\in U\right\};$$

$$K^{\mu,\mu}(\alpha;\beta,-\beta)=K(\alpha,\beta)=\left\{f\in\mathcal{A}:\left|\frac{\frac{zf''(z)}{f'(z)}}{\frac{zf''(z)}{f'(z)}+2(1-\alpha)}\right|<\beta;0\le\alpha<1,0<\beta\le1,z\in U\right\}$$

The subclasses $T(\alpha,\beta)$ and $C(\alpha,\beta)$ are the well-known subclasses of starlike and convex functions of order α and type

β, respectively, see [8].

(iii) $S^{\mu,\gamma}(\alpha;1,-1)=S^{\mu,\gamma}(\alpha)=\left\{f\in\mathcal{A}:\operatorname{Re}\left\{\frac{z\left(\mathfrak{T}_z^{\mu,\gamma}f(z)\right)'}{\mathfrak{T}_z^{\mu,\gamma}f(z)}\right\}>\alpha\right\}$ and

$$K^{\mu,\gamma}(\alpha;1,-1)=K^{\mu,\gamma}(\alpha)=\left\{f\in\mathcal{A}:\operatorname{Re}\left\{1+\frac{z\left(\mathfrak{T}_z^{\mu,\gamma}f(z)\right)''}{\left(\mathfrak{T}_z^{\mu,\gamma}f(z)\right)'}\right\}>\alpha\right\};$$

where $0<\gamma\le1, 0<\mu\le1, 0\le\mu-\gamma<1, \mu>\gamma, 0\le\alpha<1, z\in U$.

(iv) $S^{\mu,\mu}(\alpha;1,-1)=S^*(\alpha)$ *and* $K^{\mu,\mu}(\alpha;1,-1)=K(\alpha)$.

For the functions subclasses $\mathcal{S}^{\mu,\gamma}(\alpha;A,B)$ and $\mathcal{K}^{\mu,\gamma}(\alpha;A,B)$ we begin with deducing two sufficient coefficients bounds, and the consequences are some subordination properties and the partial sums inequalities.

In the sequel, it is assumed that $1\le B<A\le1$, $0\le\alpha<1$, $0<\gamma\le1$, $0<\mu\le1$, $0\le\mu-\gamma<1$, $\mu>\gamma$ and $z\in U$.

2. Coefficients estimates

Within this section we deduce sufficient conditions on the coefficients in (1.1) sothat the function f belonging to the subclasses $\mathcal{S}^{\mu,\gamma}(\alpha;A,B)$ and $\mathcal{K}^{\mu,\gamma}(\alpha;A,B)$ respectively, in the following two theorems.

Theorem 2.1. *If the function f be given by (1.1). Then $f\in\mathcal{S}^{\mu,\gamma}(\alpha;A,B)$, if*

$$\sum_{k=2}^{\infty}\left[(1-B)(k-1)+(A-B)(1-\alpha)\right]\frac{\Gamma(\mu+k)}{\Gamma(\gamma+k)}\left|a_k\right|\le(A-B)(1-\alpha)\frac{\Gamma(\mu+1)}{\Gamma(\gamma+1)}. \tag{2.1}$$

Proof. It suffices to show that

$$\left|\frac{\frac{z(\mathfrak{T}_z^{\mu,\gamma}f(z))'}{\mathfrak{T}_z^{\mu,\gamma}f(z)}-1}{B\frac{z(\mathfrak{T}_z^{\mu,\gamma}f(z))'}{\mathfrak{T}_z^{\mu,\gamma}f(z)}-((A-B)(1-\alpha)+B)}\right|<1.$$

