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Subclasses of κ–uniformly starlike and κ-uniformly convex functions associated with Pascal distribution series

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Abstract

The purpose of the present paper is to find the necessary and sufficient conditions and inclusion relations for Pascal distribution series to be in some subclasses of analytic univalent functions of κ–starlike and κ–uniformly convex functions of order ϑ in the open unit disk $\mathbb{U}$. Further, we consider an integral operator related to Pascal distribution series, and several corollaries and consequences of the main results are also considered.

Keywords: Univalent, Starlike, Convex, Uniformly starlike functions, Uniformly convex functions, Hadamard product, Pascal distribution series
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1 Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(\zeta)=\zeta+\sum_{n=2}^{\infty}a_n\zeta^n,\ \zeta\in\mathbb{U}, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}:=\{\zeta\in\mathbb{C}:|\zeta|<1\}$ and normalized by the conditions $f(0)=0=f'(0)-1$. Also, denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(\zeta)=\zeta-\sum_{n=2}^{\infty}a_n\zeta^n,\ \zeta\in\mathbb{U},\ \text{with } a_n\geq 0, \tag{1.2}$$

introduced and studied extensively by Silverman [27].

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For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n$, we recall that the well-known *Hadamard (or convolution) product* of f and g is given by

$$(f * g)(\zeta) := \zeta + \sum_{n=2}^{\infty} a_n b_n \zeta^n, \ \zeta \in \mathbb{U}.$$

A function $f \in \mathcal{A}$ is said to be *starlike of order* ϑ $(0 \le \vartheta < 1)$, if

$$\operatorname{Re}\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right) > \vartheta, \ \zeta \in \mathbb{U}.$$

This function class is denoted by $\mathcal{S}^*(\vartheta)$, and we also write $\mathcal{S}^* \equiv \mathcal{S}^*(0)$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ such that $f(\mathbb{U})$ is a starlike domain with respect to the origin.

A function $f \in \mathcal{A}$ is said to be *convex of order* ϑ $(0 \le \vartheta < 1)$, if

$$\operatorname{Re}\left(1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right) > \vartheta, \ \zeta \in \mathbb{U}.$$

This class is denoted by $\mathcal{K}(\vartheta)$, and $\mathcal{K} \equiv \mathcal{K}(0)$ represents the well-known standard class of convex functions. It is an established fact that

$$f \in \mathcal{K}(\vartheta) \Leftrightarrow \zeta f'(\zeta) \in \mathcal{S}^*(\vartheta).$$

Definition 1.1. A function f is *uniformly convex (starlike)* in $\mathbb{U}$ if f is in the class $\mathcal{CV}(\mathcal{ST})$ and has the property that for every circular arc γ contained in $\mathbb{U}$, with center μ also in $\mathbb{U}$, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\mu)$. The class of uniformly convex functions is denoted by $\mathcal{UCV}$ and the class of uniformly starlike functions by $\mathcal{UST}$.

Goodman [15, 16] gave the following analytic characterization of these classes:

Theorem 1.2. A function f of the form (1.1) is in the class $\mathcal{UCV}$ if and only if

$$\operatorname{Re}\left(1 + (\zeta - \mu)\frac{f''(\zeta)}{f'(\zeta)}\right) \ge 0, \ (\zeta, \mu) \in \mathbb{U} \times \mathbb{U},$$

and is in the class $\mathcal{UST}$ if and only if

$$\operatorname{Re}\left(\frac{f(\zeta) - f(\mu)}{(\zeta - \mu) f'(\zeta)}\right) \ge 0, \ (\zeta, \mu) \in \mathbb{U} \times \mathbb{U}.$$

For a more detailed on uniformly convex and starlike functions, we refer the reader to [17, 18, 21, 9, 26, 12, 10].

We recall the following notions of subclasses of *κ–uniformly convex functions* and the corresponding subclass of *κ–uniformly starlike functions* due to [26].

