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On some properties of elements in hypergroup algebras

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Abstract

Let H be a hypergroup with left Haar measure and let $L^1(H)$ be the complex Lebesgue space associated with it. Let $L^\infty(H)$ be the set of all locally measurable functions that are bounded except on a locally null set, modulo functions that are zero locally a.e. Let $\mu \in M(H)$. We want to find out when $\mu F \in L^\infty(H)^*$ implies that $F \in L^1(H)$. Some necessary and sufficient conditions is found for a measure μ for which if $\mu F \in L^1(H)$ for every $F \in L^\infty(H)^*$, then $F \in L^1(H)$.

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1 Introduction

Locally compact hypergroups were independently introduced around the 1970's by Dunkl, Jewett and Spector. They generalize the concepts of locally compact groups with the purpose of doing standard harmonic analysis. Similar structures had been studied earlier in the 1950's by Berezansky and colleagues, and even earlier in works of Delsarte and Levitan, for more information see [1], [11] and [13].
Hypergroups are a suitable generalization of classical locally compact groups. Hypergroups arise as double coset spaces of locally compact groups. In classical setting, the convolution of two point mass measures is a point mass measure, while in hypergroup structure, the convolution of two point mass measures is a probability measure with compact support.
We introduce our notations briefly; for other ideas used here we refer the reader to [5] and [2] . Let H be a locally compact Hausdorff space. $M(H)$ denotes the space of all bounded Radon measures, $M^1(H)$ the subset of all probability measures and δ_x the point measure of $x \in H$. The support of a measure μ is denoted by suppμ. The space H is called a hypergroup if the following conditions are satisfied:

(i) There exists a map: $H \times H \to M^1(H)$, $(x, y) \mapsto \delta_x * \delta_y$, called convolution, which is continuous, where $M^1(H)$ bears the vague topology. The linear extension to $M(H)$, satisfies $\delta_x * (\delta_y * \delta_z) = (\delta_x * \delta_y) * \delta_z$;

(ii) supp$\delta_x * \delta_y$ is compact;

(iii) There exists a homeomorphism $H \to H$, $x \mapsto \overline{x}$, called involution, such that $x = \overline{\overline{x}}$ and $\overline{\delta_x * \delta_y} = \delta_{\overline{y}} * \delta_{\overline{x}}$.

(iv) There exists an element $e \in H$, called unit element, such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$;

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(v) $e \in \text{supp}\delta_x * \delta_{\overline{y}}$ if and only if $x = y$;

(vi) The map $(x, y) \mapsto \text{supp}\delta_x * \delta_y$ of $H \times H$ into the space of nonvoid compact subsets of H is continuous,.

It is still unknown if an arbitrary hypergroup admits a left Haar measure. It particular, it remains unknown whether every amenable hypergroup admits a left Haar measure. But all the known examples such as commutative hypergroups and central hypergroups do, for more information see [3] and [12]. In this case, one can define the convolution algebra $L^1(H)$ with multiplication $f*g(x) = \int f(x*y)g(\overline{y})d\lambda(y)$ for all $f, g \in L^1(H)$. Recall that $L^1(H)$ is a Banach subalgebra and an ideal in $M(H)$ with a bounded approximate identity [5]. It should be noted that these algebras include not only the group algebra $L^1(G)$ but also most of the semigroup algebras. Throughout this paper, unless explicitly stated otherwise, H will denote a hypergroup with a left Haar measure.
In this paper, among the other things, we present a few results in the theory of measures. We want to find out when $\mu \in M(H)$ and $\mu F \in L^\infty(H)^*$, imply $\mu \in L^1(H)$. Some necessary and sufficient conditions is found for a measure μ for which if $\mu F \in L^1(H)$ for every $F \in L^\infty(H)^*$, then $F \in L^1(H)$.

2 Main results

Let H be a hypergroup with left Haar measure λ. The first Arens product on $L^\infty(H)^*$ is defined in stages as follows.

(i) For each $\mu \in L^1(H)$ and for each $f \in L^\infty(H)$ we define $f\mu \in L^\infty(H)$ by $\langle f\mu, \nu\rangle = \langle f, \mu * \nu\rangle$;

(ii) For each $F \in L^\infty(H)^*$ and for each $f \in L^\infty(H)$ we define $Ff \in L^\infty(H)$ by $\langle Ff, \mu\rangle = \langle F, f\mu\rangle$;

(iii) Lastly, the first Arens product on $L^\infty(H)^*$ is the multiplication on $L^\infty(H)^*$ defined for all $F, G \in L^\infty(H)^*$ by $\langle GF, f\rangle = \langle G, Ff\rangle$.