We have

$$\begin{aligned}
&\left|\frac{\frac{z(\mathfrak{T}_z^{\mu,\gamma}f(z))'}{\mathfrak{T}_z^{\mu,\gamma}f(z)}-1}{B\frac{z(\mathfrak{T}_z^{\mu,\gamma}f(z))'}{\mathfrak{T}_z^{\mu,\gamma}f(z)}-[B+(A-B)(1-\alpha)]}\right|\\
&=\left|\frac{z\mathfrak{T}_z^{\mu,\gamma}f'(z)-\mathfrak{T}_z^{\mu,\gamma}f(z)}{Bz\mathfrak{T}_z^{\mu,\gamma}f'(z)-[B+(A-B)(1-\alpha)]\,\mathfrak{T}_z^{\mu,\gamma}f(z)}\right|\\
&=\left|\frac{\frac{\Gamma(\gamma+1)}{\Gamma(\mu+1)}\sum_{k=2}^{\infty}(k-1)\frac{\Gamma(\mu+k)}{\Gamma(\gamma+k)}a_kz^k}{(A-B)(1-\alpha)z-\frac{\Gamma(\gamma+1)}{\Gamma(\mu+1)}\sum_{k=2}^{\infty}[-Bk+B+(A+B)(1-\alpha)]\frac{\Gamma(\mu+k)}{\Gamma(\gamma+k)}a_kz^k}\right|\\
&\le\frac{\frac{\Gamma(\gamma+1)}{\Gamma(\mu+1)}\sum_{k=2}^{\infty}(k-1)\frac{\Gamma(\mu+k)}{\Gamma(\gamma+k)}\left|a_k\right|\left|z\right|^k}{(A-B)(1-\alpha)\left|z\right|-\frac{\Gamma(\gamma+1)}{\Gamma(\mu+1)}\sum_{k=2}^{\infty}[-Bk+B+(A+B)(1-\alpha)]\frac{\Gamma(\mu+k)}{\Gamma(\gamma+k)}\left|a_k\right|\left|z\right|^k}\\
&\le\frac{\frac{\Gamma(\gamma+1)}{\Gamma(\mu+1)}\sum_{k=2}^{\infty}(k-1)\frac{\Gamma(\mu+k)}{\Gamma(\gamma+k)}\left|a_k\right|}{(A-B)(1-\alpha)-\frac{\Gamma(\gamma+1)}{\Gamma(\mu+1)}\sum_{k=2}^{\infty}[-Bk+B+(A+B)(1-\alpha)]\frac{\Gamma(\mu+k)}{\Gamma(\gamma+k)}\left|a_k\right|}
\end{aligned}$$

This last expression is b ounded ab ove by 1 if

$$\frac{\Gamma(\gamma+1)}{\Gamma(\mu+1)}\sum_{k=2}^{\infty}(k-1)\frac{\Gamma(\mu+k)}{\Gamma(\gamma+k)}\left|a_k\right| \le (A-B)(1-\alpha)$$
$$-\frac{\Gamma(\gamma+1)}{\Gamma(\mu+1)}\sum_{k=2}^{\infty}[-Bk+B+(A+B)(1-\alpha)]\frac{\Gamma(\mu+k)}{\Gamma(\gamma+k)}\left|a_k\right|,$$

which coincides with inequality (2.1), and the proof is completed.

Further, it is easily by using (1.16) in Theorem 1.1, we obtain the coefficient estimates theorem of the subclass $\mathcal{K}^{\mu,\gamma}(\alpha; A, B)$ and the proof is omitted.

Theorem 2.2. *If the function f be given by (1.1), then $f \in \mathcal{K}^{\mu,\gamma}(\alpha; A, B)$, if*

$$\sum_{k=2}^{\infty}[(1-B)(k-1)+(A-B)(1-\alpha)]\frac{\Gamma(\mu+k)}{\Gamma(\gamma+k)}k\left|a_k\right| \le (A-B)(1-\alpha)\frac{\Gamma(\mu+1)}{\Gamma(\gamma+1)}. \tag{2.5}$$

3. Partial sums

In this section we will study the ratio of a function of the form (1.1) to its sequence of partial sums defined by $f_1(z) = z$ and $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$ when the coefficients of $f(z)$ are sufficiently small to satisfy the condition (1.9). We will determine sharp lower bounds for $\operatorname{Re}\left(\frac{f(z)}{f_n(z)}\right), \operatorname{Re}\left(\frac{f_n(z)}{f(z)}\right), \operatorname{Re}\left(\frac{f'(z)}{f_n'(z)}\right)$ and $\operatorname{Re}\left(\frac{f_n'(z)}{f'(z)}\right)$.

In what follows, we will use the well known result

$$\operatorname{Re}\left(\frac{1-w(z)}{1+w(z)}\right) > 0 \qquad (z \in U),$$

if and only if$w(z) = \sum\limits_{k=1}^{\infty} D_k z^k$, satisfies the inequality $|w(z)| \le |z| < 1$.

