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Global behavior of positive solutions of a third order difference equations system

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Abstract

In this paper, we investigate the global behavior of positive solutions of the system of difference equations

$$x_{n+1} = \alpha + \frac{y_n^p}{y_{n-2}^p},\ y_{n+1} = \alpha + \frac{x_n^q}{x_{n-2}^q},\ n = 0, 1, 2, ...$$

where parameters $\alpha, p, q \in (0, \infty)$ and the initial values $x_i,\ y_i$ are arbitrary positive numbers for $i = -2, -1, 0$. Moreover, the rate of convergence of positive solutions is established and some numerical examples are given to demonstrate our theoretical results.

Keywords: Difference equation, semi-cycle, equilibrium, boundedness, global asymptotic stability, rate of convergence
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1 Introduction

Recently, nonlinear difference equations and systems have been received great attention due to their practical applications [1-20]. These equations appear naturally as mathematical models that describe biological, physical, and economic phenomena. It has been observed that, difference equations might sometimes have a very simple form, however, there have not been any effective general methods to deal with the global behavior of their solutions so far. Therefore, the study of a certain form of difference equations is worth further consideration.

In [12], Stević et al. studied the boundedness character of the positive solutions of the following system of difference equations

$$x_{n+1} = A + \frac{y_n^p}{x_{n-1}^q}, y_{n+1} = A + \frac{x_n^p}{y_{n-1}^q},\ n = 0, 1, \dots,$$

where $A, p, q \in (0, \infty)$ and the initial values x_i, y_i are positive numbers for $i = -1, 0$.

In [5], Gümüş introduced the system of two recursive sequences

$$u_{n+1} = A + \frac{v_{n-k}}{v_n},\ v_{n+1} = A + \frac{u_{n-k}}{u_n},\ n = 0, 1, \dots,$$

where $A \in (0, \infty)$, u_i, v_i are arbitrary positive numbers for $i = -k, -k+1, \dots, 0$ and $k \in \mathbb{Z}^+$. The global asymptotic stability of the unique positive equilibrium point and the rate of convergence of positive solutions of the system were

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examined by the author.

In [17], Taşdemir suggested the following system of difference equations contained quadratic terms

$$y_{n+1} = A + B\frac{z_n}{z_{n-1}^2},\ z_{n+1} = A + B\frac{y_n}{y_{n-1}^2},\ n = 0, 1, \ldots,$$

where $A, B \in (0, \infty)$ and the initial values y_i, z_i, are positive numbers for $i = -1, 0$. The author studied the global asymptotic stability and the rate of convergence of solutions of mentioned system.

All the above-mentioned systems have motivated us to consider the following system of difference equations

$$x_{n+1} = \alpha + \frac{y_n^p}{y_{n-2}^p},\ y_{n+1} = \alpha + \frac{x_n^q}{x_{n-2}^q},\ n = 0, 1, 2, ... \tag{1.1}$$

where $\alpha, p, q \in (0, \infty)$ and the initial values $x_i, y_i \in (0, \infty), i = -2, -1, 0$. To our knowledge, no papers have been published about the dynamics of this system. In this work, we investigate semi-cycle analysis of solutions of (1.1). We also study the boundedness of the positive solutions and the global asymptotic stability of the unique equilibrium point in case $\alpha > 1$, $0 < p \leq 1$, $0 < q \leq 1$. Furthermore, the rate of convergence of the solutions of (1.1) is established and some numerical examples are given to verify our theoretical results.

2 Semi-cycle analysis of (1.1)

In this section, we study the behavior of positive solutions of system (1.1) with the help of semi-cycle analysis. Obviously, the system (1.1) has a unique possitive equilibrium point $(\bar{x}, \bar{y}) = (\alpha + 1, \alpha + 1)$.

Lemma 2.1. Assume that $\{(x_n, y_n)\}_{n=-2}^{\infty}$ is a solution of the system (1.1). Then, either $\{(x_n, y_n)\}_{n=-2}^{\infty}$ does not oscillate or it oscillates about the equilibrium $(\bar{x}, \bar{y}) = (\alpha + 1, \alpha + 1)$ with semi-cycles that if there is a semi-cycle with at least two terms, then each successive semi-cycle will have at least three terms.

Proof . Here we limit our consideration to the situation of oscillatory solutions of the system (1.1). Suppose that $\{(x_n, y_n)\}_{n=-2}^{\infty}$ is a solution of (1.1) that oscilates about the equilibrium, and there is $n_0 \geq 0$ with (x_{n_0}, y_{n_0}) is the last term of a semi-cycle that has at least two terms. Then, the following two cases can occur.
Case 1: $\ldots, x_{n_0-1}, x_{n_0} < \alpha + 1 \leq x_{n_0+1}$ and $\ldots, y_{n_0-1}, y_{n_0} < \alpha + 1 \leq y_{n_0+1}$.
Case 2: $\ldots, x_{n_0-1}, x_{n_0} \geq \alpha + 1 > x_{n_0+1}$ and $\ldots, y_{n_0-1}, y_{n_0} \geq \alpha + 1 > y_{n_0+1}$.
Now we look at the first case, the second case is similar and is neglected. With the happening of the first case and from (1.1), we obtain

$$x_{n_0+2} = \alpha + \left(\frac{y_{n_0+1}}{y_{n_0-1}}\right)^p > \alpha + 1,\ y_{n_0+2} = \alpha + \left(\frac{x_{n_0+1}}{x_{n_0-1}}\right)^q > \alpha + 1. \tag{2.1}$$