$L^\infty(H)^*$ is a Banach algebra, for more details see [6] and [7]. For an element F fixed in $L^\infty(H)^*$, the mapping $G \to GF$ is weak*-weak* continuous. However, for an element F fixed in $L^\infty(H)^*$, the mapping $F \to GF$ is in general not weak*-weak* continuous unless F is in $L^1(H)$. Hence ,by making use of these explanations, the topological center of $L^\infty(H)^*$ with respect to the first Arens multiplication is defined as follows:

$$Z_t(L^\infty(H)^*) = \{F \in L^\infty(H)^*;\ \text{The mapping } G \to FG \text{ is weak}^* - \text{weak}^* \text{ continuous on } L^\infty(H)^*\}.$$

It is known that $Z_t(L^\infty(H)^*) = L^1(H)$, see [9].

Proposition 2.1. Let H be a hypergroup with left Haar measure λ. Then the following conditions are equivalent:

(i) there exists $0 \neq \mu \in L^1(H)$ such that if $F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$, then $F \in L^1(H)$;

(ii) H is discrete.

Proof . (i) implies (ii). Let μ be a nonzero element in $L^1(H)$ such that if $F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$, then $F \in L^1(H)$. Let $\{e_\alpha\}$ be an approximate identity for $L^1(H)$ of bound 1 [5]. Then we may suppose that $\{e_\alpha\}$ converges in the weak*-topology on $L^\infty(H)^*$, say to E [10]. It is easy to see that $FE = F$ for every $F \in L^\infty(H)^*$. Since $\mu E = \mu \in L^1(H)$, hence so is $E \in L^1(H)$. Thus the Banach algebra $L^1(H)$ admits a norm one identity, hence it is discrete.
(ii) implies (i). If H is a discrete hypergroup, then $\delta_e \in L^1(H)$ [8]. It is clear that $\delta_e F = F$ for every $F \in L^\infty(H)^*$. The result is immediate if we choose $\mu = \delta_e$. □

The following corollary is a direct consequence of proposition 2.1.

Corollary 2.2. Let H be a compact hypergroup. Then the following conditions are equivalent:

(i) there exists $0 \neq \mu \in L^1(H)$ such that if $F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$, then $F \in L^1(H)$;

(ii) H is finite.

Let H be a compact hypergroup. Let $\mu \in L^1(H)$. The mapping $x \mapsto \delta_x * \mu$ is weakly continuous [5]. Since H is compact, $\{\delta_x * \mu;\ x \in H\}$ is relatively weakly compact. By the Krein-Smulian theorem the closed, convex, circled hull of $\{\delta_x * \mu; x \in H\}$ is also weakly compact [4]. It follows that $\{\nu * \mu; \nu \in L^1(H), \|\nu\| \leq 1\}$ is relatively weakly compact. It is easy to see that $\{\mu F; F \in L^\infty(H)^*, \|F\| \leq 1\}$ is relatively weakly compact. Suppose that $F \in \{F \in L^\infty(H)^*; \|F\| \leq 1\}$ and $\{\nu_\alpha\}$ is a net in $\{\nu \in L^1(H); \|\nu\| \leq 1\}$ which converges to F in the weak*-topology. Therefore $\{\mu * \nu_\alpha\}$ converges to μF in the weak*-topology. Passing to a subnet if necessary, we can assume that $\{\mu * \nu_\alpha\}$ converges weak to a measure $\nu \in L^1(H)$. Consequently $\mu F = \nu \in L^1(H)$.
The next corollary is an immediate consequence of above explanation.

Corollary 2.3. Let H be an infinite compact hypergroup. Then $L^1(H)$ is a right ideal in $L^\infty(H)$ and $L^1(H)$ is not reflexive.

Let S be a non-empty subset of $M(H)$. The annihilator of S, denoted $Ann(S)$, is the set of all elements ν in $L^1(H)$ such that, for all μ in S, $\mu * \nu = 0$. In set notation,

$$Ann(S) = \{\nu \in L^1(H); \mu * \nu = 0 \text{ for all } \mu \in S\}.$$

Moreover, for every $\mu \in M(H)$ and $S \subseteq M(H)$, we define $\mu(S) = \{\mu * \nu; \nu \in S\}$.

Theorem 2.4. Let H be a hypergroup with left Haar measure λ. The following two properties of an element μ in $M(H)$ are equivalent:

(i) if $F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$, then $F \in L^1(H)$;

(ii) if $\{\nu_n\}$ is a bounded sequence in $L^1(H)$ such that $\{\mu * \nu_n\}$ is weakly convergent, then $\{\nu_n\}$ contains a weakly convergent subsequence.

