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On ordered theoretic controlled fuzzy metric spaces

Khalil Javed[a], Fahim Ud Din[b], Umar Ishtiaq[c,*], Choonkil Park[d,*], Muhammad Arshad[a]

[a] *Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad 44000, Pakistan*

[b] *Abdus Salam School of Mathematical Sciences, Government College University, Lahore, Pakistan*

[c] *Office of Research, Innovation and Commercialization, University of Management and Technology Lahore, Pakistan*

[d] *Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea*

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Abstract

In this article, we introduce the concept of $\Re$-control fuzzy metric spaces. We prove some fixed point results in the sense of $\Re$-control fuzzy metric spaces and furnish our work with several non-trivial examples to verify the validity of the proposed results. In the end, we incorporate this work with an application to solve an integral equation.

Keywords: ordered theoretic set; fuzzy metric space; control fuzzy metric space;integral equation
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1 Introduction

Since the inception of Banach contraction principle, many authors studied fixed point theory vividly and enriched this field with different ideas. This classical result was generalized in different spaces and different structures were attained using this topic. In this connectedness, Bakhtin [3] and Czerwik [5] gave a generalization of metric space, and named it as the b-metric space, with a triangle inequality weaker than that of metric spaces. On the other hand, Zadeh [23] introduced the concept of fuzzy sets and generalized the notion of metric space using fuzzy sets and named the new space as the fuzzy metric space. Since then, these results have become the center of interest for many researchers and in this sequel Roldan et al. [18] introduced interrelationships between fuzzy metric structures. Gupta et al. [10] presented the graphical interpretation of fuzzy metric space. Chauhan et al. [4] initiated Banach contraction theorem for fuzzy cone-metric space. Altun et al. [1] introduced the notion of Ordered non-Archidemedean fuzzy metric spaces and utilized this concept to investigate some fixed point results. Recently, Deng [6] generalized the notion of fuzzy metric and introduced fuzzy b-metric space. Many authors used fuzzy b-metric space in different formats, to investigate fixed point results, we refer the readers to please check [7, 14, 17, 20, 8, 22, 9]. Later, Mehmood [13] introduced the concept of extended fuzzy b-metric spaces and generalized fuzzy b-metric spaces. Recently, Mlaiki [16] introduced the concept of controlled metric type spaces and Sezen [21] used this concept of controlled metric spaces and introduced the notion of controlled fuzzy metric spaces, which is the generalization of fuzzy b-metric spaces and extended fuzzy b-metric spaces.

*Corresponding author
Email addresses: khaliljaved15@gmail.com (Khalil Javed), fahim@sms.edu.pk (Fahim Ud Din), umarishtiaq000@gmail.com (Umar Ishtiaq), baak@hanyang.ac.kr (Choonkil Park), marshadzia@iiu.edu.pk (Muhammad Arshad)

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Most recently, Khalehoghli et al. [12] introduced the notion of an $\Re$-set and gave a real generalization of Banach's fixed point results. Many authors used this idea to help them work on different structures. Lately, Ali et al. extended this concept to investigate fixed point in partial metric spaces. Baghani et al. [2] initiated fixed point theorem for set-valued mappings in $\Re$-complete metric spaces. Javed et al. [11] utilized this concept to find fixed point results in fuzzy b-metric space. Sawangsup [19] used this idea to discuss fixed point results for JS-quasi contraction of multi-dimensional mapping with the transitivity, and many more. For more information and results in fuzzy metric spaces and applications, see [1, 4, 10, 15, 18].

In this article, we generalize and extend the concept of control fuzzy metric spaces and introduce the concept of $\Re$-control fuzzy metric spaces. To validate our results, we impart our work with an example and towards the end, we present an application to solve integral equation.

2 Preliminaries

First, we recall some fundamental definitions related to this article.

Definition 2.1. [6] A 4-tuple $(X, \Delta, *, u)$ is called fuzzy b-metric space if X is an arbitrary (nonempty) set, $*$ is a continuous t-norm and Δ is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions for all $x, w, z \in X$ and $t, s > 0$, and a given real number $u \geq 1$,

B1) $\Delta(x, w, t) > 0$;

B2) $\Delta(x, w, t) = 1$ if and only if $x = w$;

B3) $\Delta(x, w, t) = \Delta(w, x, t)$;

B4) $\Delta(x, z, u(t + s)) \geq \Delta(x, w, t) * \Delta(w, z, s)$;

B5) $\Delta(x, w, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Definition 2.2. [13] A 4-tuple $(X, \Delta_\alpha, *, \alpha)$ is called an extended fuzzy b-metric space if X is a (nonempty) set, $\alpha : X \times X \to [1, \infty)$, $*$ is a continuous t-norm and Δ_α is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions for all $x, w, z \in X$ and $t, s > 0$,

Δ1) $\Delta_\alpha(x, w, 0) = 0$;

Δ2) $\Delta_\alpha(x, w, t) = 1$ if and only if $x = w$;

Δ3) $\Delta_\alpha(x, w, t) = \Delta_\alpha(w, x, t)$;

Δ4) $\Delta_\alpha(x, z, \alpha(x, z)(t + s)) \geq \Delta_\alpha(x, w, t) * \Delta_\alpha(w, z, s)$;

Δ5) $\Delta_\alpha(x, w, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Definition 2.3. [21] A triple $(X, \Delta_\gamma, *)$ is called a control fuzzy metric space if X is a (nonempty) set, $\gamma : X \times X \to [1, \infty)$, $*$ is a continuous t-norm and Δ_γ is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions for all $x, w, z \in X$ and $t, s > 0$,

1) $\Delta_\gamma(x, w, 0) = 0$;

2) $\Delta_\gamma(x, w, t) = 1$ if and only if $x = w$;

3) $\Delta_\gamma(x, w, t) = \Delta_\gamma(w, x, t)$;

4) $\Delta_\gamma(x, z, t + s) \geq \Delta_\gamma(x, w, \frac{t}{\gamma(x,w)}) * \Delta_\gamma(w, z, \frac{s}{\gamma(w,z)})$;

5) $\Delta_\gamma(x, w, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Definition 2.4. [21] Let X be a set and let $\zeta : X \to X$ and $O(\nu) = \{\nu_0, \zeta\nu_0, \zeta^2\nu_0, \ldots\}$ (for some $\nu_0 \in X$) be the orbit of ζ. A function $T : X \to X$ is said to be ζ-orbitally lower semi continuous at $u \in X$ if for $\nu_n \in O(\nu_0)$ such that $\nu_n \to u$, we get $T(u) \geq \lim_{n\to\infty} supT(\nu_n)$.

Example 2.5. [2] Let $\Re$ be a binary relation on X and define the binary relation $\Re$ such that $x\Re y$ if and only if $x\Re y$ or $y\Re x$.Then $\Re$ is a binary relation on X.

Example 2.6. [11] Let $X = [0, \infty)$ and define $x\Re w$ if $xw = \min\{x, w\}$. Take $x_0 = 0$. Then $(X, \Re)$ is an $\Re$-set.