Similarly to (2.1), we have $x_{n_0+3} > \alpha + 1$ and $y_{n_0+3} > \alpha + 1$. Hence, the semi-cycle begining with (x_{n_0+1}, y_{n_0+1}) has at least three terms. Suppose that the semi-cycle which starts with (x_{n_0+1}, y_{n_0+1}) has length three, then the next semi-cycle will begin with (x_{n_0+4}, y_{n_0+4}) so that $x_{n_0+1}, x_{n_0+2}, x_{n_0+3} \geq \alpha + 1 > x_{n_0+4}$ and $y_{n_0+1}, y_{n_0+2}, y_{n_0+3} \geq \alpha + 1 > y_{n_0+4}$, then

$$x_{n_0+5} = \alpha + \left(\frac{y_{n_0+4}}{y_{n_0+2}}\right)^p < \alpha + 1,\ y_{n_0+2} = \alpha + \left(\frac{x_{n_0+4}}{x_{n_0+2}}\right)^q < \alpha + 1. \tag{2.2}$$

By arguing similar to (2.2), we have $x_{n_0+6} < \alpha + 1$ and $y_{n_0+6} < \alpha + 1$. From above arguments, we can conclude that the semi-cycle containing (x_{n_0+1}, y_{n_0+1}) and every semi-cycle after that has at least three terms. The proof is completed. □

Lemma 2.2. System (1.1) does not have nontrivial two periodic solutions.

Proof . Suppose that system (1.1) has a two periodic solution. Then $(x_{n-2}, y_{n-2}) = (x_n, y_n)$, for all $n \geq 0$. Therefore,

$$x_{n+1} = \alpha + \left(\frac{y_n}{y_{n-2}}\right)^p = \alpha + 1,\ y_{n+1} = \alpha + \left(\frac{x_n}{x_{n-2}}\right)^q = \alpha + 1, \text{for all } n \geq 0.$$

So, the solution $(x_n, y_n) = (\alpha + 1, \alpha + 1)$ is the equilibrium solution of system (1.1), which is contradiction with our assumtion. □

Lemma 2.3. If system (1.1) has an increasing solution then it is non-oscillatory positive solution.

Proof . Assume that $\{(x_n, y_n)\}_{n=-2}^{\infty}$ is an increasing solution of the system (1.1). Then, either $\alpha + 1 \leq x_1$ and $\alpha + 1 \leq y_1$ or $x_1 < \alpha + 1$ and $y_1 < \alpha + 1$. The first case is trivial and therefore left out. The second case is now considered. If $x_1 < \alpha + 1$ and $y_1 < \alpha + 1$, then we can assert that the negative semi-cycle containing (x_1, y_1) has at most three terms. Assume by contradiction that the negative semi-cycle begining with (x_1, y_1) involves (x_4, y_4). Then from

$$x_4 = \alpha + \left(\frac{y_3}{y_1}\right)^p < \alpha + 1 \text{ and } y_4 = \alpha + \left(\frac{x_3}{x_1}\right)^q < \alpha + 1$$

imply that $y_3 < y_1$ and $x_3 < x_1$ which contradicts the fact that the solution is increasing, so any increasing solution of system (1.1) is non-oscillatory positive solution. □

Lemma 2.4. Every solution of system (1.1) has no infinite negative semi-cycle.

Proof . Assume contradictorily that the system (1.1) has a solution $\{(x_n, y_n)\}_{n=-2}^{\infty}$ that contains an infinite negative semi-cycle, and assume that this semi-cycle begins with (x_N, y_N), where $N \geq -2$. Then for all $n \geq N$, $(x_n, y_n) < (\alpha + 1, \alpha + 1)$. From,

$$x_{n+1} = \alpha + \left(\frac{y_n}{y_{n-2}}\right)^p < \alpha + 1, \; y_{n+1} = \alpha + \left(\frac{x_n}{x_{n-2}}\right)^q < \alpha + 1, \text{for all } n \geq N + 2,$$

we imply $y_n < y_{n-2}$ and $x_n < x_{n-2}$. So, we have $\alpha < \ldots < x_{n+2} < x_n < x_{n-2} < \alpha + 1$ and $\alpha < \ldots < y_{n+2} < y_n < y_{n-2} < \alpha + 1$ for all $n \geq N + 2$, which means that $\{x_n\}$, $\{y_n\}$ have two subsequences $\{x_{2n}\}$, $\{x_{2n+1}\}$ and $\{y_{2n}\}$, $\{y_{2n+1}\}$ that are decreasing and bounded from below. Hence, there exist a_1, a_2, b_1, b_2 such that

$$\lim_{n\to\infty} x_{2n} = a_1, \; \lim_{n\to\infty} x_{2n+1} = a_2,$$
$$\lim_{n\to\infty} y_{2n} = b_1, \; \lim_{n\to\infty} y_{2n+1} = b_2.$$

So (a_1, b_1), (a_2, b_2) is a periodic solution of period two of system (1.1), which contradicts the Lemma 2.2 unless the solution is a trivial solution. Therefore, the solution converges to the equilibrium, this will not happen because the solution moves further and further away from the equilibrium point. Hence, system (1.1) does not have any solution that includes infinite negative semi-cycle. □

Lemma 2.5. System (1.1) does not have any decreasing non-oscillatory solution.