Proof . Let $\{\nu_n\}$ be a bounded sequence in $L^1(H)$ such that $\{\mu * \nu_n\}$ is weakly convergent to $\nu \in M(H)$. Put $X = \{\nu_n; n \in \mathbb{N}\}$. We have

$$\overline{\{\mu * \nu_n; n \in \mathbb{N}\}} = \{\mu * \nu_n; n \in \mathbb{N}\} \cup \{\nu\} \subseteq L^1(H)$$

where closure is taken in the weak*-topology. Let F be a fixed element in $\overline{X}^{w^*}$ and let $\{\nu_\alpha\}$ be a net in X converging to F in the weak*-topology. Clearly, $\{\mu * \nu_\alpha\}$ converges to μF in the weak*-topology. On the other hand, $\overline{\{\mu * \nu_n; n \in \mathbb{N}\}}^{w^*} \subseteq L^1(H)$. Therefore $\{\mu * \nu_\alpha\}$ converges to $\mu F \in L^1(H)$ in the weak-topology [6]. By hypothesis, $F \in L^1(H)$. This shows that $\overline{X}^{w^*} = \overline{X}^{w}$. The Banach-Alaoglu theorem imply that X is relatively weakly compact [10]. Consequently the Eberlein-Smulian theorem imply that $\{\nu_n\}$ contains a subsequence $\{\nu_{k_n}\}$ which converges weakly to some $\eta \in L^1(H)$ [4].
To prove the converse, we show first that if $F \in L^\infty(H)^*$, $\|F\| \leq 1$ and $\mu F = 0$, then $F \in L^1(H)$. Let V be a neighborhood of F in the weak*-topology. Choose a convex neighborhood U of F such that $\overline{U}^{w^*} \subseteq V$. We have

$$F \in \overline{U \cap \overline{B}^{w^*}}^{w^*} = \overline{U \cap B}^{w^*},$$

where B is the unit ball of $L^1(H)$. Now, let $\eta \in \overline{\mu(U \cap B)}^{w^*} \cap L^1(H)$. There exists a net $\{\eta_\alpha\}$ in $U \cap B$ such that $\{\mu * \eta_\alpha\}$ converging to η in the weak*-topology. Since $\eta, \eta_\alpha \in L^1(H)$ for all α, so that $\{\mu * \eta_\alpha\}$ converging to η in the weak-topology. We can write

$$0 = \mu F \in \overline{\mu(U \cap B)}^{w^*} \cap L^1(H) \subseteq \overline{\mu(U \cap B)}^{w}.$$

Let $\{\nu_n\}$ be a bounded sequence in $U \cap B$ such that $\{\mu * \nu_n\}$ is weakly convergent to 0. By assumption, let $\{\nu_{n_k}\}$ be a subsequence in $\{\nu_n\}$ that converges to some $\nu \in L^1(H)$ in the weak-topology. Therefore $\{\mu * \nu_{n_k}\}$ converges to $\mu * \nu$ in the weak-topology. Thus $\mu * \nu = 0$. On the other hand, $\nu_{n_k} \in U \cap B$ for all $k \in \mathbb{N}$, and so $\nu \in \overline{U \cap B}^{w^*} \subseteq \overline{U}^{w^*} \subseteq V$. This shows that $\nu \in V \cap Ann(\mu)$. Consequently $F \in \overline{Ann(\mu)}^{w^*}$. By the Eberlein-Smulian theorem $Ann(\mu) \cap \{F; \|F\| \leq 1\}$ is relatively weakly compact [6]. Hence

$$\overline{Ann(\mu) \cap \{F; \|F\| \leq 1\}}^{w^*} = \overline{Ann(\mu) \cap \{F; \|F\| \leq 1\}}^{w}.$$

Therefore $\{\nu_{n_k}\}$ converges to F in the weak-topology. Thus $F \in L^1(H)$.

Now, let $F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$. An argument similar to the proof of above shows that

$$\mu F \in \overline{\mu(U \cap B)}^{w^*} \cap L^1(H) \subseteq \overline{\mu(U \cap B)}^{w}.$$

There exists a sequence $\{\nu_n\}$ in $U \cap B$ such that the sequence $\{\mu * \nu_n\}$ converging to μF in the weak-topology. By assumption, without loss of generality we may assume that $\{\nu_n\}$ converges to ν in the weak-topology. Therefore $\mu * \nu = \mu F$, i.e. $\mu(F - \nu) = 0$. We can write $F - \nu \in Ann(\mu)$, and so $F = \nu + \eta \in L^1(H)$ for some $\eta \in Ann(\mu)$. This completes the proof. □
Let S be non-empty subset of $L^1(H)$. The unite ball of S, denoted by B_S, is the set of all element μ in S such that $\|\mu\| < 1$. In set notation, $B_S = \{\mu \in S; \|\mu\| < 1\}$.