Definition 2.7. [2] Suppose that $(X, \Re)$ is an $\Re$-set. A sequence $\{x_n\}$ is said to be an $\Re$-sequence if $(\forall n; x_n \Re x_{n+1})$ or $(\forall n; x_{n+1} \Re x_n)$.

Definition 2.8. [12] (a) A metric space (X, d) is called an $\Re$-metric space if $(X, \Re)$ is an $\Re$-set.

(b) A mapping $T : X \to X$ is called $\Re$-continuous at $x \in X$ if for each $\Re$-sequence $\{x_n\}$ in X, $\lim_{n\to\infty} d(x_n, x) = 0$, $\lim_{n\to\infty} d(Tx_n, Tx) = 0$. Furthermore, T is called $\Re$-continuous on X if T is $\Re$-continuous at each $x \in X$.

(c) A mapping $T : X \to X$ is called $\Re$-preserving if $x\Re w$, then $Tx\Re Tw$ for all $x, w \in X$.

(d) An $\Re$-sequence $\{x_n\}$ in X is said to be an $\Re$-Cauchy sequence if for every $\epsilon > 0$ there exists an integer n such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq \mathbb{N}$. It is clear that $x_n \Re x_m$ or $x_m \Re x_n$.

(e) X is called $\Re$-complete if every $\Re$-Cauchy sequence is convergent.

3 Main Results

In this section we introduce the notion of $\Re$-control fuzzy metric space and prove some fixed point results.

Definition 3.1. A 4-tuple $(X, \theta, *, \Re)$ is called an $\Re$-control fuzzy metric space if for a nonempty set X, a reflexive binary relation $\Re$ on X, a mapping $\gamma : X \times X \to [1, \infty)$, continuous t-norm $*$ and a fuzzy set θ on $X \times X \times (0, \infty)$, the following conditions hold for all $t, s > 0$ and for all $x, w, z \in X$ with either ($x\Re z$ or $z\Re x$) or ($x\Re w$ or $w\Re x$) or ($w\Re z$ or $z\Re w$),

(θ_1) $\theta(x, w, t) > 0$;

(θ_2) $\theta(x, w, t) = 1$ if and only if $x = w$;

(θ_3) $\theta(x, w, t) = \theta(w, x, t)$;

(θ_4) $\theta(x, z, t + s) \geq \theta(x, w, \frac{t}{\gamma(x,w)}) * \theta(w, z, \frac{s}{\gamma(w,z)})$;

(θ_5) $\theta(x, w, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Example 3.2. Let $X = \mathbb{Z} = A \cup B$, where $A = \{-1, -2, -3, \ldots\}$ and $B = \{0, 1, 2, 3, \ldots\}$. Define a binary relation $\Re$ by $x\Re w \iff x + w \geq 0$. Define $\theta : X \times X \times (0, \infty) \to [0, 1]$ as

$$\theta(x, w, t) = \begin{cases} 1 \text{ if } x = w, \\ \frac{t}{t+\max\{x,w\}} \text{ otherwise.} \end{cases}$$

For all $t > 0$ and $x, w \in X$ with continuous t-norm $*$ defined as $t_1 * t_2 = t_1 \cdot t_2$, define $\gamma : X \times X \to [1, \infty)$ as

$$\gamma(x, w) = \begin{cases} 1, & x, w \in A \text{ or } x = 0 \text{ or } w = 0 \\ \max\{x, w\}, & \text{otherwise.} \end{cases}$$

It can be easily seen that $(X, \theta, *, \Re)$ is an $\Re$-control fuzzy metric space but it is not a control fuzzy metric space.

First, we will check that $(X, \theta, *, \Re)$ is an $\Re$-control fuzzy metric space.

It is clear that (θ_1), (θ_2) and (θ_5) hold.

(θ_3) $\theta(x, w, t) = \theta(w, x, t)$ for all $x, w \in X, t > 0$ such that ($x\Re z$ or $z\Re x$), since

$$\theta(x, w, t) = \frac{t}{t + \max\{x, w\}} = \frac{t}{t + \max\{w, x\}} = \theta(w, x, t).$$

(θ_4) $\theta(x, z, \gamma(x, w)\gamma(w, z)(t + s) \geq \theta(x, w, t) * \theta(w, z, s)$ for all $x, w, z \in X, t, s > 0$ with either ($x\Re z$ or $z\Re x$) or

$(x\Re w$ or $w\Re x)$ or $(w\Re z$ or $z\Re w)$, since

$$\begin{aligned}
& max\{x,z\} \leq \gamma(x,w)[\max\{x,w\}] + \gamma(w,z)[\max\{w,z\}] \\
\Longrightarrow\quad & ts\max\{x,z\} \leq \gamma(x,w)(ts+s^2)[\max\{x,w\}] + \gamma(w,z)(ts+t^2)\,[\max\{w,z\}] \\
\Longrightarrow\quad & ts\max\{x,z\} \leq \gamma(x,w)(t+s)s[\max\{x,w\}] + \gamma(w,z)(s+t)t[\max\{w,z\}] \\
\Longrightarrow\quad & ts\max\{x,z\} \leq \gamma(x,w)\gamma(w,z)(t+s)[\frac{s\max\{x,w\}}{\gamma(w,z)} + \frac{t\max\{w,z\}}{\gamma(x,w)}] \\
\Longrightarrow\quad & ts\max\{x,z\} \leq \gamma(x,w)\gamma(w,z)(t+s)[s\max\{x,w\} + t\max\{w,z\}] \\
\Longrightarrow\quad & ts\max\{x,z\} \leq \gamma(x,w)\gamma(w,z)(t+s)[s\max\{x,w\} + t\max\{w,z\} \\
& + \max\{x,w\}\max\{w,z\}] \\
\Longrightarrow\quad & \gamma(x,w)\gamma(w,z)(t+s)ts + ts\max\{x,z\} \leq \gamma(x,w)\gamma(w,z)(t+s)ts \\
& +\gamma(x,w)\gamma(w,z)(t+s)[s\max\{x,w\} + t\max\{w,z\} + \max\{x,w\}\max\{w,z\}] \\
\Longrightarrow\quad & \gamma(x,w)\gamma(w,z)(t+s)ts + ts\max\{x,z\} \leq \gamma(x,w)\gamma(w,z)(t+s) \\
& [ts + s\max\{x,w\} + t\max\{w,z\} + \max\{x,w\}\max\{w,z\}] \\
\Longrightarrow\quad & ts[\gamma(x,w)\gamma(w,z)(t+s) + \max\{x,z\}] \leq \gamma(x,w)\gamma(w,z)(t+s)[t+\max\{x,w\}] \\
& [s+\max\{w,z\}] \\
\Longrightarrow\quad & \frac{\gamma(x,w)\gamma(w,z)(t+s)}{\gamma(x,w)\gamma(w,z)(t+s)+\max\{x,z\}} \geq \frac{ts}{[t+\max\{x,w\}][s+\max\{w,z\}]} \\
\Longrightarrow\quad & \frac{\gamma(x,w)\gamma(w,z)(t+s)}{\gamma(x,w)\gamma(w,z)(t+s)+\max\{x,z\}} \geq \frac{t}{t+\max\{x,w\}}\cdot\frac{s}{s+\max\{w,z\}} \\
\Longrightarrow\quad & \theta(x,z,\gamma(x,w)\gamma(w,z)(t+s)) \geq \theta(x,w,t)*\theta(w,z,s).
\end{aligned}$$