Proof . Assume that $\{(x_n, y_n)\}_{n=-2}^{\infty}$ is a decreasing non-oscillatory solution of system (1.1). Then, either

$$x_1 > \alpha + 1 \text{ and } y_1 > \alpha + 1$$

or

$$x_1 \leq \alpha + 1 \text{ and } y_1 \leq \alpha + 1$$

We now consider the first case. If $x_1 \leq \alpha + 1$ and $y_1 \leq \alpha + 1$, then we can say that the positive semi-cycle starts with (x_1, y_1) have at most three terms. Suppose contradictorily that the positive semi-cycle containing (x_1, y_1) ends with (x_4, y_4). Then from

$$x_4 = \alpha + \left(\frac{y_3}{y_1}\right)^p > \alpha + 1 \text{ and } y_4 = \alpha + \left(\frac{x_3}{x_1}\right)^q > \alpha + 1$$

imply that $y_3 > y_1$ and $x_3 > x_1$, which contradicts our assumption that the solution is decreasing. In this case, the solution has an infinite negative semi-cycle which begins latest from (x_4, y_4).
If the second case occurs, then we have

$$\ldots \leq x_3 \leq x_2 \leq x_1 \leq \alpha + 1 \text{ and } \ldots \leq y_3 \leq y_2 \leq y_1 \leq \alpha + 1$$

In both cases, the solution of system (1.1) has an infinite negative semi-cycle, which contradicts the Lemma 2.4. Therefore, system (1.1) has no decreasing non-oscillatory solution. □

Lemma 2.6. If $0 < p \leq 1$, $0 < q \leq 1$, then every semi-cycle of system (1.1) has at most five terms and every solution osillates about equilibrium point $(\bar{x}, \bar{y}) = (\alpha + 1, \alpha + 1)$.

Proof . Let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be a solution of system (1.1). We now study the case of positive semi-cycle, the case of negative semi-cycle is similar and is not considered here. Let (x_{n_0}, y_{n_0}) be the beginning term in a positive semi-cycle, and assume that this semi-cycle has five terms. Then,

$$x_{n_0}, x_{n_0+1}, x_{n_0+2}, x_{n_0+3}, x_{n_0+4} > \alpha + 1$$

and

$$y_{n_0}, y_{n_0+1}, y_{n_0+2}, y_{n_0+3}, y_{n_0+4} > \alpha + 1.$$

We have

$$\begin{aligned} x_{n_0+4} &= \alpha + \left(\frac{y_{n_0+3}}{y_{n_0+1}}\right)^p < \alpha + \frac{y^p_{n_0+3}}{(\alpha+1)^p} = \frac{\alpha(\alpha+1)^p + y^p_{n_0+3}}{(\alpha+1)^p} \\ &< \frac{(\alpha+1)y^p_{n_0+3}}{(\alpha+1)^p} = y^p_{n_0+3}(\alpha+1)^{1-p} < y_{n_0+3}, \\ y_{n_0+3} &= \alpha + \left(\frac{x_{n_0+2}}{x_{n_0}}\right)^q < \alpha + \frac{x^q_{n_0+2}}{(\alpha+1)^q} = \frac{\alpha(\alpha+1)^q + x^q_{n_0+2}}{(\alpha+1)^q} \\ &< \frac{(\alpha+1)x^q_{n_0+2}}{(\alpha+1)^q} = x^q_{n_0+2}(\alpha+1)^{1-q} < x_{n_0+2}, \end{aligned}$$

so

$$x_{n_0+4} < y_{n_0+3} < x_{n_0+2}. \tag{2.3}$$

Similarly, we have

$$y_{n_0+4} < x_{n_0+3} < y_{n_0+2}. \tag{2.4}$$

Now, from (2.3) and (2.4), we obtain

$$x_{n_0+5} = \alpha + \left(\frac{y_{n_0+4}}{y_{n_0+2}}\right)^p < \alpha + 1 \text{ and } y_{n_0+5} = \alpha + \left(\frac{x_{n_0+4}}{x_{n_0+2}}\right)^q < \alpha + 1$$

so a positive semi-cycle has at most five terms. Therefore, we can conclude that every semi-cycle of system (1.1) has at most five terms, this also implies that the solution osillates about equilibrium point $(\bar{x}, \bar{y}) = (\alpha+1, \alpha+1)$. □

3 Boundedness and persistence of system (1.1)

In this section, we will examine the boundedness and persistence of system (1.1) in the case of $\alpha > 1$, $0 < p \le 1$, and $0 < q \le 1$.

Theorem 3.1. Assume that $\alpha > 1$, $0 < p \le 1$, and $0 < q \le 1$. Then every positive solution of system (1.1) is bounded and persists.