For $-1 < \vartheta \le 1$ and $\kappa \ge 0$ a function $f \in \mathcal{A}$ is said to be in the class of:

(i) *κ–uniformly starlike functions of order ϑ*, denoted by $\mathcal{S}_P(\vartheta, \kappa)$, if it satisfies the condition

$$\operatorname{Re}\left(\frac{\zeta f'(\zeta)}{f(\zeta)} - \vartheta\right) > \kappa \left|\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right|, \ \zeta \in \mathbb{U}, \tag{1.3}$$

and

(ii) *κ–uniformly convex functions of order ϑ*, denoted by $\mathcal{UCV}(\vartheta, \kappa)$, if it satisfies the condition

$$\operatorname{Re}\left(1 + \frac{\zeta f''(\zeta)}{f'(\zeta)} - \vartheta\right) > \kappa \left|\frac{\zeta f''(\zeta)}{f'(\zeta)}\right|, \ \zeta \in \mathbb{U}. \tag{1.4}$$

From (1.3) and (1.4) it follows that

$$f \in \mathcal{UCV}(\vartheta, \kappa) \Leftrightarrow \zeta f'(\zeta) \in \mathcal{S}_P(\vartheta, \kappa).$$

Definition 1.3. Let $0 \leq \lambda < 1$, $0 \leq \vartheta < 1$ and $\kappa \geq 0$.

(i) We denote by $\mathcal{P}_\lambda(\vartheta,\kappa)$ the subclass of $\mathcal{A}$ consisting of functions f of the form (1.1) and satisfying the inequality

$$\operatorname{Re}\left(\frac{\zeta f'(\zeta)+\lambda\zeta^2 f''(\zeta)}{(1-\lambda)f(\zeta)+\lambda\zeta f'(\zeta)}-\vartheta\right)>\kappa\left|\frac{\zeta f'(\zeta)+\lambda\zeta^2 f''(\zeta)}{(1-\lambda)f(\zeta)+\lambda\zeta f'(\zeta)}-1\right|,\ \zeta\in\mathbb{U}.$$

(ii) Also, let $\mathcal{Q}_\lambda(\vartheta,\kappa)$ be the subclass of $\mathcal{A}$ satisfying the analytic criteria

$$\operatorname{Re}\left(\frac{\lambda\zeta^2 f'''(\zeta)+(1+2\lambda)\zeta f''(\zeta)+f'(\zeta)}{f'(\zeta)+\lambda\zeta f''(\zeta)}-\vartheta\right)>\kappa\left|\frac{\lambda\zeta^2 f'''(\zeta)+(1+\lambda)\zeta f''(\zeta)}{f'(\zeta)+\lambda\zeta f''(\zeta)}\right|,\ \zeta\in\mathbb{U}.$$

Also, let denote by $\mathcal{P}^*_\lambda(\vartheta,\kappa):=\mathcal{P}_\lambda(\vartheta,\kappa)\cap\mathcal{T}$ and $\mathcal{Q}^*_\lambda(\vartheta,\kappa):=\mathcal{Q}_\lambda(\vartheta,\kappa)\cap\mathcal{T}$, and we mention that the classes $\mathcal{P}^*_\lambda(\vartheta,\kappa)$ and $\mathcal{Q}^*_\lambda(\vartheta,\kappa)$ were studied in [20].

Remark 1.4. (i) Choosing $\lambda=0$ we get the class $\mathcal{UCT}(\vartheta,\kappa)\equiv\mathcal{Q}^*_0(\vartheta,\kappa)$, and for $\lambda=1$ we get the class $\mathcal{M}(\vartheta,\kappa)\equiv\mathcal{Q}^*_1(\vartheta,\kappa)$ studied in [20].

(ii) Suitably specializing the parameters one can define various subclasses like

$$\mathcal{TS}_P(\vartheta,\kappa)\equiv\mathcal{P}^*_0(\vartheta,\kappa)\quad\text{and}\quad\mathcal{UCT}(\vartheta,\kappa)\equiv\mathcal{P}^*_1(\vartheta,\kappa),$$

studied in [3].