Theorem 3.1. *Define the partial sums $f_1(z)$ and $f_n(z)$ by*

$$f_1(z) = z \text{ and } f_n(z) = z + \sum_{k=2}^{n} a_k z^k \quad (n \in \mathbb{N}\backslash\{1\}).$$

Let the function $f \in \mathcal{S}^{\mu,\gamma}(\alpha; A, B)$ be given by (1.1) and satisfies the condition (2.1) and

$$c_k \ge \begin{cases} 1, & k = 2, 3, ..., n, \\ c_{n+1}, & k = n+1, n+2, ..., \end{cases} \tag{3.1}$$

where, for convenience,

$$c_k = \frac{[(1-B)(k-1)+(A-B)(1-\alpha)]\Gamma(\gamma+1)\Gamma(\mu+k)}{(A-B)(1-\alpha)\Gamma(\mu+1)\Gamma(\gamma+k)}. \tag{3.2}$$

Then

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} > 1 - \frac{1}{c_{n+1}} \ (z \in U; \ n \in \mathbb{N}), \tag{3.3}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} > \frac{c_{n+1}}{1+c_{n+1}}. \tag{3.4}$$

Proof. For the coefficients c_k given by (3.2) it is not difficult to verify that

$$c_{k+1} > c_k > 1. \tag{3.5}$$

Therefore, we have

$$\sum_{k=2}^{n}\left|a_k\right| + c_{n+1}\sum_{k=n+1}^{\infty}\left|a_k\right| \le \sum_{k=2}^{\infty}c_k\left|a_k\right| \le 1. \tag{3.6}$$

By setting

$$g_1(z) = c_{n+1}\left\{\frac{f(z)}{f_n(z)} - \left(1-\frac{1}{c_{n+1}}\right)\right\}$$

$$= 1 + \frac{c_{n+1}\sum\limits_{k=n+1}^{\infty} a_k z^{k-1}}{1+\sum\limits_{k=2}^{n} a_k z^{k-1}}, \tag{3.7}$$

and applying (3.3), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| < \frac{c_{n+1}\sum\limits_{k=n+1}^{\infty}|a_k|}{2-2\sum\limits_{k=2}^{n}|a_k| - c_{n+1}\sum\limits_{k=n+1}^{\infty}|a_k|}. \tag{3.8}$$

Now,

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| < 1,$$

if

$$\sum_{k=2}^{n}|a_k| + c_{n+1}\sum_{k=n+1}^{\infty}|a_k| \le 1.$$

From the condition (2.1), it is sufficient to show that

$$\sum_{k=2}^{n}|a_k| + c_{n+1}\sum_{k=n+1}^{\infty}|a_k| \le \sum_{k=2}^{\infty} c_k |a_k|,$$

which is equivalent to

$$\sum_{k=2}^{n}(c_k-1)|a_k| + \sum_{k=n+1}^{\infty}(c_k-c_{n+1})|a_k| \ge 0, \tag{3.9}$$

which readily yields the assertion (3.3) of Theorem 3.1. In order to see that

$$f(z) = z + \frac{z^{n+1}}{c_{n+1}}, \tag{3.10}$$

gives sharp result, we observe that for $z = re^{\frac{i\pi}{n}}$ that $\frac{f(z)}{f_n(z)} = 1 + \frac{z^n}{c_{n+1}} \to 1 - \frac{1}{c_{n+1}}$ as $z \to 1^-$. Similarly, if we take

$$g_2(z) = (1+c_{n+1})\left\{\frac{f_n(z)}{f(z)} - \frac{c_{n+1}}{1+c_{n+1}}\right\}$$

$$= 1 - \frac{(1+c_{n+1})\sum\limits_{k=n+1}^{\infty} a_k z^{k-1}}{1+\sum\limits_{k=2}^{\infty} a_k z^{k-1}}, \tag{3.11}$$

and making use of (3.6), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| < \frac{(1+c_{n+1})\sum\limits_{k=n+1}^{\infty}|a_k|}{2-2\sum\limits_{k=2}^{n}|a_k| - (1-c_{n+1})\sum\limits_{k=n+1}^{\infty}|a_k|}, \tag{3.12}$$

which leads us immediately to the assertion (3.4) of Theorem 3.1.

The bound in (3.4) is sharp for each $n \in \mathbb{N}$ with the extremal function f given by (3.10). Then the proof of Theorem 3.1. is completed.