Proof . Assume that $\{(x_n, y_n)\}_{n=-2}^{\infty}$ is a positive solution of the system (1.1). Obviously, from (1.1) we have

$$x_n, y_n > \alpha \text{ for all } n \ge 1. \tag{3.1}$$

By combining (1.1) and (3.1), we obtain

$$\begin{aligned} x_n &= \alpha + \left(\frac{y_{n-1}}{y_{n-3}}\right)^p < \alpha + \frac{1}{\alpha^p} y^p_{n-1} < \alpha + \frac{1}{\alpha^p} y_{n-1}, \\ y_n &= \alpha + \left(\frac{x_{n-1}}{x_{n-3}}\right)^q < \alpha + \frac{1}{\alpha^q} x^q_{n-1} < \alpha + \frac{1}{\alpha^q} x_{n-1}, \end{aligned} \tag{3.2}$$

for all $n \ge 4$.
Assume that $\{s_n, t_n\}$ is a solution of coming system

$$s_n = \alpha + \frac{1}{\alpha^p} t_{n-1},\ t_n = \alpha + \frac{1}{\alpha^q} s_{n-1},\ \text{for all } n \ge 4, \tag{3.3}$$

such that

$$s_i = x_i,\ t_i = y_i, i = 1, 2, 3. \tag{3.4}$$

By induction we have

$$x_n < s_n,\ y_n < t_n,\ \text{for all } n \geq 4. \tag{3.5}$$

From (3.3) and (3.4), we get

$$s_{n+2} = \frac{1}{\alpha^{p+q}} s_n + \alpha^{1-p} + \alpha,\ t_{n+2} = \frac{1}{\alpha^{p+q}} t_n + \alpha^{1-q} + \alpha,\ n \geq 2, \tag{3.6}$$

for simplicity, let $a = \dfrac{1}{\alpha^{p+q}}$, $b = \alpha^{1-p} + \alpha$ and $c = \alpha^{1-q} + \alpha$. Then (3.6) turns into

$$s_{n+2} = a s_n + b,\ t_{n+2} = a t_n + c,\ n \geq 2, \tag{3.7}$$

Solve (3.7), we obtain

$$s_{2n+2} = x_2 a^n + \frac{b}{1-a}(1-a^n),\ s_{2n+3} = x_3 a^n + \frac{b}{1-a}(1-a^n),\ \text{for all } n \geq 0, \tag{3.8}$$

and

$$t_{2n+2} = y_2 a^n + \frac{c}{1-a}(1-a^n),\ t_{2n+3} = y_3 a^n + \frac{c}{1-a}(1-a^n),\ \text{for all } n \geq 0. \tag{3.9}$$

Then, from (3.1), (3.5), (3.8) and (3.9), it follows that for all $n \geq 0$, we have

$$\begin{aligned} \alpha < x_{2n+2} &\leq x_2 a^n + \frac{b}{1-a}(1-a^n), \\ \alpha < x_{2n+3} &\leq x_3 a^n + \frac{b}{1-a}(1-a^n), \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \alpha < y_{2n+2} &\leq y_2 a^n + \frac{c}{1-a}(1-a^n), \\ \alpha < y_{2n+3} &\leq y_3 a^n + \frac{c}{1-a}(1-a^n). \end{aligned} \tag{3.11}$$

The proof is finished. □

4 Global behavior of system (1.1)

In the following theorem, the global attractor for system (1.1) will be established.

Theorem 4.1. If $\alpha > 1$, $0 < p \leq 1$, and $0 < q \leq 1$, then every positive solution of system (1.1) converges to the equilibrium point $(\bar{x}, \bar{y}) = (\alpha+1, \alpha+1)$ as $n \to \infty$.

Proof . Let

$$l_1 = \liminf_{n\to\infty} x_n,\ l_2 = \liminf_{n\to\infty} y_n,$$
$$L_1 = \limsup_{n\to\infty} x_n,\ L_2 = \limsup_{n\to\infty} y_n.$$

Obviously, $1 < l_1 \leq L_1$ and $1 < l_2 \leq L_2$. From system (1.1) we indicate that

$$l_1 \geq \alpha + \left(\frac{l_2}{L_2}\right)^p \geq \alpha + \frac{l_2}{L_2},\ l_2 \geq \alpha + \frac{l_1}{L_1},\ L_1 \leq \alpha + \frac{L_2}{l_2},\ L_2 \leq \alpha + \frac{L_1}{l_1}. \tag{4.1}$$

From (4.1), we get

$$l_1 L_2 \geq \alpha L_2 + l_2, \tag{4.2}$$
$$l_2 L_1 \geq \alpha L_1 + l_1, \tag{4.3}$$
$$l_2 L_1 \leq \alpha l_2 + L_2, \tag{4.4}$$
$$l_1 L_2 \leq \alpha l_1 + L_1. \tag{4.5}$$

From (4.2) and (4.5) imply that

$$\alpha L_2 + l_2 \leq \alpha l_1 + L_1. \tag{4.6}$$

From (4.3) and (4.4) indicate that

$$\alpha L_1 + l_1 \leq \alpha l_2 + L_2. \tag{4.7}$$

From (4.6) and (4.7), we obtain

$$\alpha L_2 + l_2 - \alpha l_2 - L_2 \leq \alpha l_1 + L_1 - \alpha L_1 - l_1, \tag{4.8}$$

which is equivalent to

$$(\alpha - 1)(L_2 - l_2 + L_1 - l_1) \leq 0. \tag{4.9}$$

Since $\alpha - 1 > 0$, so we can infer from (4.9) that $(L_2 - l_2 + L_1 - l_1) \leq 0$. But $L_2 - l_2 \geq 0$ and $L_1 - l_1 \geq 0$, so $(L_2 - l_2 + L_1 - l_1) \geq 0$. Hence, $L_2 - l_2 = 0$ and $L_1 - l_1 = 0$, so $L_2 = l_2$ and $L_1 = l_1$. Back to (4.2) and (4.4), we get $l_1 \geq \alpha + 1$ and $L_1 \leq \alpha + 1$, so $\lim_{n\to\infty} x_n = L_1 = l_1 = \alpha + 1$. Similarly, we imply $\lim_{n\to\infty} y_n = L_2 = l_2 = \alpha + 1$. The proof is finished. □

Theorem 4.2. Assume that $\alpha > 1$, $0 < p \leq 1$, and $0 < q \leq 1$, then the unique positive equilibrium $(\bar{x}, \bar{y}) = (\alpha + 1, \alpha + 1)$ of system (1.1) is locally asymptotically stable.