Now, we show that X is not a control fuzzy metric space.

$$\begin{aligned}
\theta(x,z,\gamma(x,w)\gamma(w,z)(t+s)) &= \frac{\gamma(x,w)\gamma(w,z)(t+s)}{\gamma(x,w)\gamma(w,z)(t+s)+\max\{x,z\}}, \\
\theta(x,w,t) &= \frac{t}{t+\max\{x,w\}}, \\
\theta(w,z,s) &= \frac{s}{s+\max\{w,z\}}.
\end{aligned}$$

Thus

$$\frac{\gamma(x,w)\gamma(w,z)(t+s)}{\gamma(x,w)\gamma(w,z)(t+s)+\max\{x,z\}} \geq \frac{t}{t+\max\{x,w\}}\cdot\frac{s}{s+\max\{w,z\}}.$$

Now, let $x=w=z=-1$. Then $\gamma(x,w)=\gamma(w,z)=1$, $\max\{x,z\}=\max\{x,w\}=\max\{w,z\}=-1$. This implies

$$\frac{t+s}{t+s-1} \geq \frac{t}{t-1}\cdot\frac{s}{s-1} = \frac{ts}{(t-1)(s-1)}, t,s \neq 1.$$

Take $t=s=2$. This is a contradiction. Hence X is not a control fuzzy metric space.

Example 3.3. Let $X=\{-1,1,2,3,4,\ldots\}=A\cup B$, where $A=\{-1,1\}$ and $B=\mathbb{N}\setminus\{1\}$. Define a binary relation $\Re$ by $x\Re w \iff x+w\geq 0$. Define $\theta: X\times X\times[0,\infty)\to[0,1]$ as

$$\theta(x,w,t) = \begin{cases} 1, & \text{if } x=w \\ \frac{t+\frac{1}{x}}{t+\frac{1}{y}}, & \text{if } x\in B \text{ and } w\in A \\ \frac{t+\frac{1}{w}}{t+\frac{1}{x}}, & \text{if } x\in A \text{ and } w\in B \\ \frac{t+\frac{1}{\max\{x,w\}}}{t+\frac{1}{\min\{x,w\}}}, & \text{otherwise.} \end{cases}$$

With continuous t-norm $*$ defined as $t_1 * t_2 = t_1\cdot t_2$, define $\gamma: X\times X\to[1,\infty)$ as

$$\gamma(x,w) = \begin{cases} 1, & x,w\in A \text{ or } x=0 \text{ or } w=0 \\ \max\{x,w\}, & \text{otherwise.} \end{cases}$$

Then $(X, \theta, *, \Re)$ is an $\Re$-control fuzzy metric space but it is not a control fuzzy metric space. First, we show that $(X, \theta, *, \Re)$ is an $\Re$-control fuzzy metric space.

$(\theta_1), (\theta_2), (\theta_3)$ and (θ_5) are obvious.

(θ_4) We will show that $\theta(x, z, t+s) \geq \theta(x, w, \frac{t}{\gamma(x,w)}) * \theta(w, z, \frac{s}{\gamma(w,z)})$ for all $x, w, z \in X, t, s > 0$ with either ($x\Re z$ or $z\Re x$) or ($x\Re w$ or $w\Re x$) or ($w\Re z$ or $z\Re w$).

We have the following cases to prove (θ_4):

Case 1) If $z = x$, then $\theta(x, z, t+s) = 1$. Also

$$\theta(x, w, \frac{t}{\gamma(x,w)}) \leq 1 \text{ and } \theta(w, z, \frac{s}{\gamma(w,z)}) \leq 1.$$

This implies

$$\theta(x, w, \frac{t}{\gamma(x,w)}) * \theta(w, z, \frac{s}{\gamma(w,z)}) \leq 1.$$

Case 2) If $z = w$, then $\theta(w, z, \frac{s}{\gamma(w,z)}) = 1$ and clearly $\theta(x, z, t+s) \geq \theta(x, w, \frac{t}{\gamma(x,w)})$. This implies

$$\theta(x, z, t+s) \geq \theta(x, w, \frac{t}{\gamma(x,w)}) * \theta(w, z, \frac{s}{\gamma(w,z)}).$$

Case 3) If $z \neq x$, $z \neq w$ and $x = w$, then $\theta(x, w, \frac{t}{\gamma(x,w)}) = 1$ and clearly,

$$\theta(x, z, t+s) \geq \theta(w, z, \frac{s}{\gamma(w,z)}).$$

This implies

$$\theta(x, z, t+s) \geq \theta(x, w, \frac{t}{\gamma(x,w)}) * \theta(w, z, \frac{s}{\gamma(w,z)}).$$

Case 4) If $z \neq x$, $z \neq w$ and $x \neq w$, then we have the following cases:

1. $x, z \in A$ and $w \in B$;
2. $w \in A$ and $x, z \in B$;
3. $w, z \in A$ and $x \in B$;
4. $x, w \in A$ and $z \in B$;
5. $z \in A$ and $x, w \in B$;
6. $x \in A$ and $w, z \in B$;
7. $x, b, w \in A$;
8. $x, b, w \in B$.

Proof of (1): If $x, z \in A$ and $w \in B$, then

$$\theta(x, z, t+s) = \frac{t+s+\frac{1}{\max\{x,z\}}}{t+s+\frac{1}{\min\{x,z\}}}.$$

Observe that $\max\{x, z\} = \min\{x, z\} = 1$. This implies $\theta(x, z, t+s) = 1$. On the other hand,

$$\theta(x, w, \frac{t}{\gamma(x,w)}) = \frac{\frac{t}{\gamma(x,w)} + \frac{1}{w}}{\frac{t}{\gamma(x,w)} + \frac{1}{x}}.$$

Observe that $\gamma(x, w) = w$. Then

$$\theta(x, w, \frac{t}{\gamma(x,w)}) = \frac{xt+x}{xt+w} < 1, \quad \text{and} \quad \theta(w, z, \frac{s}{\gamma(w,z)}) = \frac{\frac{s}{\gamma(w,z)} + \frac{1}{w}}{\frac{s}{\gamma(w,z)} + \frac{1}{z}}.$$

Observe that $\gamma(w,z) = w$. Then

$$\theta(w,z,\frac{s}{\gamma(w,z)}) = \frac{zs+z}{zs+w} < 1.$$

This implies

$$\theta(x,z,t+s) \geq \theta(x,w,\frac{t}{\gamma(x,w)}) * \theta(w,z,\frac{s}{\gamma(w,z)}).$$

Similarly, for the other cases, we can prove it. Thus $(X,\theta,*,\Re)$ is an $\Re$-control fuzzy metric space. Now, we show that θ is not a control fuzzy metric space. Let $x,w,z \in A$ and also let $x = z = 1$, $w = -1$ and $t,s > 1$. Then $\theta(x,z,t+s) = 1$. On the other hand,

$$\theta(x,w,\frac{t}{\gamma(x,w)}) = \frac{t+1}{t-1}, (t \neq 1),$$

$$\theta(w,z,\frac{s}{\gamma(w,z)}) = \frac{s+1}{s-1}, (s \neq 1).$$

This implies

$$1 \geq \frac{t+1}{t-1} + \frac{s+1}{s-1}.$$

This contradicts to the condition (4) in Definition 2.3 and hence θ is not a control fuzzy metric space.