(iii) Further, by taking $\kappa=0$ we note that the classes $\mathcal{T}^*(\vartheta)\equiv\mathcal{TS}_P(\vartheta,0)$ and $\mathcal{C}(\vartheta)\equiv\mathcal{UCT}(\vartheta,0)$ have been studied in [27].

(iv) Also, by taking $\vartheta=0$ we get the classes

$$\mathcal{TS}_P(\kappa)\equiv\mathcal{P}^*_0(0,\kappa)\quad\text{and}\quad\mathcal{UCT}(\kappa)\equiv\mathcal{P}^*_1(0,\kappa),$$

studied in [30].

A variable x is said to be *Pascal distribution* if it takes the values $0,1,2,3,\ldots$ with probabilities

$$(1-q)^m,\ \frac{qm(1-q)^m}{1!},\ \frac{q^2m(m+1)(1-q)^m}{2!},\ \frac{q^3m(m+1)(m+2)(1-q)^m}{3!},\ \ldots$$

respectively, where q and m are called the parameters, and thus

$$P(x=k)=\binom{k+m-1}{m-1}q^k(1-q)^m, k=0,1,2,3,\ldots.$$

Very recently, El-Deeb et al. [7] (see also [14, 5]) introduced a power series whose coefficients are probabilities of Pascal distribution, that is

$$\Psi^m_q(\zeta):=\zeta+\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\zeta^n,\ \zeta\in\mathbb{U},$$

where $m\geq 1$, $0\leq q\leq 1$, and we note that, by ratio test the radius of convergence of above series is infinity. We also define the series

$$\Phi^m_q(\zeta):=2\zeta-\Psi^m_q(\zeta)=\zeta-\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\zeta^n,\ \zeta\in\mathbb{U}. \tag{1.5}$$

Let consider the linear operator $\mathcal{I}^m_q:\mathcal{A}\to\mathcal{A}$ defined by the convolution or Hadamard product

$$\mathcal{I}^m_q f(\zeta):=\Psi^m_q(\zeta)*f(\zeta)=\zeta+\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m a_n\zeta^n,\ \zeta\in\mathbb{U},$$

where $m\geq 1$ and $0\leq q\leq 1$.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions, by using *hypergeometric functions* (see for example, [6, 13, 19, 28, 29]) and by the recent investigations (see, for example [1, 4, 2, 11, 20, 22, 24, 23, 25]), in the present paper we determine the necessary and sufficient conditions for Φ_q^m to be in our classes $\mathcal{P}_\lambda^*(\vartheta,\kappa)$ and $\mathcal{Q}_\lambda^*(\vartheta,\kappa)$. We give connections of these subclasses with $\mathcal{R}^\tau(A,B)$, and finally, we give sufficient conditions for the function f such that its image by the integral operator $\mathcal{G}_q^m f(\zeta)=\int\limits_0^\zeta \frac{\mathcal{I}_q^m f(t)}{t}dt$ belongs to the above classes.

To establish our main results, we need the following lemmas.

Lemma 1.5. [20, Theorem 1.3] A function f of the form (1.2) is in the class $\mathcal{P}_\lambda^*(\vartheta,\kappa)$ if and only if

$$\sum_{n=2}^{\infty}(1+n\lambda-\lambda)\left[n(1+\kappa)-(\vartheta+\kappa)\right]|a_n|\leq 1-\vartheta. \tag{1.6}$$

Lemma 1.6. [20, Theorem 1.4] A function f of the form (1.2) is in the class $\mathcal{Q}_\lambda^*(\vartheta,\kappa)$ if and only if

$$\sum_{n=2}^{\infty}n(1+n\lambda-\lambda)\left[n(1+\kappa)-(\vartheta+\kappa)\right]|a_n|\leq 1-\vartheta.$$

2 Necessary and sufficient conditions for $\Phi_q^m\in\mathcal{P}_\lambda^*(\vartheta,\kappa)$ and $\Phi_q^m\in\mathcal{Q}_\lambda^*(\vartheta,\kappa)$