Proposition 2.5. Let H be a hypergroup with left Haar measure λ. The following two properties of an element μ in $M(H)$ are equivalent:

(i) if $F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$, then $F \in L^1(H)$;

(ii) $Ann(\mu)$ is reflexive and $\overline{\mu B_S} \subseteq \mu S$ for every closed subspace S of $L^1(H)$.

Proof . (i) implies (ii). Let $\{\nu_n\}$ be a sequence contained in ball of $Ann(\mu)$. Clearly $\mu * \nu_n = 0$ for all $n \in \mathbb{N}$. In accordance with condition (ii) in theorem 2.4, the sequence $\{\nu_n\}$ contains a subsequence $\{\nu_{n_k}\}$ which is weakly convergence to some $\nu \in L^1(H)$. Obviously $\{\mu * \nu_{n_k}\}$ converges to $\mu * \nu$ in the weak-topology, and so $\mu * \nu = 0$. This shows that $\nu \in Ann(\mu)$. It follows easily that the closed ball in $Ann(\mu)$ is relatively weakly compact. Consequently $Ann(\mu)$ is reflexive [10].
A similar argument show that if S is a closed subspace of $L^1(H)$, $\overline{\mu B_S} \subseteq \mu S$.
(ii) implies (i). Let $Ann(\mu)$ be reflexive and $\overline{\mu B_S} \subseteq \mu S$ for every closed subspace S of $L^1(H)$. In particular, $\overline{\mu B} \subseteq \mu L^1(H)$. Assume that there exist $F \in L^\infty(H)^*$ such that $\mu F \in L^1(H)$ and $F \notin L^1(H)$. Thus theorem 2.4 yields a sequence $\{\nu_n\}$ in $B \cap Ann(\mu)$ with no weakly convergent subsequence such that $\{\mu * \nu_n\}$ converges to 0 in the weak-topology. Since $Ann(\mu)$ is reflexive, passing to a subsequence of $\{\nu_n\}$ if necessary, the two sequences $\{\nu_n + \nu_0\}$ and $\{\nu_n + \nu_0 + Ann(\mu)\}$ can be assumed to be linearly independent. Let S be the closed subspace spanned by $\{\nu_1 + \nu_0, \nu_2 + \nu_0, ...\}$. Note that $\{\mu * (\nu_0 + \nu_n)\}$ converges to $\mu * \nu_0$ in the weak-topology. Since the weak closure $\overline{B_S}^{w}$ of B is equal to its original closure $\overline{B_S}$ [10], it follows that $\mu * \nu_0 \in 2\overline{\mu B_S}$. We shall show that $\mu * \nu_0 \notin \mu S$. If $\mu * \nu_0 = \mu * \nu$ for some $\nu = \sum_{i=1}^\infty \alpha_i(\nu_i - \nu_0) \in S$, we would have $\nu - \nu_0 \in Ann(\mu)$. Therefore

$$\begin{aligned} Ann(\mu) &= \sum_{i=1}^\infty \alpha_i(\nu_i - \nu_0) + Ann(\mu) - (\nu_0 + Ann(\mu)) \\ &= \Big[\sum_{i=1}^\infty \alpha_i - 1\Big]\nu_0 + \sum_{i=1}^\infty \alpha_i \nu_i + Ann(\mu). \end{aligned}$$

On the other hand, the set $\{\nu_n + \nu_0 + Ann(\mu)\}$ is linearly independent, and so $\sum_{i=1}^\infty \alpha_i = 1$ and $\alpha_i = 0$ for all i. This is a contradiction . □

Let $\mathbb{C}$ be the multiplicative group of all complex numbers. Let $\mu \in M(\mathbb{C})$. Consider the following assertions:

(i) if $\mu F \in L^1(\mathbb{C})$, then $F \in L^1(\mathbb{C})$;

(ii) $\nu \in M(\mathbb{C})$ and $\mu * \nu \in L^1(\mathbb{C})$ imply $\nu \in L^1(\mathbb{C})$.

Clearly (i) impies (ii). Are the converse implication true?

Proposition 2.6. Assume that H is a commutative hypergroup. Let $\mu \in M(H)$, and let $\{\mu F; F \in L^\infty(H)^*\} + L^1(H)$ be a dense subspace of $L^\infty(H)^*$. If $\mu F \in L^1(H)$, then $F \in L^1(H)$.