Theorem 3.2. *Let $f \in \mathcal{S}^{\mu,\gamma}(\alpha; A, B)$, then*

$$\operatorname{Re}\left(\frac{f'(z)}{f_n'(z)}\right) \geq 1 - \frac{n+1}{c_{n+1}} \qquad (z \in U, n \in \mathbb{N}), \tag{3.13}$$

and

$$\operatorname{Re}\left(\frac{f_n'(z)}{f'(z)}\right) \geq \frac{c_{n+1}}{n+1+c_{n+1}} \qquad (z \in U, n \in \mathbb{N}), \tag{3.14}$$

where c_k is given by (3.2). The estimates in (3.13) and (3.14) are sharp with the extremal function f is as defined in (3.10).

The corresponding partial sums results of the subclass $\mathcal{K}^{\mu,\gamma}(\alpha; A, B)$ is can be obtained using the same technique and is omitted.

4. Subordination results

In this section, we derive subordination results for the subclasses $\mathcal{S}^{\mu,\gamma}(\alpha; A, B)$ and $\mathcal{K}^{\mu,\gamma}(\alpha; A, B)$.

Definition 4.1 (Subordination Factor Sequence). *A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever f of the form (1.1) is analytic, univalent and convex in U, we have the subordination given by*

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec f(z) \quad (z \in U;\ a_1 = 1). \tag{4.1}$$

To prove our main result we need the following lemma due to Wilf [23].

Lemma 4.1. *The sequence $\{c_k\}_{k=1}^{\infty}$ is a subordinating factor sequence, if and only if*

$$\operatorname{Re}\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0 \quad (z \in U). \tag{4.2}$$

Let $\mathcal{S}^{*\mu,\gamma}(\alpha; A, B)$ denotes the subclass of $\mathcal{A}$ whose coefficients satisfies the condition (2.1). We note that $\mathcal{S}^{*\mu,\gamma}(\alpha; A, B) \subseteq \mathcal{S}^{\mu,\gamma}(\alpha; A, B)$.

Employing the technique used earlier by Attiya [3] and Srivastava and Attiya [17], we introduce the following theorem:

Theorem 4.1. *Let f of the form (1.1) and let $f \in \mathcal{S}^{*\mu,\gamma}(\alpha; A, B)$. Then*

$$\frac{[(1-B)+(A-B)(1-\alpha)](\mu+1)}{2([(1-B)+(A-B)(1-\alpha)](\mu+1)+(1-\alpha)(A-B)(\gamma+1))} (f * h)(z) \prec h(z) \ (z \in U), \tag{4.3}$$

for every function h in $\mathcal{K}$, and

$$\operatorname{Re}\{f(z)\} > -\frac{[(1-B)+(A-B)(1-\alpha)](\mu+1)+(1-\alpha)(A-B)(\gamma+1)}{[(1-B)+(A-B)(1-\alpha)](\mu+1)} \ (z \in U). \tag{4.4}$$

The constant factor $\frac{[(1-B)+(A-B)(1-\alpha)](\mu+1)}{2([(1-B)+(A-B)(1-\alpha)](\mu+1)+(1-\alpha)(A-B)(\gamma+1))}$ in the subordination result (4.3) cannot be replaced by a larger one.

Proof. Let $f \in \mathcal{S}^{*\mu,\gamma}(\alpha; A, B)$ and let a function $h \in \mathcal{K}$, such that $h(z) = z + \sum_{k=2}^{\infty} c_k z^k$. Then we have

$$\frac{[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(\gamma+1)\Gamma(\mu+2)}{\Gamma(\gamma+2)\Gamma(\mu+1)}}{2\left[[(1-B)+(A-B)(1-\alpha)]\frac{\Gamma(\gamma+1)\Gamma(\mu+2)}{\Gamma(\gamma+2)\Gamma(\mu+1)} + (1-\alpha)(A-B)\right]} (f * h)(z)$$

$$= \frac{[(1-B)+(A-B)(1-\alpha)](\mu+1)}{2\left[[(1-B)+(A-B)(1-\alpha)](\mu+1)+(1-\alpha)(\gamma+1)(A-B)\right]} (f * h)(z)$$

$$= \frac{[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)}{2\,[[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)+(1-\alpha)(\gamma+1)\,(A-B)]}\left(z+\sum_{k=2}^{\infty}a_kc_kz^k\right). \tag{4.5}$$