For convenience throughout in the sequel, we use the following identities that hold at least for $m\geq 1$ and $0\leq q<1$:

$$\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n=\frac{1}{(1-q)^m},\quad \sum_{n=0}^{\infty}\binom{n+m-2}{m-2}q^n=\frac{1}{(1-q)^{m-1}},$$
$$\sum_{n=0}^{\infty}\binom{n+m}{m}q^n=\frac{1}{(1-q)^{m+1}},\quad \sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n=\frac{1}{(1-q)^{m+2}},$$
$$\sum_{n=0}^{\infty}\binom{n+m+2}{m+2}q^n=\frac{1}{(1-q)^{m+3}}.$$

By simple calculations we derive the following relations:

$$\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}=\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n-1=\frac{1}{(1-q)^m}-1,$$
$$\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}=qm\sum_{n=0}^{\infty}\binom{n+m}{m}q^n=\frac{qm}{(1-q)^{m+1}},$$
$$\sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1}=q^2m(m+1)\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n=\frac{q^2m(m+1)}{(1-q)^{m+2}}.$$

and

$$\begin{aligned}\sum_{n=4}^{\infty}(n-1)(n-2)(n-3)\binom{n+m-2}{m-1}q^{n-1}&=q^3m(m+1)(m+2)\sum_{n=0}^{\infty}\binom{n+m+2}{m+2}q^n\\&=\frac{q^3m(m+1)(m+2)}{(1-q)^{m+3}}.\end{aligned}$$

Unless otherwise mentioned, we shall assume in this paper that $0\leq\lambda<1$, $0\leq\vartheta<1$, $\kappa\geq 0$ and $0\leq q<1$.

In the first two results we obtain the necessary and sufficient conditions for Φ_q^m to be in the classes $\mathcal{P}_\lambda^*(\vartheta,\kappa)$ and $\mathcal{Q}_\lambda^*(\vartheta,\kappa)$, respectively.

Theorem 2.1. Let $m\geq 1$. Then $\Phi_q^m\in\mathcal{P}_\lambda^*(\vartheta,\kappa)$, if and only if

$$\lambda(1+\kappa)\frac{q^2m(m+1)}{(1-q)^{m+2}}+\left[(2\lambda+1)(1+\kappa)-\lambda(\vartheta+\kappa)\right]\frac{qm}{(1-q)^{m+1}}\leq 1-\vartheta. \tag{2.1}$$

Proof . Since Φ_q^m is defined by (1.5), in view of Lemma 1.5 it is sufficient to show that

$$\sum_{n=2}^{\infty}(1+n\lambda-\lambda)\left[n(1+\kappa)-(\vartheta+\kappa)\right]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \leq 1-\vartheta. \tag{2.2}$$

Writing in left hand side of (2.2)

$$\begin{aligned} n &= (n-1)+1, \\ n^2 &= (n-1)(n-2)+3(n-1)+1, \end{aligned}$$

we get

$$\begin{aligned} &\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[n(1+\kappa)-(\vartheta+\kappa)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\ = \ & \sum_{n=2}^{\infty}\left\{n^2\lambda(1+\kappa)+n\left[(1-\lambda)(1+\kappa)-\lambda(\vartheta+\kappa)\right]-(1-\lambda)(\vartheta+\kappa)\right\}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\ = \ & \lambda(1+\kappa)\sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\ & +\left[(2\lambda+1)(1+\kappa)-\lambda(\vartheta+\kappa)\right]\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\ & +(1-\vartheta)\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\ = \ & \lambda(1+\kappa)\frac{q^2m(m+1)}{(1-q)^2}+\left[(2\lambda+1)(1+\kappa)-\lambda(\vartheta+\kappa)\right]\frac{qm}{1-q}+(1-\vartheta)\left(1-(1-q)^m\right), \end{aligned}$$

but this last expression is upper bounded by $1-\vartheta$ if and only if (2.1) holds. □