Proof . Let $F \in L^\infty(H)^*$ such that $\mu F \in L^1(H)$. Let $G \in L^\infty(H)^*$ and $\{\nu_\alpha\}$ be a net in $L^1(H)$ such that $\nu_\alpha \to G$ in the weak*-topology [10]. We can write

$$\mu F G = \lim_\alpha \mu F \nu_\alpha = \lim_\alpha \nu_\alpha * \mu F = G \mu F,$$

because H is commutative. This shows that $\mu FG = G\mu F$ for all $G \in L^\infty(H)^*$. Fix $G \in L^\infty(H)^*$. By assumption, $\{\mu F; F \in L^\infty(H)^*\} + L^1(H)$ is a dense subspace of $L^\infty(H)^*$. Consequently, we can find sequences $\{F_n\} \subseteq L^\infty(H)^*$ and $\{\mu_n\} \subseteq L^1(H)$ with $\{\mu F_n + \mu_n\}$ norm-convergent to G. Therefore

$$\begin{aligned} FG &= \lim_n F(\mu F_n + \mu_n) = \lim_n F\mu F_n + F\mu_n \\ &= \lim_n \mu F_n F + \mu_n F = \lim_n (\mu F_n + \mu_n)F = GF. \end{aligned}$$

Therefore $FG = GF$ for all $G \in L^\infty(H)^*$. We next show that $F \in Z_t(L^\infty(H)^*) = L^1(H)$ [9]. Indeed, if $\{G_\alpha\}$ is a net in $L^\infty(H)^*$ and $G_\alpha \to G$ in the weak*-topology, then

$$\begin{aligned} \lim_\alpha \langle FG_\alpha, f\rangle &= \lim_\alpha \langle G_\alpha F, f\rangle = \lim_\alpha \langle G_\alpha, Ff\rangle \\ &= \langle G, Ff\rangle = \langle GF, f\rangle, \end{aligned}$$

for all $f \in L^\infty(H)$. On the other hand, $\langle GF, f\rangle = \langle FG, f\rangle$. Hence $FG_\alpha \to FG$ (in the weak*-topology) implies that F is in the topological center of $L^\infty(H)^*$. This completes our proof. □

Recall that a basic sequence $\{x_n\}$ in a Banach space X is said to be boundedly complete if for each sequence of scalars $\{\alpha_n\}$, $\sum_{n=1}^\infty \alpha_n x_n$ is convergent whenever $\sup\{\| \sum_{i=1}^n \alpha_i x_i\|; n \in \mathbb{N}\} < \infty$.

Proposition 2.7. Let H be a hypergroup with a left Haar measure, and let $\mu \in M(H)$. Consider the following assertions:

(i) If $\{\mu_n\}$ is a basic sequence in B and $\sum_{i=1}^\infty \|\mu * \mu_n\| < \infty$, then $\{\mu_n\}$ is boundedly complete;

(ii) $F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$ imply $F \in L^1(H)$.

Then the implication $(i) \to (ii)$ hold.

Proof . Let us assume that (i) holds but there exists $F \in L^\infty(H)^*$ such that $\mu F \in L^1(H)$ and $F \notin L^1(H)$. Then, by Theorem 2.4, there exist $\nu \in L^1(H)$ and a bounded sequence $\{\nu_n\}$ with no weakly convergent subsequence such that $\{\mu * \nu_n\}$ converges to ν in the weak-topology. There exists $f \in L^\infty(H)$ such that $0 < \alpha_n = \langle f, \nu_n\rangle \to \alpha$ (after passage to a subsequence). It is clear that $\{\alpha_n^{-1}\nu_n\}$ contains a basic subsequence $\{\mu_n\}$ with no weakly convergent subsequence. Obviously the sequence $\{\mu * \mu_n\}$ tends to $\alpha^{-1}\nu$. Pick $\epsilon_n > 0$ so that $\sum_{i=1}^\infty \epsilon_n \leq 1$. Assume $i \geq 1$ and μ_{n_i} is picked. There exists $\mu_{n_{i+1}}$ such that $\|\mu * \mu_{n_i} - \mu * \mu_{n_{i+1}}\| < \epsilon_{i+1}$. Put $\eta_i = \mu_{n_i} - \mu_{n_{i+1}}$. By induction, this process define a basic sequence η_i in $L^1(H)$. Moreover $\sum_{i=1}^\infty \|\mu * \eta_i\| \leq 1$, and so $\{\eta_i\}$ is boundedly complete by hypothesis. On the other hand, $\sum_{i=1}^k \eta_i - \eta_{i+1} = \eta_{n_1} - \eta_{n_k}$ a bounded non-convergent sequence. This is a contradiction. □

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