Proof . Set

$$u_n^{(1)} = x_n, u_n^{(2)} = x_{n-1}, u_n^{(3)} = x_{n-2}, u_n^{(4)} = y_n, u_n^{(5)} = y_{n-1}, u_n^{(6)} = y_{n-2},$$
$$U_n = \left(u_n^{(1)}, u_n^{(2)}, u_n^{(3)}, u_n^{(4)}, u_n^{(5)}, u_n^{(6)}\right)^T.$$

Then the linearized equation of system (1.1) about the equilibrium point $(\bar{x}, \bar{y}) = (\alpha + 1, \alpha + 1)$ is

$$U_{n+1} = AU_n,$$

where

$$\begin{aligned} U_{n+1} &= \left(u_{n+1}^{(1)}, u_{n+1}^{(2)}, u_{n+1}^{(3)}, u_{n+1}^{(4)}, u_{n+1}^{(5)}, u_{n+1}^{(6)}\right)^T \\ &= \left(\alpha + \left(\frac{u_n^{(4)}}{u_n^{(6)}}\right)^p, u_n^{(1)}, u_n^{(2)}, \alpha + \left(\frac{(u_n^{(1)})}{(u_n^{(3)})}\right)^q, u_n^{(4)}, u_n^{(5)}\right)^T, \end{aligned}$$

and A is the Jacobian matrix, which is determined by

$$A = \begin{pmatrix} 0 & 0 & 0 & \frac{p}{\alpha+1} & 0 & -\frac{p}{\alpha+1} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{q}{\alpha+1} & 0 & -\frac{q}{\alpha+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_6$ denote the eigenvalues of matrix A and let

$$D = diag(d_1, d_2, \ldots, d_6)$$

be a diagonal matrix in which

$$d_1 = d_4 = 1, d_2 = d_5 = 1 - 2\epsilon, d_3 = d_6 = 1 - 3\epsilon, \tag{4.10}$$

with $0 < \epsilon < \frac{1}{3}$, so $\det D = (1 - 2\epsilon)(1 - 3\epsilon) > 0$. Therefore, D is an invertible matrix. Computing matrix DAD^{-1}, we have

$$DAD^{-1} = \begin{pmatrix} 0 & 0 & 0 & \frac{d_1}{d_4}\frac{p}{\alpha+1} & 0 & -\frac{d_1}{d_6}\frac{p}{\alpha+1} \\ \frac{d_2}{d_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{d_3}{d_2} & 0 & 0 & 0 & 0 \\ \frac{d_4}{d_1}\frac{q}{\alpha+1} & 0 & -\frac{d_4}{d_3}\frac{q}{\alpha+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{d_5}{d_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{d_6}{d_5} & 0 \end{pmatrix}.$$

We know that A and DAD^{-1} are two similar matrices so they have the same eigenvalues as DAD^{-1}. Therefore, we imply

$$\rho(DAD^{-1}) = \max_{i\in\{1,\dots,6\}} |\lambda_i| \le \|DAD^{-1}\|_\infty \tag{4.11}$$

where $\rho(DAD^{-1})$ is a spectral radius of DAD^{-1} and

$$\|DAD^{-1}\|_\infty = \max\left\{\frac{p}{\alpha+1}(\frac{d_1}{d_4}+\frac{d_1}{d_6}),\ \frac{d_2}{d_1},\ \frac{d_3}{d_2},\ \frac{q}{\alpha+1}(\frac{d_4}{d_1}+\frac{d_4}{d_3}),\ \frac{d_5}{d_4},\ \frac{d_6}{d_5}\right\}. \tag{4.12}$$

From (4.10), we imply that

$$\frac{d_2}{d_1} < 1,\ \frac{d_3}{d_2} < 1,\ \frac{d_5}{d_4} < 1,\ \frac{d_6}{d_5} < 1. \tag{4.13}$$

From (4.12) and (4.13), in oder to show $\|DAD^{-1}\|_\infty < 1$, we only need

$$\frac{p}{\alpha+1}(\frac{d_1}{d_4}+\frac{d_1}{d_6}) < 1$$

and

$$\frac{q}{\alpha+1}(\frac{d_4}{d_1}+\frac{d_4}{d_3}) < 1.$$

Now, we consider

$$\frac{p}{\alpha+1}(\frac{d_1}{d_4}+\frac{d_1}{d_6}) = \frac{p}{\alpha+1}\frac{(2-3\epsilon)}{(1-3\epsilon)} < \frac{2p}{(\alpha+1)(1-3\epsilon)} < 1 \tag{4.14}$$

it follows

$$\epsilon < \frac{\alpha+1-2p}{3(\alpha+1)} \tag{4.15}$$

Similarly, from

$$\frac{q}{\alpha+1}(\frac{d_4}{d_1}+\frac{d_4}{d_3}) < 1 \tag{4.16}$$

we imply

$$\epsilon < \frac{\alpha+1-2q}{3(\alpha+1)} \tag{4.17}$$

From (4.14)-(4.17), we can choose ϵ such that

$$0 < \epsilon < \max\left\{\frac{\alpha+1-2p}{3(\alpha+1)},\ \frac{\alpha+1-2q}{3(\alpha+1)}\right\}, \tag{4.18}$$

then

$$\max\left\{\frac{p}{\alpha+1}(\frac{d_1}{d_4}+\frac{d_1}{d_6}), \frac{q}{\alpha+1}(\frac{d_4}{d_1}+\frac{d_4}{d_3})\right\} < 1. \tag{4.19}$$