Theorem 2.2. Let $m \geq 1$. Then $\Phi_q^m \in \mathcal{Q}_\lambda^*(\vartheta,\kappa)$, if and only if

$$\begin{aligned} \lambda(1+\kappa)\frac{q^3m(m+1)(m+2)}{(1-q)^{m+3}}&+\left[(5\lambda+1)(1+\kappa)-\lambda(\vartheta+\kappa)\right]\frac{q^2m(m+1)}{(1-q)^{m+2}} \\ &+\left[(1+\kappa)(4\lambda+3)-(2\lambda+1)(\vartheta+\kappa)\right]\frac{qm}{(1-q)^{m+1}} \leq 1-\vartheta. \end{aligned} \tag{2.3}$$

Proof . In view of Lemma 1.6 we must to show that

$$\sum_{n=2}^{\infty}n(1+n\lambda-\lambda)\left[n(1+\kappa)-(\vartheta+\kappa)\right]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \leq 1-\vartheta. \tag{2.4}$$

Substituting

$$\begin{aligned} n^3 &= (n-1)(n-2)(n-3)+6(n-1)(n-2)+7(n-1)+1, \\ n^2 &= (n-1)(n-2)+3(n-1)+1, \\ n &= (n-1)+1, \end{aligned}$$

in (2.4), we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(1+n\lambda-\lambda)[n(1+\kappa)-(\vartheta+\kappa)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\\
&=\lambda(1+\kappa)\sum_{n=4}^{\infty}(n-1)(n-2)(n-3)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\\
&+[(5\lambda+1)(1+\kappa)-\lambda(\vartheta+\kappa)]\sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\\
&+[(1+\kappa)(4\lambda+3)-(2\lambda+1)(\vartheta+\kappa)]\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\\
&+(1-\vartheta)\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\\
&=\lambda(1+\kappa)\frac{q^3m(m+1)(m+2)}{(1-q)^3}+[(5\lambda+1)(1+\kappa)-\lambda(\vartheta+\kappa)]\frac{q^2m(m+1)}{(1-q)^2}\\
&+[(1+\kappa)(4\lambda+3)-(2\lambda+1)(\vartheta+\kappa)]\frac{qm}{(1-q)}+(1-\vartheta)\left(1-(1-q)^m\right),
\end{aligned}$$

therefore, the last expression is upper bounded by $1-\vartheta$ if the inequality (2.3) is satisfied. □

3 Sufficient conditions for $\mathcal{I}_q^m\left(\mathcal{R}^\tau(A,B)\right)\subset\mathcal{P}_\lambda^*(\vartheta,\kappa)$ and $\mathcal{I}_q^m\left(\mathcal{R}^\tau(A,B)\right)\subset\mathcal{Q}_\lambda^*(\vartheta,\kappa)$

In [8] Dixit and Pad introduced the following subclass of $\mathcal{A}$:

Definition 3.1. [8] A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(A,B)$, with $\tau\in\mathbb{C}\setminus\{0\}$ and $-1\le B<A\le 1$, if it satisfies the inequality

$$\left|\frac{f'(\zeta)-1}{(A-B)\tau-B\left[f'(\zeta)-1\right]}\right|<1,\ \zeta\in\mathbb{U}.$$

Also, they proved the next sharp estimations regarding the coefficients of the power expansions for the functions belonging to this class, as follows:

Lemma 3.2. [8] If $f\in\mathcal{R}^\tau(A,B)$ is of the form (1.1), then

$$|a_n|\le(A-B)\frac{|\tau|}{n},\ n\in\mathbb{N}\setminus\{1\},$$

and the result is sharp.

Making use of Lemma 3.2, we will study the action of the Pascal distribution series on the class $\mathcal{P}_\lambda^*(\vartheta,\kappa)$.

