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APPROXIMATELY HIGHER HILBERT C^*-MODULE DERIVATIONS

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Dedicated to the 70th Anniversary of S.M.Ulam's Problem for Approximate Homomorphisms

ABSTRACT. We show that higher derivations on a Hilbert C^*-module associated with the Cauchy functional equation satisfying generalized Hyers–Ulam stability.

1. INTRODUCTION

Let A be a C^*-algebra and M be a linear space that is a left $A-$module with a scalar multiplication satisfying $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for $x \in M, a \in A, \lambda \in \mathbb{C}$. The space M is called a pre-Hilbert $A-$module or inner product $A-$module if there exists an inner product $< .,. >$: $M \times M \to A$ with the following properties:

1.$< x, x > \geq 0$; and $< x, x >= 0$ iff $x = 0$;
2. $< \lambda x + y, z >= \lambda < x, z > + < y, z >$;
3.$< ax, y >= a < x, y >$;
4. $< x, y >^* =< y, x >$.

M is called a (left) Hilbert $A-$module, or a Hilbert C^*-module over the C^*-algebra A if it is complete with respect to the norm $\|x\| = \| < x, x > \|_A^{\frac{1}{2}}$. We always assume that the linear structure of A and M are compatible.

(i) The C^*- algebra A itself can be reorganized to become a Hilbert $A-$module if we define the inner product $< a, b >= ab^*$. The Hilbert Submodules of A are precisely its closed (left) ideals.

(ii) Every inner product space is a left Hilbert $\mathbb{C}-$module; cf [9, 27].

A linear mapping $d : M \to M$ is called a derivation on the Hilbert C^*-module M if it satisfies the condition $d(< x, y > z) =< d(x), y > z+ < x, d(y) > z+ < x, y > d(z)$ for every $x, y, z \in M$ (see [1, 10]). It is clear that every adjointable mapping T satisfying $T^* = -T$ is a derivation. The converse is not true in general; see [1].

Let $\mathbb{N}$ be the set of natural numbers. For $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A sequence $H = \{h_0, h_1, ..., h_m\}$ (resp. $H = \{h_o, h_1, ..., h_n, ...\}$) of linear maps from Hilbert

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$A-$module M into Hilbert $A-$module N is called a higher derivation of rank m (resp. infinite rank) from M into N if

$$h_n(< x, y > z) = \sum_{i+j+k=n} < h_i(x), h_j(y) > h_k(z)$$

holds for each $n \in \{0, 1, ..., m\}$ (resp. $n \in \mathbb{N}_0$) and all $x, y, z \in M$. A higher derivation of rank 0 from M into N is a homomorphism; that is, h_0 is linear and $h_o(< x, y > z) =< h_0(x), h_o(y) > h_o(z)$. The higher derivation H from M into N is said to be onto if $h_o : M \to N$ is onto. The higher derivation H on M is called strong if h_0 is an identity mapping on M. A strong higher derivation of rank 1 on M is a derivation. Thus, a higher derivation is a generalization of both a homomorphism and a derivation (for similar definitions on algebras, see [7]).

The stability of functional equations was first introduced by S. M. Ulam [26] in 1940. In 1941, D. H. Hyers [5] gave a partial solution of *Ulam's* problem for the case of approximate additive mappings in the context of Banach spaces. In 1978, Th. M. Rassias [24] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences $\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p), (\epsilon > 0, p \in [0, 1))$. This phenomenon of stability that was introduced by Th. M. Rassias [24] is called the Hyers–Ulam–Rassias stability (or the generalized Hyers-Ulam stability). In 1992, Găvruta [4] generalized the Th.M. Rassias theorem as follows:

Suppose $(G, +)$ is an ablian group and X is a Banach space $\varphi : G \times G \longrightarrow [0, \infty)$ satisfying

$$\tilde{\varphi}(x, y) = \frac{1}{2} \sum_{n=0}^{\infty} 2^{-n} \varphi(2^n x, 2^n y) < \infty$$

for all $x, y \in G$. If $f : G \to X$ is a mapping with

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$, then there exists a unique mapping $T : G \to X$ such that $T(x+y) = T(x) + T(y)$ and $\|f(x) - T(x)\| \leq \tilde{\varphi}(x, x)$ for all $x, y \in G$.