Combining (4.13) and (4.19), we have

$$\max|\lambda_i| \le \|DAD^{-1}\|_\infty < 1.$$

It means that all eigenvalues of A are in the unit disk. This indicates that the unique positive equilibrium $(\bar{x},\bar{y}) = (\alpha+1,\alpha+1)$ of system (1.1) is locally asymptotically stable. Thus, the proof is finished. □

Connecting Theorem 4.1 and Theorem 4.2, we get the last theorem in this section.

Theorem 4.3. Assume that $\alpha > 1$, $0 < p \le 1$, and $0 < q \le 1$. Then the unique positive equilibrium point $(\bar{x},\bar{y}) = (\alpha+1,\alpha+1)$ of system (1.1) is globally asymptotically stable.

5 Rate of convergence

In this section we give the rate of convergence of a solution that converges to the equilibrium point $(\bar{x},\ \bar{y}) = (\alpha+1, \alpha+1)$ of the systems (1.1) for $\alpha > 1$, $0 < p \leq 1$, and $0 < q \leq 1$.

The following results provide the convergence rate of solutions for a system of difference equations

$$\mathbf{V}_{n+1} = [A + B(n)]\mathbf{V}_n \tag{5.1}$$

where $\mathbf{V}_n$ is a k-dimensional vector, $A \in \mathbb{C}^{k\times k}$ is a constant matrix, and $B:\ \mathbb{Z}^+ \longrightarrow \mathbb{C}^{k\times k}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \text{ when } n \to\ \infty, \tag{5.2}$$

where $\|.\|$ denotes any matrix norm associated with the vector norm.

Theorem 5.1. (Perron's Theorem, [9]) Consider system (5.1) and assume that condition (5.2) holds. If $\mathbf{V}_n$ is a solution of system (5.1), then either $\mathbf{V}_n = 0$ for all large n, or

$$\rho = \lim_{n\to\infty} \sqrt[n]{\|\mathbf{V}_n\|}$$

or

$$\rho = \lim_{n\to\infty} \frac{\|\mathbf{V}_{n+1}\|}{\|\mathbf{V}_n\|}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

We now present and prove the main result of this section.

Theorem 5.2. Assume that $\alpha > 1$, $0 < p \leq 1$, $0 < q \leq 1$, and $\{(x_n, y_n)\}$ is a solution of the system (1.1) such that $\lim\limits_{n\to\infty} x_n = \bar{x}$, $\lim\limits_{n\to\infty} y_n = \bar{y}$. Then, the error vector

$$\xi_n = \begin{pmatrix} \xi_n^1 \\ \xi_{n-1}^1 \\ \xi_{n-2}^1 \\ \xi_n^2 \\ \xi_{n-1}^2 \\ \xi_{n-2}^2 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ x_{n-1} - \bar{x} \\ x_{n-2} - \bar{x} \\ y_n - \bar{y} \\ y_{n-1} - \bar{y} \\ y_{n-2} - \bar{y} \end{pmatrix}$$

of every solution of (1.1) satisfies both of the following asymptotic relations:

$$\lim_{n\to\infty} \sqrt[n]{\|\xi_n\|} = |\lambda_i J_F(\bar{x}, \bar{y})| \text{ for some } i \in \{1,\ 2, \ldots, 6\}$$

or

$$\lim_{n\to\infty} \frac{\|\xi_{n+1}\|}{\|\xi_n\|} = |\lambda_i J_F(\bar{x}, \bar{y})| \text{ for some } i \in \{1,\ 2, \ldots, 6\}$$

where $|\lambda_i J_F(\bar{x}, \bar{y})|$ is equal to the modulus of one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium $(\bar{x}, \bar{y})$.

Proof . In order to find a system satisfied by the error terms, we set

$$\begin{aligned} x_{n+1} - \bar{x} &= \sum_{i=0}^{2} a_i(x_{n-i} - \bar{x}) + \sum_{i=0}^{2} b_i(y_{n-i} - \bar{x}) \\ y_{n+1} - \bar{y} &= \sum_{i=0}^{2} c_i(x_{n-i} - \bar{x}) + \sum_{i=0}^{2} d_i(y_{n-i} - \bar{x}) \end{aligned} \tag{5.3}$$