Theorem 3.3. If $f\in\mathcal{R}^\tau(A,B)$, then $\mathcal{I}_q^m f\in\mathcal{P}_\lambda^*(\vartheta,\kappa)$ if

$$\begin{aligned}
&(A-B)|\tau|\left\{\frac{qm\lambda(1+\kappa)}{1-q}+[(1+\kappa)-\lambda(\vartheta+\kappa)]\left(1-(1-q)^m\right)\right.\\
&\left.-\frac{(1-\lambda)(\vartheta+\kappa)}{q(m-1)}\left[(1-q)-(1-q)^m-q(m-1)(1-q)^m\right]\right\}\le 1-\vartheta. \qquad (3.1)
\end{aligned}$$

Proof . According to Lemma 1.5 it is sufficient to show that

$$\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[n(1+\kappa)-(\vartheta+\kappa)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\,|a_n|\le 1-\vartheta.$$

Since $f \in \mathcal{R}^\tau(A,B)$, using Lemma 3.2 we have

$$|a_n| \leq \frac{(A-B)\,|\tau|}{n},\ n \in \mathbb{N}\setminus\{1\},$$

therefore

$$\begin{aligned}
&\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[n(1+\kappa)-(\vartheta+\kappa)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\,|a_n| \\
\leq&(A-B)\,|\tau|\left\{\sum_{n=2}^{\infty}\frac{1}{n}\left[n^2\lambda(1+\kappa)+n\left[(1-\lambda)(1+\kappa)-\lambda(\vartheta+\kappa)\right]-(1-\lambda)(\vartheta+\kappa)\right]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\right\} \\
=&(A-B)\,|\tau|\left\{\sum_{n=2}^{\infty}\left[n\lambda(1+\kappa)+[(1-\lambda)(1+\kappa)-\lambda(\vartheta+\kappa)]-\frac{1}{n}(1-\lambda)(\vartheta+\kappa)\right]\right. \\
&\left.\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\right\} \\
=&(A-B)\,|\tau|\,(1-q)^m\left[\lambda(1+\kappa)\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}\right. \\
&\left.+[(1+\kappa)-\lambda(\vartheta+\kappa)]\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}-(1-\lambda)(\vartheta+\kappa)\sum_{n=2}^{\infty}\frac{1}{n}\binom{n+m-2}{m-1}q^{n-1}\right] \\
=&(A-B)\,|\tau|\,(1-q)^m\left\{\frac{qm\lambda(1+\kappa)}{(1-q)^{m+1}}+[(1+\kappa)-\lambda(\vartheta+\kappa)]\left(\frac{1}{(1-q)^m}-1\right)\right. \\
=&-\left.\frac{(1-\lambda)(\vartheta+\kappa)}{q(m-1)}\left[\sum_{n=0}^{\infty}\binom{n+m-2}{m-2}q^n-1-(m-1)q\right]\right\} \\
=&(A-B)|\tau|\left\{\frac{qm\lambda(1+\kappa)}{1-q}+[(1+\kappa)-\lambda(\vartheta+\kappa)]\left(1-(1-q)^m\right)\right. \\
&\left.-\frac{(1-\lambda)(\vartheta+\kappa)}{q(m-1)}\left[(1-q)-(1-q)^m-q(m-1)(1-q)^m\right]\right\}.
\end{aligned}$$

But this last expression is upper bounded by $1-\vartheta$ if (3.1) holds, which completes our proof. □

Applying Lemma 1.6 and using the same technique as in the proof of Theorem 3.3, we have the following result:

Theorem 3.4. If $f \in \mathcal{R}^\tau(A,B)$, then $\mathcal{I}_q^m f \in \mathcal{Q}_\lambda^*(\vartheta,\kappa)$ if

$$\begin{aligned}
(A-B)\,|\tau|\,\Bigg[\lambda(1+\kappa)\frac{q^2m(m+1)}{(1-q)^2}&+[(2\lambda+1)(1+\kappa)-\lambda(\vartheta+\kappa)]\,\frac{qm}{1-q} \\
&+(1-\vartheta)\left(1-(1-q)^m\right)\Bigg]\leq 1-\vartheta. \qquad (3.2)
\end{aligned}$$

Proof . According to Lemma 1.6 it is sufficient to show that

$$\sum_{n=2}^{\infty}n(1+n\lambda-\lambda)[n(1+\kappa)-(\vartheta+\kappa)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\,|a_n| \leq 1-\vartheta. \qquad (3.3)$$