Remark 3.4. Every control fuzzy metric space is an $\Re$-control fuzzy metric space but the converse is not true.

Remark 3.5. Note that Example 3.2 also holds for t-norm $t_1 * t_2 = \min\{t_1, t_2\}$.

Definition 3.6. Let $(X,\theta,*,\Re)$ be an $\Re$-control fuzzy metric space. Then an $\Re$-sequence $\{x_n\}$ in X is said to be $\Re$-convergent to x if $\lim_{n\to\infty} \theta(x_n,x,t) = 1$ for all $t > 0$.

Definition 3.7. Let $(X,\theta,*,\Re)$ be an $\Re$-control fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be an $\Re$-Cauchy sequence if for each $\epsilon > 0$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\theta(x_n,x_m,t) > 1-\epsilon$ for all n, $m > n_0$.

Definition 3.8. Let $(X,\theta,*,\Re)$ be an $\Re$-control fuzzy metric space. If each $\Re$-Cauchy *sequence* is convergent, then X is called $\Re$-complete.

Definition 3.9. A mapping $T : X \to X$ is called $\Re$-continuous in an $\Re$-control fuzzy metric space $(X,\theta,*,\Re)$ if for each $\Re$-sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} \theta(x_n,x,t) = 1$ for all $t > 0$, $\lim_{n\to\infty} \theta(Tx_n,Tx,t) = 1$ for all $t > 0$. Furthermore, T is called $\Re$-continuous if T is $\Re$-continuous in each $x \in X$. Also, T is called $\Re$-preserving if $Tx\Re Tw$ whenever $x\Re w$.

Remark 3.10. It is necessary that the limit of a convergent sequence is unique in an $\Re$-control fuzzy metric space.

Remark 3.11. It is necessary that the convergent sequence is $\Re$-Cauchy sequence in an $\Re$-control fuzzy metric space.

Definition 3.12. Let $x,w \in [0,1]$ with $x\Re w$ and Ψ be the class of all mappings $\psi : [0,1] \to [0,1]$ such that ψ is $\Re$-continuous, nondecreasing and $\psi(\vartheta) > \vartheta$ for all $\vartheta \in (0,1)$.

If $\psi \in \Psi$, then $\psi(1) = 1$ and $\lim_{n\to\infty} \psi^n(\vartheta) = 1$ for all $\vartheta \in (0,1)$.

Theorem 3.13. Let $(X,\theta,*,\Re)$ be an $\Re$-complete control fuzzy metric space with $\gamma : X \times X \to [1,\infty)$ and suppose that

$$\lim_{t\to\infty} \theta(x,w,t) = 1 \tag{3.1}$$

for all $x \in X$. Assume that $T : X \to X$ is $\Re$-continuous, $\Re$-contractive and $\Re$-preserving and satisfies

$$\theta(Tx,Tw,kt) \geq \theta(x,w,t) \tag{3.2}$$

for all $x, w \in X$ with $x\Re w$, $t > 0$, where $k \in (0, 1)$. Also assume that for every $x \in X$,

$$\lim_{n\to\infty} \gamma(x_n, w) \quad \text{and} \quad \lim_{n\to\infty} \gamma(w, x_n)$$

are exist. Then T has a unique fixed point in X. Furthermore,

$$\lim_{n\to\infty} \theta(T^n u, u, t) = \theta(u, u, t) = 1$$

for all $u \in X$ and $t > 0$.

Proof . Since $(X, \theta, *, \Re)$ is an $\Re$-complete control fuzzy metric space, there exists $x_0 \in X$ such that $x_0\Re w$ for all $w \in X$. This implies $x_0\Re Tx_0$. Assume

$$x_1 = Tx_0, x_2 = T^2x_0 = Tx_1, \dots, x_n = T^n x_0 = Tx_{n-1}.$$

If $x_n = x_{n-1}$ then x_n is a fixed point of T. Suppose that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Since T is $\Re$-preserving, $\{x_n\}$ is an $\Re$-sequence and T is an $\Re$-contraction, we have

$$\begin{aligned} \theta(x_n, x_{n+1}, t) &= \theta(Tx_{n-1}, Tx_n, t) \geq \theta(x_{n-1}, x_n, \frac{t}{k}) \\ &\geq \dots \geq \theta(x_0, x_1, \frac{t}{k^{n-1}}). \end{aligned} \tag{3.3}$$

Now, from (θ_4), we have

$$\begin{aligned} \theta(x_n, x_{n+m}, t) &\geq \theta(x_n, x_{n+1}, \frac{t}{2\gamma(x_n, x_{n+1})}) * \theta(x_{n+1}, x_{n+m}, \frac{t}{2\gamma(x_{n+1}, x_{n+m})}) \\ &\geq \theta(x_n, x_{n+1}, \frac{t}{2\gamma(x_n, x_{n+1})}) * \theta(x_{n+1}, x_{n+2}, \frac{t}{(2)^2\gamma(x_{n+1}, x_{n+m})\gamma(x_{n+1}, x_{n+2})}) \\ &\quad *\theta(x_{n+2}, x_{n+m}, \frac{t}{(2)^2\gamma(x_{n+1}, x_{n+m})\gamma(x_{n+2}, x_{n+m})}) \\ &\geq \theta(x_n, x_{n+1}, \frac{t}{2\gamma(x_n, x_{n+1})}) * \theta(x_{n+1}, x_{n+2}, \frac{t}{(2)^2\gamma(x_{n+1}, x_{n+m})\gamma(x_{n+1}, x_{n+2})}) \\ &\quad *\theta(x_{n+2}, x_{n+3}, \frac{t}{(2)^3\gamma(x_{n+1}, x_{n+m})\gamma(x_{n+2}, x_{n+m})\gamma(x_{n+2}, x_{n+3})}) \\ &\quad *\theta(x_{n+3}, x_{n+m}, \frac{t}{(2)^3\gamma(x_{n+1}, x_{n+m})\gamma(x_{n+2}, x_{n+m})\gamma(x_{n+3}, x_{n+m})}) \\ &\geq \dots \geq \theta(x_n, x_{n+1}, \frac{t}{2\gamma(x_n, x_{n+1})}) \\ &\quad *[*_{i=n+1}^{n+m-2}\theta(x_i, x_{i+1}, \frac{t}{(2)^{m-2}(\prod_{j=n+1}^{i} (\gamma(x_j, x_{n+m})\gamma(x_i, x_{i+1}))})] \\ &\quad *[\theta(x_{n+m-1}, x_{n+m}, \frac{t}{(2)^{m-1}(\prod_{i=n+1}^{n+m-1} \gamma(x_i, x_{n+m}))})] \\ &\geq \theta(x_0, x_1, \frac{t}{2k^{n-1}\gamma(x_n, x_{n+1})}) \\ &\quad *[*_{i=n+1}^{n+m-2}\theta(x_0, x_1, \frac{t}{(2)^{m-1}k^{i-1}(\prod_{j=n+1}^{i} (\gamma(x_j, x_{n+m})\gamma(x_i, x_{i+1}))})] \\ &\quad *[\theta(x_0, x_1, \frac{t}{(2)^{m-1}k^{n+m-1}(\prod_{i=n+1}^{n+m-1} \gamma(x_i, x_{n+m}))})]. \end{aligned} \tag{3.4}$$