R. Badora [2] and T. Miura et al. [11] proved the Ulam-Hyers stability and the Isaac and Rassias-type stability of derivations [6]; M. Bavand Savadkouhi, M. Eshaghi Gorrdji, J. M. Rassias, and N. Ghobadipour [3] have contributed works regarding the stability of ternary Jordan derivations. Yong-Soo Jung and Ick-Soon Chang [7] investigated the stability and superstability of higher derivations on rings. Amyari and M. S. Moslehian [1] studied the stability of derivations on Hilbert C^*-modules (see also [12]–[25]).

2. Main results

We start our work with a known fixed point theorem.

Theorem 2.1. *(The alternative of fixed point). Suppose* (X, d) *be a generalized complete metric space and* $J : X \to X$ *is a strictly contractive mapping; that is ,*

$$d(Jx, Jy) \leq Ld(x, y)(x, y \in X),$$

for some $L < 1$. Then, for each given element $x \in X$, either

$$d(J^n x, T^{n+1} x) = \infty, \forall n \geq 0,$$

or

$$d(J^n x, J^{n+1} x) < \infty, \forall n \geq n_o,$$

for some natural n_0. Moreover, if the second alternative holds, then:
(i) The sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;
(ii) y^* is a unique fixed point of J in $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$; and $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)(x, y \in Y)$.

Lemma 2.2. *([lemma 2, 1]) Let X be a linear space and Y be a Banach space $0 \leq L < 1$ and $\lambda \geq 0$ are given numbers and $\psi : X \to [0, \infty)$ has the property*

$$\psi(x) \leq \lambda L \psi(\frac{x}{\lambda}),$$

for all $x \in X$. Assume that $S = \{g : X \to Y : g(0) = 0\}$ and the generalized metric d on S is defined by

$$d(g, h) = inf\{c \in (o, \infty) : \|g(x) - h(x)\| \leq c\psi(x), \forall x \in X\}.$$

Then the mapping $J : S \to S$ given by $Jg(x) = \frac{1}{\lambda} g(\lambda x)$ is a strictly contractive mapping.

Theorem 2.3. *Let $\varphi : M^5 \to [0, \infty)$ be a control function such that*

$$lim_{n \to \infty} \frac{\varphi(2^n x, 2^n y, 2^n u, 2^n t, 2^n z)}{2^n} = 0$$

for all $x, y, u, t, z \in M$. Suppose that $F = \{f_0, f_1, ..., f_n, ...\}$ is a sequence of mappings from M into N such that $f_n(0) = 0$ and

$$\|f_n(\lambda x + y + < u, t > z) - \lambda f_n(x) - f_n(y) - \sum_{i+j+k=n} < f_i(u), f_j(t) > f_k(z)\| \leq \varphi(x, y, u, t, z), \tag{2.1}$$

for all $x, y, u, t, z \in M, n \in \mathbb{N}_0, \lambda \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Assume that there exists $0 \leq L < 1$ such that the mapping $\psi(x) = \varphi(\frac{x}{2}, \frac{x}{2}, 0, 0, 0)$ has the property

$$\psi(x) \leq 2L\psi(\frac{x}{2}), \tag{2.2}$$

for all $x \in M$. Then there exists a unique higher derivation $H = \{h_0, h_1, ..., h_n, ...\}$ of any rank from M into N such that

$$\|f_n(x) - h_n(x)\| \leq \frac{L}{1-L} \psi(x),$$

for each $n \in \mathbb{N}_0$ and for all $x \in M$.