Let $\xi_n^1 = x_n - \bar{x}$ and $\xi_n^2 = y_n - \bar{y}$, then system (5.3) can be written as following form

$$\begin{aligned} \xi_{n+1}^1 &= \sum_{i=0}^{2} a_i \xi_{n-i}^1 + \sum_{i=0}^{2} b_i \xi_{n-i}^2, \\ \xi_{n+1}^2 &= \sum_{i=0}^{2} c_i \xi_{n-i}^1 + \sum_{i=0}^{2} d_i \xi_{n-i}^2, \end{aligned}$$

where

$$\begin{aligned}
&a_0 = a_1 = a_2 = 0,\\
&b_0 = \frac{py_n^{p-1}}{y_{n-2}^{p}}, b_1 = 0, b_2 = -\frac{py_n^{p}}{y_{n-2}^{p+1}},\\
&c_0 = \frac{qx_n^{q-1}}{x_{n-2}^{q}}, c_1 = 0, c_2 = -\frac{qx_n^{q}}{x_{n-2}^{q+1}},\\
&d_0 = d_1 = d_2 = 0.
\end{aligned}$$

Taking the limits of $a_i,\ b_i,\ c_i$ and d_i as $n \to \infty$ for $i = 0, 1, 2$, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} a_i = 0,\ \text{for } i = 0, 1, 2,\\
&\lim_{n\to\infty} b_0 = \frac{p}{\bar{y}}, \lim_{n\to\infty} b_1 = 0, \lim_{n\to\infty} b_2 = -\frac{p}{\bar{y}},\\
&\lim_{n\to\infty} c_0 = \frac{q}{\bar{x}}, \lim_{n\to\infty} c_1 = 0, \lim_{n\to\infty} c_2 = -\frac{q}{\bar{x}},\\
&\lim_{n\to\infty} d_i = 0,\ \text{for } i = 0, 1, 2.
\end{aligned}$$

That is

$$\begin{aligned}
&b_0 = \frac{p}{\bar{y}} + \beta_n,\ b_2 = -\frac{p}{\bar{y}} + \gamma_n,\\
&c_0 = \frac{q}{\bar{x}} + \delta_n,\ c_2 = -\frac{q}{\bar{x}} + \eta_n,
\end{aligned}$$

where $\beta_n \to 0,\ \gamma_n \to 0, \delta_n \to 0$ and $\eta_n \to 0$ as $n \to \infty$.

Now, we have the following system of the form (5.1):

$$\xi_{n+1} = [A + B(n)]\xi_n,$$

where $\xi_n = \left(\xi_n^1, \xi_{n-1}^1, \xi_{n-2}^1, \xi_n^2, \xi_{n-1}^2, \xi_{n-2}^2\right)^T$ and

$$A = J_F(\bar{x}, \bar{y}) = \begin{pmatrix}
0 & 0 & 0 & \frac{p}{\alpha+1} & 0 & -\frac{p}{\alpha+1}\\
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
\frac{q}{\alpha+1} & 0 & -\frac{q}{\alpha+1} & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},$$

$$B(n) = \begin{pmatrix}
0 & 0 & 0 & \beta_n & 0 & \gamma_n\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
\delta_n & 0 & \eta_n & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

and

$$\|B(n)\| \to 0 \text{ as } n \to\ \infty.$$

Thus, the limiting system of error terms can be written as:

$$\begin{pmatrix}
\xi_{n+1}^1\\ \xi_n^1\\ \xi_{n-1}^1\\ \xi_{n+1}^2\\ \xi_n^2\\ \xi_{n-1}^2
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & \frac{p}{\alpha+1} & 0 & -\frac{p}{\alpha+1}\\
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
\frac{q}{\alpha+1} & 0 & -\frac{q}{\alpha+1} & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}\begin{pmatrix}
\xi_n^1\\ \xi_{n-1}^1\\ \xi_{n-2}^1\\ \xi_n^2\\ \xi_{n-1}^2\\ \xi_{n-2}^2
\end{pmatrix}.$$

The above system is an exactly linearized system of (1.1) calculated at the equilibrium $(\bar{x},\ \bar{y}) = (\alpha+1,\ \alpha+1)$. From Theorem 5.1, we deduce the result. □

6 Examples

To verify our theoretical results, let us look at some interesting numerical examples. These examples perform different kinds of qualitative behavior of solutions of the system (1.1). All the graphics in this section are drawn using MATLAB.

Example 6.1. Consider $\alpha = 1.3, p = 0.9, q = 0.8$ and initial conditions $x_{-2} = 2.6$, $x_{-1} = 1.8$, $x_0 = 3$, $y_{-2} = 3$, $y_{-1} = 5$, $y_0 = 1$. Then system (1.1) can be written as

$$x_{n+1} = 1.3 + \left(\frac{y_n}{y_{n-2}}\right)^{0.9}, \; y_{n+1} = 1.3 + \left(\frac{x_n}{x_{n-2}}\right)^{0.8} \tag{6.1}$$

In this case, the unique positive equilibrium $(\bar{x}, \bar{y}) = (2.3, 2.3)$ is globally asymptotically stable (see Figure 1, Theorem 4.3).