Now, taking the limit as $n \to \infty$ in (3.4) and (3.3) together with (3.1), we have

$$\lim_{n\to\infty} \theta(x_n, x_{n+m}, t) \geq 1 * 1 * \dots * 1 = 1$$

for all $t > 0$ and $n, m \in \mathbb{N}$. Thus $\{x_n\}$ is an $\Re$-Cauchy sequence in X. From the completeness of $(X, \theta, *, \Re)$, there exists $u \in X$ such that

$$\lim_{n\to\infty} \theta(x_n, u, t) = 1 \tag{3.5}$$

for all $t > 0$. Now, since T is an $\Re$-continuous mapping, $\theta(x_{n+1}, Tu, t) = \theta(Tx_n, Tu, t) \to 1$ as $n \to \infty$. For $t > 0$ and from (θ_4), we have

$$\begin{aligned} \theta(u, Tu, t) &\geq \theta(u, x_{n+1}, \frac{t}{2\gamma(u, x_{n+1})}) * \theta(x_{n+1}, Tu, \frac{t}{2\gamma(x_{n+1}, Tu)}) \\ &= \theta(u, x_{n+1}, \frac{t}{2\gamma(u, x_{n+1})}) * \theta(Tx_n, Tu, \frac{t}{2\gamma(x_{n+1}, Tu)}). \end{aligned} \tag{3.6}$$

Taking $n \to \infty$ in (3.6) and using (3.5), we get $\theta(u, Tu, t) = 1$ for all $t > 0$, that is, $Tu = u$. Now, let $w \in X$ be another fixed point for T. Then there exists $t > 0$ such that $\theta(u, w, t) \neq 1$. We have $x_0 \Re u$ and $x_0 \Re w$. Since T is $\Re$-preserving,

$$T^n x_0 \Re T^n u \text{ and } T^n x_0 \Re T^n w \text{ for all } n \in \mathbb{N}.$$

From (3.2), we have

$$\theta(T^n x_0, T^n u, t) \geq \theta(T^n x_0, T^n u, kt) \geq \theta(x_0, u, \frac{t}{k^n})$$

and

$$\theta(T^n x_0, T^n w, t) \geq \theta(T^n x_0, T^n w, kt) \geq \theta(x_0, w, \frac{t}{k^n}).$$

Thus we have

$$\begin{aligned} \theta(u, w, t) &= \theta(T^n u, T^n w, t) \geq \theta(T^n x_0, T^n u, \frac{t}{2\gamma(x_0, u)}) * \theta(T^n x_0, T^n w, \frac{t}{2\gamma(x_0, w)}) \\ &\geq \theta(x_0, u, \frac{t}{k^n 2\gamma(x_0, u)}) * \theta(x_0, w, \frac{t}{k^n 2\gamma(x_0, w)}) \end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we get $\theta(u, w, t) = 1$ for all $t > 0$ and hence $u = w$. □

Corollary 3.14. Let $(X, \theta, *, \Re)$ be an $\Re$-complete control fuzzy metric space. Let $T : X \to X$ be $\Re$-contractive and $\Re$-preserving. Also, if $\{x_n\}$ is an $\Re$-sequence with $x_n \to x \in X$, then $x \Re x_n$ for all $n \in \mathbb{N}$. Furthermore, T has a unique fixed point $x_* \in X$ and $\lim_{n\to\infty} \theta(T^n x, x_*, t) = \theta(x_*, x_*, t)$ for all $x \in X$ and $t > 0$.

Proof . The proof of this corollary runs along the same line as the proof of Theorem 3.13 that $\{x_n\}$ is an $\Re$-Cauchy sequence and converges to $x_* \in X$. Hence $x_* \Re x_n$ for all $n \in \mathbb{N}$. From (3.1), we have

$$\theta(Tx_*, x_{n+1}, t) = \theta(Tx_*, Tx_n, t) \geq \theta(Tx_*, Tx_n, kt) \geq \theta(x_*, x_n, t)$$

and

$$\lim_{n\to\infty} \theta(Tx_*, x_{n+1}, t) = 1.$$

Then we have

$$\theta(x_*, Tx_*, t) \geq \theta(x_*, x_{n+1}, \frac{t}{2\gamma(x_*, x_{n+1})}) * \theta(x_{n+1}, Tx_*, \frac{t}{2\gamma(x_{n+1}, Tx_*)}).$$

Taking the limit as $n \to \infty$, we get $\theta(x_*, Tx_*, t) = 1 * 1 = 1$ and hence $Tx_* = x_*$. The rest of the proof is similar to the proof of Theorem 3.13. □

Theorem 3.15. Let $(X, \theta, *, \Re)$ be an $\Re$-complete control fuzzy metric space with $\gamma : X \times X \to [1, \infty)$ and suppose that

$$\lim_{t\to\infty} \theta(x, w, t) = 1$$

for all $x \in X$. If $T : X \to X$ is $\Re$-contractive and $\Re$-preserving and satisfies

$$\theta(Tx, T^2 x, kt) \geq \theta(x, Tx, t)$$

for all $x \in O(x), t > 0$, where $k \in (0, 1)$, then $T^n x_0 \to u$. Furthermore, u is a fixed point of T if and only if $Tx = \theta(x, Tx, t)$ is ζ-orbitally lower semi continuous at u.