Proof. Setting $\lambda = 1, y = x$, and $u = t = z = 0$ in (2.1) implies

$$\|f_n(2x) - 2f_n(x)\| \leq \varphi(x, x, 0, 0, 0), \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\|\frac{1}{2}f_n(2x) - f_n(x)\| \leq \frac{1}{2}\psi(2x) \leq L\psi(x).$$

for each $n \in \mathbb{N}_0$ and $x \in M$. So $d(f_n, Tf_n) \leq L < \infty$, where the mapping T defined on $S = \{g_n : M \to N : g_n(0) = 0\}$ by $(Tg_n)(x) = \frac{1}{2}g_n(2x)$ is a strictly contractive function as in lemma 2.2. Applying the fixed point alternative, we deduce the existence of a mapping $h_n : M \to N$ such that h_n is a fixed point of T that is $h_n(2x) = 2h_n(x)$ for all $x \in M$. Since $lim_{m\to\infty}d(T^m f_n, h_n) = 0$, it follows that

$$lim_{m\to\infty}\frac{f_n(2^m x)}{2^m} = h_n(x), \tag{2.4}$$

for all $x \in M, n \in \mathbb{N}_0$. The mapping h_n is the unique fixed point of T in the set $U = \{g_n \in S : d(f_n, g_n) < \infty\}$. Hence h_n is the unique fixed point of T such that $\|f_n(x) - h_n(x)\| \leq K\psi(x)$ for some $K > 0$ and for all $x \in M$. Again, by applying the fixed point alternative theorem, we infer that

$$d(f_n, h_n) \leq \frac{1}{1-L}d(f_n, Tf_n) \leq \frac{L}{1-L},$$

so

$$\|f_n(x) - h_n(x)\| \leq \frac{L}{1-L}\varphi(\frac{x}{2}, \frac{x}{2}, 0, 0, 0),$$

for all $x \in M, n \in \mathbb{N}_0$. It follows from (2.1) that

$$\|f_n(\lambda x + y) - \lambda f_n(x) - f_n(y)\| \leq \varphi(x, y, 0, 0, 0),$$

By replacing x and y in (2.4) by $2^n x$ and $2^n y$, respectively, dividing both sides by 2^n and taking $n \to \infty$, we get

$$h_n(\lambda x + y) = \lambda h_n(x) + h_n(y),$$

for all $\lambda \in \mathbb{T}$ and all $x, y \in M$.

Now, let $\lambda \in \mathbb{C}(\lambda \neq 0)$ and let K be a natural number greater than $4|\lambda|$. Then $|\frac{\lambda}{K}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By Theorem 1 in [8], there exist numbers $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{T}$ such that $3\frac{\lambda}{K} = \lambda_1 + \lambda_2 + \lambda_3$. By the additivity of each $h_n, n \in \mathbb{N}_0$, we get $h_n(\frac{1}{3}x) = \frac{1}{3}h_n(x)$ for each $n \in \mathbb{N}_0$ and all $x \in M$. Therefore,

$$h_n(\lambda x) = h_n(\frac{K}{3}.3.\frac{\lambda}{K}x) = \frac{K}{3}h_n(3.\frac{\lambda}{K}x) = \frac{K}{3}h_n(\lambda_1 x + \lambda_2 x + \lambda_3 x)$$
$$= \frac{K}{3}(h_n(\lambda_1 x) + h_n(\lambda_2 x) + h_n(\lambda_3 x)) = \frac{K}{3}(\lambda_1 + \lambda_2 + \lambda_3)h_n(x) = \lambda h_n(x),$$

for each $n \in \mathbb{N}_0$ and all $x \in M$, so that h_n is $\mathbb{C}-$linear for each $n \in \mathbb{N}_0$.