Thus, by Definition 4.1, the subordination result (4.3) will hold true if the sequence

$$\left\{\frac{[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)}{2\,([(1-B)+(A-B)(1-\alpha)]\,(\mu+1)+(1-\alpha)\,(A-B)\,(\gamma+1))}a_k\right\}_{k=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 4.1, this is equivalent to the following inequality:

$$\mathrm{Re}\left\{1+\sum_{k=1}^{\infty}\frac{[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)}{[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)+(1-\alpha)\,(A-B)\,(\gamma+1)}a_kz^k\right\}>0\ (z\in U). \tag{4.6}$$

Now, since

$$\Psi(k)=[(1-B)(k-1)+(A-B)(1-\alpha)]\,\frac{\Gamma(\gamma+1)\Gamma(\mu+k)}{\Gamma(\mu+1)\Gamma(\gamma+k)}$$

is an increasing function of k $(k\geq 2)$, we have

$$\begin{aligned}
&\mathrm{Re}\left\{1+\sum_{k=1}^{\infty}\frac{[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)}{[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)+(1-\alpha)\,(A-B)\,(\gamma+1)}a_kz^k\right\}\\
&=\mathrm{Re}\left\{1+\frac{[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)}{[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)+(1-\alpha)\,(A-B)\,(\gamma+1)}z\right.\\
&\left.+\tfrac{1}{[(1-B)+(A-B)(1-\alpha)](\mu+1)+(1-\alpha)(A-B)(\gamma+1)}\sum_{k=2}^{\infty}[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)a_kz^k\right\}\\
&\geq 1-\frac{[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)}{[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)+(1-\alpha)\,(A-B)\,(\gamma+1)}r\\
&-\tfrac{1}{[(1-B)+(A-B)(1-\alpha)]\frac{(\mu+1)}{(\gamma+1)}+(1-\alpha)(A-B)}\sum_{k=2}^{\infty}[(1-B)(k-1)+(A-B)(1-\alpha)]\,\tfrac{\Gamma(\gamma+1)\Gamma(\mu+k)}{\Gamma(\mu+1)\Gamma(\gamma+k)}\,|a_k|\,r^k\\
&>1-\frac{[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)}{[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)+(1-\alpha)\,(A-B)\,(\gamma+1)}r\\
&-\frac{(1-\alpha)\,(A-B)\,(\gamma+1)}{[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)+(1-\alpha)\,(A-B)\,(\gamma+1)}r\\
&=1-r>0\quad(|z|=r<1),
\end{aligned}$$

where we have also made use of assertion (4.2) of Lemma 4.1. Thus (4.6) holds true in U, this proves the inequality (4.3). The inequality (4.4) follows from (4.3) by taking the convex function $h(z)=\dfrac{z}{1-z}=z+\displaystyle\sum_{k=2}^{\infty}z^k$.

To prove the sharpness of the constant $\frac{[(1-B)+(A-B)(1-\alpha)](\mu+1)}{2([[(1-B)+(A-B)(1-\alpha)](\mu+1)+(A-B)(1-\alpha)(\gamma+1))}$, we consider the function $f_0\in\mathcal{S}^{*\mu,\gamma}(\alpha;A,B)$ given by

$$f_0(z)=z+\frac{(A-B)(1-\alpha)(\gamma+1)}{[(1-B)+(A-B)(1-\alpha)]\,(\mu+1)}z^2. \tag{4.7}$$

Thus from (4.3), we have

$$\tfrac{[(1-B)+(A-B)(1-\alpha)](\mu+1)}{2([[(1-B)+(A-B)(1-\alpha)](\mu+1)+(A-B)(1-\alpha)(\gamma+1))}f_0(z)\prec\frac{z}{1-z}\ (z\in U). \tag{4.8}$$

Moreover, it can easily be verified for the function f_0 given by (4.7) that

$$\min_{|z|\leq r}\left\{\mathrm{Re}\,\tfrac{[(1-B)+(A-B)(1-\alpha)](\mu+1)}{2([[(1-B)+(A-B)(1-\alpha)](\mu+1)+(A-B)(1-\alpha)(\gamma+1))}f_0(z)\right\}=-\frac{1}{2}. \tag{4.9}$$

This shows that the constant $\frac{[(1-B)+(A-B)(1-\alpha)](\mu+1)}{2([[(1-B)+(A-B)(1-\alpha)](\mu+1)+(A-B)(1-\alpha)(\gamma+1))}$ is the best possible. This completes the proof of Theorem 4.1.