Proof . Since $(X,\theta,*,\Re)$ is an $\Re$-complete control fuzzy metric space, there exists $x_0 \in X$ such that $x_0\Re w$ for all $w \in X$. Hence $x_0 \Re T x_0$. Assume that

$$x_1 = Tx_0, x_2 = T^2x_0 = Tx_1, \ldots, x_n = T^n x_0 = Tx_{n-1}.$$

If $x_n = x_{n-1}$ then x_n is a fixed point of T. Suppose that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Since T is $\Re$-preserving, $\{x_n\}$ is an $\Re$-sequence and T is an $\Re$-contraction, we have

$$\begin{aligned}\theta(T^n x_0, T^{n+1}x_0, kt) &= \theta(x_n, x_{n+1}, kt) \geq \theta(x_{n-1}, x_n, \frac{t}{k})\\ &\geq \cdots \geq \theta(x_0, x_1, \frac{t}{k^{n-1}}).\end{aligned}$$

Now, from (θ_4), we have

$$\begin{aligned}\theta(x_n, x_{n+m}, t) &\geq \theta(x_n, x_{n+1}, \frac{t}{2\gamma(x_n, x_{n+1})}) * \theta(x_{n+1}, x_{n+m}, \frac{t}{2\gamma(x_{n+1}, x_{n+m})})\\ &\geq \theta(x_n, x_{n+1}, \frac{t}{2\gamma(x_n, x_{n+1})}) * \theta(x_{n+1}, x_{n+2}, \frac{t}{(2)^2\gamma(x_{n+1}, x_{n+m})\gamma(x_{n+1}, x_{n+2})})\\ &\quad *\theta(x_{n+2}, x_{n+m}, \frac{t}{(2)^2\gamma(x_{n+1}, x_{n+m})\gamma(x_{n+2}, x_{n+m})})\\ &\geq \theta(x_n, x_{n+1}, \frac{t}{2\gamma(x_n, x_{n+1})}) * \theta(x_{n+1}, x_{n+2}, \frac{t}{(2)^2\gamma(x_{n+1}, x_{n+m})\gamma(x_{n+1}, x_{n+2})})\\ &\quad *\theta(x_{n+2}, x_{n+3}, \frac{t}{(2)^3\gamma(x_{n+1}, x_{n+m})\gamma(x_{n+2}, x_{n+m})\gamma(x_{n+2}, x_{n+3})})\\ &\quad *\theta(x_{n+3}, x_{n+m}, \frac{t}{(2)^3\gamma(x_{n+1}, x_{n+m})\gamma(x_{n+2}, x_{n+m})\gamma(x_{n+3}, x_{n+m})})\\ &\geq \ldots \geq \theta(x_n, x_{n+1}, \frac{t}{2\gamma(x_n, x_{n+1})})\\ &\quad *[*_{i=n+1}^{n+m-2}\theta(x_i, x_{i+1}, \frac{t}{(2)^{m-2}(\prod_{j=n+1}^{i}(\gamma(x_j, x_{n+m})\gamma(x_i, x_{i+1}))})]\\ &\quad *[\theta(x_{n+m-1}, x_{n+m}, \frac{t}{(2)^{m-1}(\prod_{i=n+1}^{n+m-1}\gamma(x_i, x_{n+m}))})]\\ &\geq \theta(x_0, x_1, \frac{t}{2k^{n-1}\gamma(x_n, x_{n+1})})\\ &\quad *[*_{i=n+1}^{n+m-2}\theta(x_0, x_1, \frac{t}{(2)^{m-1}k^{i-1}(\prod_{j=n+1}^{i}(\gamma(x_j, x_{n+m})\gamma(x_i, x_{i+1}))})]\\ &\quad *[\theta(x_0, x_1, \frac{t}{(2)^{m-1}k^{n+m-1}(\prod_{i=n+1}^{n+m-1}\gamma(x_i, x_{n+m}))})].\end{aligned} \tag{3.7}$$

Taking the limit as $n \to \infty$ in (3.7), we get

$$\lim_{n\to\infty} \theta(x_n, x_{n+m}, t) \geq 1 * 1 * \ldots * 1 = 1$$

for all $t > 0$ and $n, m \in \mathbb{N}$. Thus x_n is an $\Re$-Cauchy sequence in X. From the completeness of $(X,\theta,*,\Re)$ there is $u \in X$ such that $x_n \to T^n x_0 = u$. Assume that T is $\Re$-semi continuous at $u \in X$. Then we have

$$\begin{aligned}\theta(u, Tu, kt) &= \lim_{n\to\infty} \sup \theta(T^n x_0, T^{n+1}x_0, kt)\\ &\geq \lim_{n\to\infty} \sup \theta(x_0, x_1, \frac{t}{k^{n-1}}) = 1.\end{aligned}$$

Conversely, let $u = Tu$ and $x_n \in Z$ with $x_n \to u$. Then we get

$$T(u) = \theta(u, Tu, kt) = 1 \geq \lim_{n\to\infty} \sup T(x_n) = \theta(T^n x_0, T^{n+1}x_0, kt),$$

as desired. □

Theorem 3.16. Let $(X,\theta,*,\Re)$ be an $\Re$-complete control fuzzy metric space and $T : X \to X$ be $\Re$-continuous, $\Re$-contractive, $\Re$-preserving and satisfy

$$\theta(x,w,t) > 0 \implies \theta(Tx,Tw,t) \geq \psi(\theta(x,w,t))$$

for all $x,w \in X$ and $t > 0$. Then T has a unique fixed point in X.

Proof . Since $(X,\theta,*,\Re)$ is an $\Re$-complete control fuzzy metric space, there exists $x_0 \in X$ such that $x_0\Re w$ for all $w \in X$. Thus $x_0\Re Tx_0$. Assume

$$x_1 = Tx_0, x_2 = T^2x_0 = Tx_1, \ldots, x_n = T^nx_0 = Tx_{n-1}.$$

If $x_n = x_{n-1}$ then x_n is a fixed point of T. Suppose that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Since T is $\Re$-preserving, $\{x_n\}$ is an $\Re$-sequence and T is an $\Re$-contraction, we have

$$\begin{aligned}\theta(x_n,x_{n+1},t) &= \theta(Tx_{-1},Tx_n,t) \geq \psi(\theta(x_{n-2},x_{n-1},t))\\ &\geq \cdots \geq \psi^n(\theta(x_0,x_1,t)).\end{aligned}$$