Since $f \in \mathcal{R}^\tau(A,B)$, using Lemma 3.2 we have

$$|a_n| \leq \frac{(A-B)\,|\tau|}{n},\ n \in \mathbb{N}\setminus\{1\},$$

hence

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(1+n\lambda-\lambda)\left[n(1+\kappa)-(\vartheta+\kappa)\right]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\left|a_n\right| \\
&\leq \sum_{n=2}^{\infty} n(1+n\lambda-\lambda)\left[n(1+\kappa)-(\vartheta+\kappa)\right]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\frac{(A-B)\left|\tau\right|}{n} \\
&= (A-B)\left|\tau\right|\sum_{n=2}^{\infty}(1+n\lambda-\lambda)\left[n(1+\kappa)-(\vartheta+\kappa)\right]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m.
\end{aligned}$$

Using similar computations like in the proof of in Theorem 2.1 it follows that the inequality (3.3) is satisfied whenever (3.2) holds. □

4 Properties of a special function

Theorem 4.1. If the function $\mathcal{G}_q^m$ is given by

$$\mathcal{G}_q^m(\zeta) := \int_0^{\zeta}\frac{\Phi_q^m(t)}{t}dt,\ \zeta\in\mathbb{U}, \tag{4.1}$$

then $\mathcal{G}_q^m \in \mathcal{Q}_\lambda^*(\vartheta,\kappa)$, if and only if (2.1) holds.

Proof . According to (1.5) it follows that

$$\mathcal{G}_q^m(\zeta) = \zeta - \sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\frac{\zeta^n}{n},\ \zeta\in\mathbb{U},$$

and using Lemma 1.6, by a similar proof like those of Theorem 2.1 we get that $\mathcal{G}_q^m f \in \mathcal{Q}_\lambda^*(\vartheta,\kappa)$ if and only if (2.1) holds. □

Theorem 4.2. Let $m>1$. Then the function $\mathcal{G}_q^m$ given by (4.1) is in the class $\mathcal{P}_\lambda^*(\vartheta,\kappa)$ if and only if

$$\begin{aligned}
&\frac{qm\lambda(1+\kappa)}{1-q} + \left[(1+\kappa)-\lambda(\vartheta+\kappa)\right]\left(1-(1-q)^m\right) \\
&-\frac{(1-\lambda)(\vartheta+\kappa)}{q(m-1)}\left[(1-q)-(1-q)^m-q(m-1)(1-q)^m\right] \leq 1-\vartheta.
\end{aligned} \tag{4.2}$$

Proof . According to Lemma 1.5, the function

$$\mathcal{G}_q^m(\zeta) = \zeta - \sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\frac{\zeta^n}{n}$$

belongs to $\mathcal{P}_\lambda^*(\vartheta,\kappa)$ if and only if the condition (1.6) is satisfied, where the coefficient a_n are

$$a_n := -\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\frac{1}{n},\ n\geq 2.$$