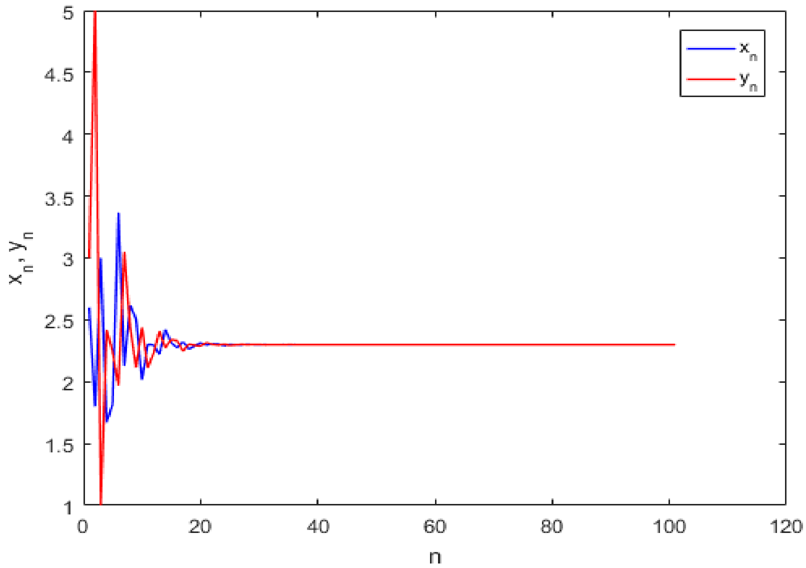

Figure 1: The plot of system (6.1).

Example 6.2. Consider $\alpha = 2$, $p = 1$, $q = 1$ and initial conditions $x_{-2} = 2.5$, $x_{-1} = 6$, $x_0 = 2$, $y_{-2} = 4$, $y_{-1} = 2$, $y_0 = 5$. Then system (1.1) can be written as

$$x_{n+1} = 2 + \frac{y_n}{y_{n-2}}, \; y_{n+1} = 2 + \frac{x_n}{x_{n-2}}. \tag{6.2}$$

In this case, the unique positive equilibrium point $(\bar{x}, \; \bar{y}) = (3, 3)$ is also globally asymptotically stable (see Figure 2, Theorem 4.3).

Example 6.3. Consider $\alpha = 0.6, p = 0.8, q = 1.9$ and initial conditions $x_{-2} = 1.6$, $x_{-1} = 2.8$, $x_0 = 4$, $y_{-2} = 4$, $y_{-1} = 1.5$, $y_0 = 6$. Then system (1.1) can be written as

$$x_{n+1} = 0.6 + \left(\frac{y_n}{y_{n-2}}\right)^{0.8}, \; y_{n+1} = 0.6 + \left(\frac{x_n}{x_{n-2}}\right)^{1.9} \tag{6.3}$$

In this system, since $\alpha < 1$ and $q > 1$ are not satisfied conditions of Theorem 4.3 so the unique positive equilibrium point $(\bar{x}, \; \bar{y}) = (1.6, 1.6)$ is not globally asymptotically stable, see Figure 3.

Example 6.4. Consider $\alpha = 0.3$, $p = 1.2$, $q = 1.5$ and initial conditions $x_{-2} = 6$, $x_{-1} = 8$, $x_0 = 3$, $y_{-2} = 3$, $y_{-1} = 5$, $y_0 = 1$. Then system (1.1) can be written as

$$x_{n+1} = 0.3 + \left(\frac{y_n}{y_{n-2}}\right)^{1.2}, \; y_{n+1} = 0.3 + \left(\frac{x_n}{x_{n-2}}\right)^{1.5}. \tag{6.4}$$

In this case, the conditions for α, p and q in Theorem 4.3 are not satisfied, so the unique positive equilibrium point $(\bar{x}, \; \bar{y}) = (1.3, 1.3)$ is not globally asymptotically stable, see Figure 4.

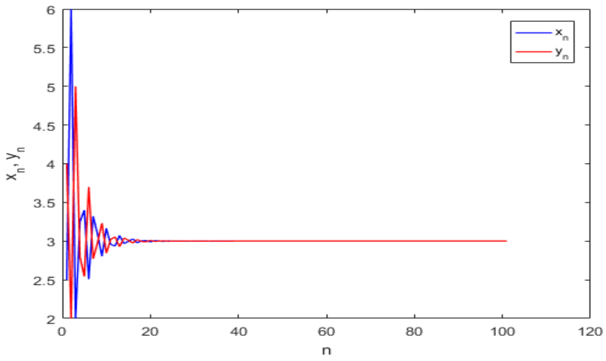

Figure 2: The plot of system (6.2)

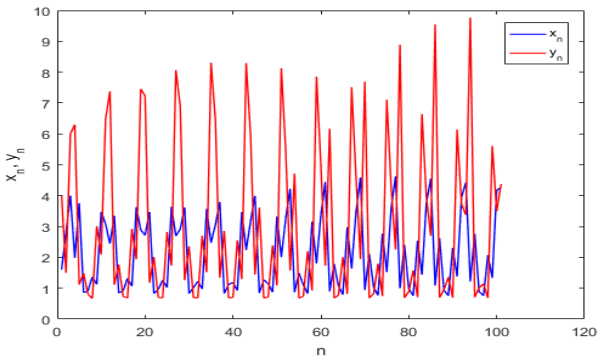

Figure 3: The plot of system (6.3)

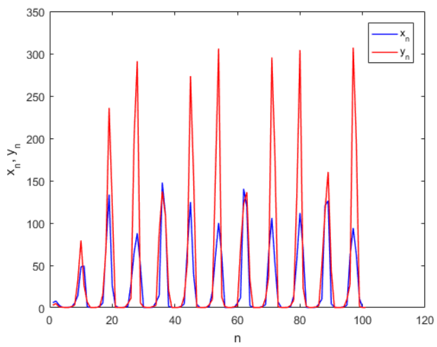

Figure 4: The plot of system (6.4)

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