Now, from (θ_4), we have

$$\begin{aligned}
\theta(x_n,x_{n+m},t) &\geq \theta(x_n,x_{n+1},\frac{t}{2\gamma(x_n,x_{n+1})})*\theta(x_{n+1},x_{n+m},\frac{t}{2\gamma(x_{n+1},x_{n+m})})\\
&\geq \theta(x_n,x_{n+1},\frac{t}{2\gamma(x_n,x_{n+1})})*\theta(x_{n+1},x_{n+2},\frac{t}{(2)^2\gamma(x_{n+1},x_{n+m})\gamma(x_{n+1},x_{n+2})})\\
&\quad *\theta(x_{n+2},x_{n+m},\frac{t}{(2)^2\gamma(x_{n+1},x_{n+m})\gamma(x_{n+2},x_{n+m})})\\
&\geq \theta(x_n,x_{n+1},\frac{t}{2\gamma(x_n,x_{n+1})})*\theta(x_{n+1},x_{n+2},\frac{t}{(2)^2\gamma(x_{n+1},x_{n+m})\gamma(x_{n+1},x_{n+2})})\\
&\quad *\theta(x_{n+2},x_{n+3},\frac{t}{(2)^3\gamma(x_{n+1},x_{n+m})\gamma(x_{n+2},x_{n+m})\gamma(x_{n+2},x_{n+3})})\\
&\quad *\theta(x_{n+3},x_{n+m},\frac{t}{(2)^3\gamma(x_{n+1},x_{n+m})\gamma(x_{n+2},x_{n+m})\gamma(x_{n+3},x_{n+m})})\\
&\geq \ldots \geq \theta(x_n,x_{n+1},\frac{t}{2\gamma(x_n,x_{n+1})})\\
&\quad *[*_{i=n+1}^{n+m-2}\theta(x_i,x_{i+1},\frac{t}{(2)^{m-2}(\prod_{j=n+1}^{i}(\gamma(x_j,x_{n+m})\gamma(x_i,x_{i+1}))})]\\
&\quad *[\theta(x_{n+m-1},x_{n+m},\frac{t}{(2)^{m-1}(\prod_{i=n+1}^{n+m-1}\gamma(x_i,x_{n+m}))})]\\
&\geq \psi^n[\theta(x_0,x_1,\frac{t}{2k^{n-1}\gamma(x_n,x_{n+1})})]\\
&\quad *[*_{i=n+1}^{n+m-2}\psi^i(\theta(x_0,x_1,\frac{t}{(2)^{m-1}k^{i-1}(\prod_{j=n+1}^{i}(\gamma(x_j,x_{n+m})\gamma(x_i,x_{i+1}))}))]\\
&\quad *[\psi^{n+m-1}(\theta(x_0,x_1,\frac{t}{(2)^{m-1}k^{n+m-1}(\prod_{i=n+1}^{n+m-1}\gamma(x_i,x_{n+m}))}))].
\end{aligned} \tag{3.8}$$

Taking the limit as $n \to \infty$ in (3.8), we get

$$\lim_{n\to\infty}\theta(x_n,x_{n+m},t) \geq 1*1*\ldots*1 = 1$$

for all $t > 0$ and $n,m \in \mathbb{N}$. Thus $\{x_n\}$ is an $\Re$-Cauchy sequence in X. From the completeness of $(X,\theta,*,\Re)$, there exists $u \in X$ such that

$$\lim_{n\to\infty}\theta(x_n,u,t) = 1 \tag{3.9}$$

for all $t > 0$. Now, since T is an $\Re$-continuous mapping, $\theta(x_{n+1}, Tu, t) = \theta(Tx_n, Tu, t) \to 1$ as $n \to \infty$. For $t > 0$ and from (θ_4), we have

$$\begin{aligned} \theta(u, Tu, t) &\geq \theta(u, x_{n+1}, \frac{t}{2\gamma(u, x_{n+1})}) * \theta(x_{n+1}, Tu, \frac{t}{2\gamma(x_{n+1}, Tu)}) \\ &= \theta(u, x_{n+1}, \frac{t}{2\gamma(u, x_{n+1})}) * \theta(Tx_n, Tu, \frac{t}{2\gamma(x_{n+1}, Tu)}) \\ &\geq \theta(u, x_{n+1}, \frac{t}{2\gamma(u, x_{n+1})}) * \psi(\theta(x_n, u, \frac{t}{2\gamma(x_{n+1}, Tu)})). \end{aligned} \tag{3.10}$$

Taking $n \to \infty$ in (3.10) and using (3.9), we get $\theta(u, Tu, t) = 1$ for all $t > 0$, that is, $Tu = u$. Now, let $w \in X$ be another fixed point for T. Then we have

$$x_0 \Re u \text{ and } x_0 \Re w.$$

Since T is $\Re$-preserving, $T^n x_0 \Re T^n u$ and $T^n x_0 \Re T^n w$ for all $n \in \mathbb{N}$. Thus we have

$$\theta(T^n x_0, T^n u, t) \geq \theta(T^n x_0, T^n u, kt) \geq \psi(\theta(x_0, u, t))$$

and

$$\theta(T^n x_0, T^n w, t) \geq \theta(T^n x_0, T^n w, kt) \geq \psi(\theta(x_0, w, t)).$$

So we get

$$\begin{aligned} \theta(u, w, t) &= \theta(T^n u, T^n w, t) \geq \theta(T^n x_0, T^n u, kt) * \theta(T^n x_0, T^n w, kt) \\ &\geq \psi(\theta(x_0, u, t)) * \psi(\theta(x_0, w, t)) \geq \theta(x_0, u, t) * \theta(x_0, w, t) \end{aligned}$$

for all $n \in \mathbb{N}$. This is a contradiction and hence $u = w$. □

Example 3.17. Let $X = \mathbb{Z} = A \cup B$, where $A = \{-1, -2, -3, \ldots\} \cup \{0, 1\}$ and $B = \{2, 3, 4, \ldots\}$. Define a binary relation $\Re$ by $x \Re w \iff x + w \geq 0$. Define $\theta : X \times X \times [0, \infty) \to [0, 1]$ as

$$\theta(x, w, t) = \left\{ \begin{array}{c} 1 \text{ if } x = w, \\ \frac{t}{t + \max\{x, w\}} \text{ otherwise} \end{array} \right\}$$

for all $t > 0$ and $x, w \in X$ with continuous t-norm $*$ defined as $t_1 * t_2 = t_1 \cdot t_2$. Define $\gamma : X \times X \to [1, \infty)$ as

$$\gamma(x, w) = \begin{cases} 1, & x, w \in A \text{ or } x = 0 \text{ or } w = 0 \\ \max\{x, w\}, & \text{otherwise.} \end{cases}$$

Then $(X, \theta, *, \Re)$ is an $\Re$-complete control fuzzy metric space. Observe that $\lim_{t \to \infty} \theta(x, w, t) = 1$. Now, we define $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{2} \text{ if } x \in A \\ 1 \text{ if } x \in B \end{cases}$$

for all $x \in X$. Observe that if $x \Re w$ then clearly $Tx \Re Tw$. Now there are some cases to prove that T is an $\Re$-contraction for $k \in \left[\frac{1}{2}, 1\right)$.

1. If $x, w \in A$ then $Tx = \frac{x}{2}$ and $Tw = \frac{w}{2}$. This implies

$$\begin{aligned} \theta(Tx, Tw, kt) &= \theta(\frac{x}{2}, \frac{w}{2}, kt) = \frac{kt}{kt + \max\{\frac{x}{2}, \frac{w}{2}\}} \\ &\geq \frac{t}{t + \max\{x, w\}} = \theta(x, w, t). \end{aligned}$$

2. If $x, w \in B$ then $Tx = 1$ and $Tw = 1$. This implies

$$\begin{aligned} \theta(Tx, Tw, kt) &= \theta(1, 1, kt) = \frac{kt}{kt + \max\{1, 1\}} \\ &\geq \frac{t}{t + \max\{x, w\}} = \theta(x, w, t). \end{aligned}$$

3. If $x \in A$ and $w \in B$ then $Tx = \frac{x}{2}$ and $Tw = 1$. This implies

$$\begin{aligned}\theta(Tx, Tw, kt) &= \theta(\frac{x}{2}, 1, kt) = \frac{kt}{kt + \max\{\frac{x}{2}, 1\}} \\ &\geq \frac{t}{t + \max\{x, w\}} = \theta(x, w, t).\end{aligned}$$