By using the same technique as used above in Theorem 4.1 and considering the subclass $\mathcal{K}^{*\mu,\gamma}(\alpha;A,B)\subset\mathcal{K}^{\mu,\gamma}(\alpha;A,B)$ satisfying condition (2.5), we give the following theorem and the proof is omitted.

Theorem 4.2. *Let f of the form (1.1) and let $f \in \mathcal{K}^{*\mu,\gamma}(\alpha; A, B)$. Then*

$$\frac{[(1-B)+(A-B)(1-\alpha)](\mu+1)}{2[(1-B)+(A-B)(1-\alpha)](\mu+1)+(A-B)(1-\alpha)(\gamma+1)}\left(f*h\right)(z) \prec h(z)\ (z\in U), \tag{4.10}$$

for every function h in $\mathcal{K}$, and

$$\operatorname{Re}\{f(z)\} > -\frac{2\left[(1-B)+(A-B)(1-\alpha)\right](\mu+1)+(A-B)\left(1-\alpha\right)(\gamma+1)}{2\left[(1-B)+(A-B)(1-\alpha)\right](\mu+1)}\ (z\in U). \tag{4.11}$$

The constant factor $\frac{[(1-B)+(A-B)(1-\alpha)](\mu+1)}{2[(1-B)+(A-B)(1-\alpha)](\mu+1)+(A-B)(1-\alpha)(\gamma+1)}$ in the subordination result (4.10) cannot be replaced by a larger one.

Remark 4.1.

(i) Taking $\mu=\gamma$ and $A=\beta\,(0<\beta\le 1)$, and $B=-\beta\,(0<\beta\le 1)$ in Theorem 4.1, we get the result obtained by Frasin [6, Corollary 2.2];

(ii) Taking $\mu=\gamma$ and $A=1$, and $B=-1$ in Theorem 4.1, we get the result obtained by Srivastava and Eker [18, Corollary 1] and Frasin [6, Corollary 2.3];

(iii) Taking $\mu=\gamma$, $\alpha=0$ and $A=1$, and $B=-1$ in Theorem 4.1, we get the result obtained by Frasin [6, Corollary 2.4];

(iv) Taking $\mu=\gamma$ and $A=\beta\,(0<\beta\le 1)$, and $B=-\beta\,(0<\beta\le 1)$ in Theorem 4.2, we get the result obtained by Frasin [6, Corollary 2.5];

(v) Taking $\mu=\gamma$ and $A=1$, and $B=-1$ in Theorem 4.2, we get the result obtained by Srivastava and Eker [18, Corollary 2] and Frasin [6, Corollary 2.6];

(vi) Taking $\mu=\gamma$, $\alpha=0$ and $A=1$, and $B=-1$ in Theorem 4.2, we get the result obtained by Frasin [6, Corollary 2.7];

(vii) Taking $\mu=\gamma$ and $\alpha=0$ in Theorem 4.1, we get the result obtained by Aouf et al. [2, Corollary 4];

(viii) Taking $\mu=\gamma$ and $\alpha=0$ in Theorem 4.2, we get the result obtained by Aouf et al. [2, Corollary 5].

Also, we establish subordination results for the associated subclasses $\mathcal{S}^{*\mu,\gamma}(\alpha,\beta)$, $\mathcal{K}^{*\mu,\gamma}(\alpha,\beta)$, $\mathcal{S}^{*\mu,\gamma}(\alpha)$ and $\mathcal{K}^{*\mu,\gamma}(\alpha)$, whose coefficients satisfy the inequality (2.1) and (2.5) in the special cases as mentioned.

Putting $A=\beta\,(0<\beta\le 1)$, and $B=-\beta\,(0<\beta\le 1)$ in Theorem 4.1, we get the following corollary.