Using similar computations like in the proof of Theorem 3.3 we get

$$\begin{aligned}
&\sum_{n=2}^{\infty}(1+n\lambda-\lambda)\left[n(1+\kappa)-(\vartheta+\kappa)\right]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\frac{1}{n} \\
&\leq \sum_{n=2}^{\infty}\frac{1}{n}\left[n^2\lambda(1+\kappa)+n\left[(1-\lambda)(1+\kappa)-\lambda(\vartheta+\kappa)\right]-(1-\lambda)(\vartheta+\kappa)\right]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&= \sum_{n=2}^{\infty}\left[n\lambda(1+\kappa)+\left[(1-\lambda)(1+\kappa)-\lambda(\vartheta+\kappa)\right]-\frac{1}{n}(1-\lambda)(\vartheta+\kappa)\right]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&= (1-q)^m\left\{\lambda(1+\kappa)\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}\right. \\
&\left.+\left[(1+\kappa)-\lambda(\vartheta+\kappa)\right]\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}-(1-\lambda)(\vartheta+\kappa)\sum_{n=2}^{\infty}\frac{1}{n}\binom{n+m-2}{m-1}q^{n-1}\right\} \\
&= (1-q)^m\left\{\frac{qm\lambda(1+\kappa)}{(1-q)^{m+1}}+\left[(1+\kappa)-\lambda(\vartheta+\kappa)\right]\left(\frac{1}{(1-q)^m}-1\right)\right. \\
&\qquad\qquad\left.-\frac{(1-\lambda)(\vartheta+\kappa)}{q(m-1)}\left[\sum_{n=0}^{\infty}\binom{n+m-2}{m-2}q^n-1-(m-1)q\right]\right\} \\
&= \frac{qm\lambda(1+\kappa)}{1-q}+\left[(1+\kappa)-\lambda(\vartheta+\kappa)\right]\left(1-(1-q)^m\right) \\
&\qquad\qquad-\frac{(1-\lambda)(\vartheta+\kappa)}{q(m-1)}\left[(1-q)-(1-q)^m-q(m-1)(1-q)^m\right].
\end{aligned}$$

It follows that (1.6) is satisfied if and only if the assumption (4.2) holds, which proves our result. □

5 Corollaries and consequences

By specializing the parameter $\lambda = 0$ in Theorem 2.1, Theorem 3.3, and Theorem 4.2 we obtain the following special cases, respectively.

Corollary 5.1. Let $m \geq 1$. Then $\Phi_q^m \in \mathcal{TS}_P(\vartheta,\kappa)$, if and only if

$$(1+\kappa)\frac{qm}{(1-q)^{m+1}} \leq 1-\vartheta. \tag{5.1}$$

Corollary 5.2. If $f \in \mathcal{R}^\tau(A,B)$, then $\mathcal{I}_q^m f \in \mathcal{TS}_P(\vartheta,\kappa)$ if

$$(A-B)|\tau|\left\{(1+\kappa)\left(1-(1-q)^m\right)-\frac{\vartheta+\kappa}{q(m-1)}\left[(1-q)-(1-q)^m-q(m-1)(1-q)^m\right]\right\} \leq 1-\vartheta.$$

Corollary 5.3. Let $m > 1$. Then the function $\mathcal{G}_q^m$ given by (4.1) is in the class $\mathcal{TS}_P(\vartheta,\kappa)$ if and only if

$$(1+\kappa)\left(1-(1-q)^m\right)-\frac{\vartheta+\kappa}{q(m-1)}\left[(1-q)-(1-q)^m-q(m-1)(1-q)^m\right] \leq 1-\vartheta.$$

Putting the parameter $\lambda = 0$ in Theorem 2.2, Theorem 3.4, and Theorem 4.1 we obtain the following corollaries, respectively.

Corollary 5.4. Let $m \geq 1$. Then $\Phi_q^m \in \mathcal{UCT}(\vartheta,\kappa)$, if and only if

$$(1+\kappa)\frac{q^2m(m+1)}{(1-q)^{m+2}}+\left[3(1+\kappa)-(\vartheta+\kappa)\right]\frac{qm}{(1-q)^{m+1}} \leq 1-\vartheta.$$

Corollary 5.5. If $f \in \mathcal{R}^{\tau}(A,B)$, then $\mathcal{I}_q^m f \in \mathcal{UCT}(\vartheta,\kappa)$ if

$$(A-B)|\tau|\left[(1+\kappa)\frac{qm}{1-q}+(1-\vartheta)\left(1-(1-q)^m\right)\right]\leq 1-\vartheta.$$

Corollary 5.6. Let $m \geq 1$. Then the function $\mathcal{G}_q^m$ given by (4.1) in the class $\mathcal{UCT}(\vartheta,\kappa)$ if and only if (5.1) holds.

Concluding Remarks. Specializing the parameter $\lambda=0$ and $\lambda=1$ we can state various interesting inclusion results (as proved in above theorems) for the subclasses studied in many other paper [3, 30, 31].

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