4. If $x \in B$ and $w \in A$ then $Tx = 1$ and $Tw = \frac{w}{2}$. This implies

$$\begin{aligned}\theta(Tx, Tw, kt) &= \theta(1, \frac{w}{2}, kt) = \frac{kt}{kt + \max\{1, \frac{w}{2}\}} \\ &\geq \frac{t}{t + \max\{x, w\}} = \theta(x, w, t).\end{aligned}$$

Hence it is an $\Re$-contraction. Now we show that it is not a contraction. If $x, w \in A$ then $Tx = \frac{x}{2}$ and $Tw = \frac{w}{2}$. Thus

$$\theta(Tx, Tw, kt) = \theta(\frac{x}{2}, \frac{w}{2}, kt) = \frac{kt}{kt + \max\{\frac{x}{2}, \frac{w}{2}\}}.$$

Let $x = w = -2, k = \frac{9}{10}$ and $t = 10$. This implies

$$\theta(Tx, Tw, kt) = \frac{9}{9 + \max\{-1, -1\}} \leq \frac{10}{10 + \max\{-2, -2\}} = \theta(x, w, t)$$

which implies $\theta(Tx, Tw, kt) \leq \theta(x, w, t)$. This is a contradiction.

If $\lim_{n\to\infty} \theta(x_n, x, t)$ exists, then $\lim_{n\to\infty} \theta(Tx_n, Tx, t)$ exists. This implies that it is $\Re$-continuous. Also observe that $\lim_{n\to\infty} \gamma(x_n, w)$ and $\lim_{n\to\infty} \gamma(w, x_n)$ exist. Hence all the conditions of Theorem 3.13 are satisfied and 0 is a unique fixed point of T.

4 Application

In this section, we apply Theorem 3.13 to fuzzy Fredholm type integral equation and investigate the existence and uniqueness of fixed point.

Let $X = C([e, g], \Re)$ be the set of all continuous real valued functions defined on $[e, g]$. Consider the fuzzy Fredholm type integral equation:

$$x(l) = f(t) + \beta \int_e^g F(l, t)x(l)dt \text{ for all } l, t \in [e, g], \tag{4.1}$$

where $\beta > 0, f(t)$ is a fuzzy function of t and $F \in X$. Define θ by

$$\theta(x(l), w(l), t) = sup_{l\in[e,g]} \frac{t}{t + \max\{x(l), w(l)\}} \text{ for all } x, w \in X \text{ and } t > 0$$

with continuous t-norm $*$ defined as $t_1 * t_2 = t_1 \cdot t_2$. Define $\gamma : X \times X \to [1, \infty)$ as

$$\gamma(x, w) = \begin{cases} 1, & x, w \in A \text{ or } x = 0 \text{ or } w = 0 \\ \max\{x, w\}, & \text{otherwise.} \end{cases}$$

Then $(X, \theta, \Re)$ is an $\Re$-complete control fuzzy metric space.

Theorem 4.1. Assume that $\max\{F(l, t)x(l), F(l, t)w(l)\} \leq \max\{x(l), w(l)\}$ for $x, w \in X$, $k \in (0, 1)$ and for all $l, t \in [e, g]$. Also consider $\int_e^g dt = g - e \leq k < 1$. Let $T : X \to X$ be

1. $\Re$-preserving;
2. $\Re$-contraction;
3. $\Re$-continuous.

Then the fuzzy Fredholm type integral equation (4.1) has a unique solution.

Proof . Define $T : X \to X$ by

$$Tx(l) = f(t) + \beta \int_e^g F(l,t)x(l)dt \text{ for all } l,t \in [e,g].$$

Define $\Re$ as $x(l)\Re w(l)$ if and only if $x(l)w(l) \in \{|x(l)|, |w(l)|\}$. We see that $x(l)$ and $Tx(l)$ belong to X. So observe that if $x(l)\Re w(l)$ then clearly $Tx(l)\Re Tw(l)$. Observe that the existence of a fixed point of the operator T is equivalent to the existence of solution of the fuzzy Fredholm type integral equation. Now, for all $x, w \in X$, we have

$$\begin{aligned}
\theta(Tx(l), Tw(l), kt) &= \sup_{l\in[e,g]} \frac{kt}{kt + \max\{Tx(l), Tw(l)\}} \\
&= \sup_{l\in[e,g]} \frac{kt}{kt + \max\{\int_e^g F(l,t)x(l)dt, \int_e^g F(l,t)w(l)dt\}} \\
&= \sup_{l\in[e,g]} \frac{kt}{kt + \int_e^g \max\{F(l,t)x(l), F(l,t)w(l)\}dt} \\
&\geq \sup_{l\in[e,g]} \frac{kt}{kt + \int_e^g \max\{x(l), w(l)\}dt} \\
&= \sup_{l\in[e,g]} \frac{kt}{kt + \max\{x(l), w(l)\}\int_e^g dt} \\
&\geq \sup_{l\in[e,g]} \frac{kt}{kt + k\max\{x(l), w(l)\}} \\
&\geq \sup_{l\in[e,g]} \frac{t}{t + \max\{x(l), w(l)\}} \\
&= \theta(x(l), w(l), t).
\end{aligned}$$

Hence T is an $\Re$-contraction. Suppose $\{x_n\}$ is an $\Re$-sequence in X such that $\{x_n\}$ converges to $x \in X$. Since T is $\Re$-preserving and $\{Tx_n\}$ is an $\Re$-sequence, from (2), we have

$$\theta(x(l), w(l), kt) \geq \theta(x(l), w(l), t).$$

So $\lim_{n\to\infty} \theta(x(l), w(l), t)$ is finite for all $t > 0$. It is clear that $\lim_{n\to\infty} \theta(x(l), w(l), kt)$ is finite. Hence T is $\Re$-continuous. Therefore, all the conditions of Theorem 3.13 are satisfied. Hence the operator T has a unique fixed point. This means that the fuzzy Fredholm type integral equation (4.1) has a unique solution. □

Corollary 4.2. Let $(X, \theta, *)$ be an $\Re$-complete control fuzzy metric space. Define $T : X \to X$ as

$$Tx(l) = f(t) + \beta \int_e^g F(l,t)x(l)dt \text{ for all } l,t \in [e,g].$$

Suppose the following conditions hold:

1. $\max\{F(l,t)x(l), F(l,t)w(l)\} \leq \max\{x(l), w(l)\}$ for $x, w \in X$, $k \in (0,1)$ and for all $l, t \in [e,g]$;
2. $\int_e^g dt = g - e \leq k < 1$.

Then the integral equation (4.1) has a solution.

5 Conclusion

In this note, we introduced the notion of $\Re$-controlled fuzzy metric space and some new type of fixed point theorems in this new setting. Moreover, we provided a non-trivial example to demonstrate the viability of the proposed methods. We have supplemented this work with an application that demonstrates how the built method outperforms those found in the literature. Since our structure is more general than the class of fuzzy and controlled fuzzy spaces, our results and notions expand and generalize a number of previously published results.

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