Corollary 4.1. *Let the function f defined by (1.1) be in the class $\mathcal{S}^{*\mu,\gamma}(\alpha,\beta)$ and suppose that $h\in\mathcal{K}$. Then*

$$\frac{[1+\beta+2\beta(1-\alpha)](\mu+1)}{2[[1+\beta+2\beta(1-\alpha)](\mu+1)+2\beta(1-\alpha)(\gamma+1)]}\left(f*h\right)(z) \prec h(z)\ (z\in U), \tag{4.12}$$

for every function h in $\mathcal{K}$, and

$$\operatorname{Re}\{f(z)\} > -\frac{\left[1+\beta+2\beta(1-\alpha)\right](\mu+1)+2\beta(1-\alpha)(\gamma+1)}{\left[1+\beta+2\beta(1-\alpha)\right](\mu+1)}\ (z\in U). \tag{4.13}$$

The constant factor $\frac{[1+\beta+2\beta(1-\alpha)](\mu+1)}{2[[1+\beta+2\beta(1-\alpha)](\mu+1)+2\beta(1-\alpha)(\gamma+1)]}$ in the subordination result (4.12) cannot be replaced by a larger one.

Putting $A=\beta\,(0<\beta\le 1)$, and $B=-\beta\,(0<\beta\le 1)$ in Theorem 4.2, we get the following corollary.

Corollary 4.2. *Let the function f defined by (1.1) be in the class $\mathcal{K}^{*\mu,\gamma}(\alpha,\beta)$ and suppose that $h\in\mathcal{K}$. Then*

$$\frac{[1+\beta+2\beta(1-\alpha)](\mu+1)}{2[1+\beta+2\beta(1-\alpha)](\mu+1)+2\beta(1-\alpha)(\gamma+1)}\left(f*h\right)(z) \prec h(z)\ (z\in U), \tag{4.14}$$

for every function h in $\mathcal{K}$, and

$$\operatorname{Re}\{f(z)\} > -\frac{[1+\beta+2\beta(1-\alpha)](\mu+1)+\beta(1-\alpha)(\gamma+1)}{[1+\beta+2\beta(1-\alpha)](\mu+1)}\ (z\in U). \tag{4.15}$$

The constant factor $\frac{[1+\beta+2\beta(1-\alpha)](\mu+1)}{2[1+\beta+2\beta(1-\alpha)](\mu+1)+2\beta(1-\alpha)(\gamma+1)}$ in the subordination result (4.14) cannot be replaced by a larger one.

Putting $A=1$, and $B=-1$ in Theorem 4.1, we get the following corollary.

Corollary 4.3. *Let the function f defined by (1.1) be in the class $\mathcal{S}^{*\mu,\gamma}(\alpha)$ and suppose that $h \in \mathcal{K}$. Then*

$$\tfrac{(2-\alpha)(\mu+1)}{2[(2-\alpha)(\mu+1)+(1-\alpha)(\gamma+1)]}\,(f*h)(z) \prec h(z) \ (z \in U), \tag{4.16}$$

for every function h in $\mathcal{K}$, and

$$\operatorname{Re}\{f(z)\} > -\frac{(2-\alpha)(\mu+1)+(1-\alpha)(\gamma+1)}{(2-\alpha)(\mu+1)} \ (z \in U). \tag{4.17}$$

The constant factor $\frac{(2-\alpha)(\mu+1)}{2[(2-\alpha)(\mu+1)+(1-\alpha)(\gamma+1)]}$ in the subordination result (4.16) cannot be replaced by a larger one.

Putting $A = 1$, and $B = -1$ in Theorem 4.2, we get the following corollary.

Corollary 4.4. *Let the function f defined by (1.1) be in the class $\mathcal{K}^{*\mu,\gamma}(\alpha)$ and suppose that $h \in \mathcal{K}$. Then*

$$\frac{(2-\alpha)(\mu+1)}{2(2-\alpha)(\mu+1)+(1-\alpha)(\gamma+1)}\,(f*h)(z) \prec h(z) \ (z \in U), \tag{4.18}$$

for every function h in $\mathcal{K}$, and

$$\operatorname{Re}\{f(z)\} > -\frac{2[2-\alpha](\mu+1)+(1-\alpha)(\gamma+1)}{2[2-\alpha](\mu+1)} \ (z \in U). \tag{4.19}$$

The constant factor $\frac{(2-\alpha)(\mu+1)}{2(2-\alpha)(\mu+1)+(1-\alpha)(\gamma+1)}$ in the subordination result (4.18) cannot be replaced by a larger one.

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