Next, we need to show that the sequence $H = \{h_0, h_1, ..., h_n, ...\}$ satisfies the identity

$$h_n(< u, t > z) = \sum_{i+j+k=n} < h_i(u), h_j(t) > h_k(z)$$

for each $n \in \mathbb{N}_0$ and all $x, y, z \in M$. Putting $x = y = 0$ in (2.1) and

$$D_n(u, t, z) = f_n(< u, t > z) - \sum_{i+j+k=n} < f_i(u), f_j(t) > f_k(z), \tag{2.5}$$

for each $n \in \mathbb{N}_o$ and all $u, t, z \in A$, we see that

$$lim_{r\to\infty}\frac{D_n(2^r u, 2^r t, 2^r z)}{2^r} = 0, \tag{2.6}$$

for each $n \in \mathbb{N}_0$ and all $u, t, z \in M$. By using (2.4), (2.5), and (2.6), we get

$$h_n(< u, t > z) = lim_{r\to\infty}\frac{f_n(2^r < u, t > z)}{2^r} = lim_{r\to\infty}\frac{f_n(< (2^r u), (2^r t) > (2^r z)}{2^{3r}}$$

$$= lim_{r\to\infty}\frac{\sum_{i+j+k=n} < f_i(2^r u), f_j(2^r t) > f_k(2^r z) + D_n(2^r u, 2^r t, 2^r z)}{2^{3r}}$$

$$= lim_{r\to\infty}\sum_{i+j+k=n} < \frac{1}{2^r}f_i(2^r u), \frac{1}{2^r}f_j(2^r t) > \frac{1}{2^r}f_k(2^r u)$$

$$+lim_{r\to\infty}\frac{D_n(2^r u, 2^r t, 2^r z)}{2^{3r}} = \sum_{i+j+k=n} < h_i(u), h_j(t) > h_k(z)$$

This completes the proof of the theorem. □

As a consequence of the previous theorem, we show the Hyers-Ulam-Rassias stability of higher derivations.

Corollary 2.4. *Let $0 \leq p < 1, \alpha, \beta > 0$ and $F = \{f_o, f_1, ..., f_n, ...\}$ is a sequence of mappings from M into N satisfying $f(0) = 0$ and*

$$\|f_n(\lambda x + y + < u, t > z) - \lambda f_n(x) - f_n(y) - \sum_{i+j+k=n} < f_i(u), f_j(t) > f_k(z)]\|$$

$$\leq \alpha + \beta(\|x\|^p + \|y\|^p + \|u\|^p + \|t\|^p + \|z\|^p)$$

for all $\lambda \in \mathbb{T}$ and all $x, y, u, t, z \in M$.

Then there exists a unique higher derivation $H = \{h_0, h_1, ..., h_n, ...\}$ of any rank from M into N such that

$$\|f_n(x) - h_n(x)\| \leq \frac{\alpha + \beta 2^{1-p}\|x\|^p}{2^{1-p} - 1},$$

for all $x \in M$.

Proof. Put $\varphi(x, y, u, t, z) = \alpha + \beta(\|x\|^p + \|y\|^p + \|u\|^p + \|t\|^p + \|z\|^p)$, and let $L = \frac{1}{2^{1-p}}$ in the previous theorem. Then $\psi(x) = \alpha + 2^{1-p}\beta\|x\|^p$, and there exists a sequence $H = \{h_0, h_1, ..., h_n, ...\}$ with required properties. □

In a similar fashion to theorem 2.3, we can prove the following theorem:

Theorem 2.5. *Let $\varphi : M^5 \to [0, \infty)$ be a control function with the property*

$$lim_{n\to\infty}2^n\varphi(2^{-n}x, 2^{-n}y, 2^{-n}u, 2^{-n}t, 2^{-n}z) = 0$$

for all $x, y, u, t, z \in A$. Assume that $F = \{f_o, f_1, ..., f_n, ...\}$ is a sequence of mappings from M into N satisfying $f(0) = 0$ and

$$\|f_n(\lambda x + y + < u, t > z) - \lambda f_n(x) - f_n(y) - \sum_{i+j+k=n} < f_i(u), f_j(t) > f_k(z)\| \leq \varphi(x, y, u, t, z), \tag{2.7}$$

for all $x, y, u, t, z \in M, \lambda \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$*. Assume that there exists* $0 \leq L < 1$ *such that the mapping* $\psi(x) = \varphi(\frac{x}{2}, \frac{x}{2}, 0, 0, 0)$ *has the property*

$$\psi(x) \leq \frac{1}{2} L\psi(2x),$$

for all $x \in M$*. Then there exists a unique higher derivation* $H = \{h_0, h_1, ..., h_n, ...\}$ *of any rank from* M *into* N *such that*

$$\|f_n(x) - h_n(x)\| \leq \frac{1}{1-L}\psi(x),$$

for each $n \in \mathbb{N}_0$ *and for all* $x \in M$*.*

Proof. Setting $\lambda = 1, y = x$, and $u = t = z = 0$ in (2.7) implies

$$\|f_n(2x) - 2f_n(x)\| \leq \varphi(x, x, 0, 0, 0), \tag{2.8}$$

Replacing x by $\frac{x}{2}$ in (2.8), we obtain

$$\|f_n(x) - 2f_n(\frac{x}{2})\| \leq \psi(x).$$

for each $n \in \mathbb{N}_0$ and $x \in M$. Thus, $d(f_n, Tf_n) \leq L < \infty$, where the mapping T defined on $S = \{g_n : M \to N : g_n(0) = 0\}$ by $(Tg_n)(x) = 2g_n(\frac{1}{2}x)$ is a strictly contractive function, as in lemma 2.2. Applying the fixed point alternative, we deduce the existence of a mapping $h_n : M \to N$ such that h_n is a fixed point of T that is $h_n(\frac{1}{2}x) = \frac{1}{2}h_n(x)$ for all $x \in M$. Since $lim_{m\to\infty} d(T^m f_n, h_n) = 0$, it follows that

$$lim_{m\to\infty} 2^m f_n(2^{-m}x) = h_n(x)$$

for all $x \in M, n \in \mathbb{N}_0$. The mapping h_n is the unique fixed point of T in the set $U = \{g_n \in S : d(f_n, g_n) < \infty\}$. Hence, h_n is the unique fixed point of T such that $\|f_n(x) - h_n(x)\| \leq K\psi(x)$ for some $K > 0$ and for all $x \in M$. Again, by applying the fixed point alternative theorem, we infer that

$$d(f_n, h_n) \leq \frac{1}{1-L} d(f_n, Tf_n) \leq \frac{1}{1-L},$$

so

$$\|f_n(x) - h_n(x)\| \leq \frac{1}{1-L}\varphi(\frac{x}{2}, \frac{x}{2}, 0, 0, 0),$$

for all $x \in M, n \in \mathbb{N}_0$. The rest is similar to the proof of theorem 2.3. □

The following corollary is similar to corollary 2.4 for the case where $p > 1$.

Corollary 2.6. *Let* $p > 1, \alpha, \beta > 0$ *and* $F = \{f_o, f_1, ..., f_n, ...\}$ *is a sequence of mappings from* M *into* N *satisfying* $f(0) = 0$ *and*

$$\|f_n(\lambda x + y + < u, t > z) - \lambda f_n(x) - f_n(y) - \sum_{i+j+k=n} < f_i(u), f_j(t) > f_k(z)\|$$

$$\leq \alpha + \beta(\|x\|^p + \|y\|^p + \|u\|^p + \|t\|^p + \|z\|^p)$$

for all $\lambda \in \mathbb{T}$ *and all* $x, y, u, t, z \in M$. *Then there exists a unique higher derivation* $H = \{h_0, h_1, ..., h_n, ...\}$ *of any rank from* M *into* N *such that*

$$\|f_n(x) - h_n(x)\| \leq \frac{\alpha 2^{p-1} + \beta\|x\|^p}{2^{1-p} - 1},$$

for all $x \in M$.

Proof. Put $\varphi(x, y, u, t, z) = \alpha + \beta(\|x\|^p + \|y\|^p + \|u\|^p + \|t\|^p + \|z\|^p)$, and let $L = \frac{1}{2^{p-1}}$ in the previous theorem. Then $\psi(x) = \alpha + 2^{1-p}\beta\|x\|^p$ and there exists a sequence $H = \{h_0, h_1, ..., h_n, ...\}$ with required properties. □

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