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Approximate symmetries and conservation laws of forced fractional oscillator

Mehdi Nadjafikhah[a], Mansoureh Mirala[b,*], Mohamad Chaichi[b]

[a]School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, 16846–13114, Iran

[b]Department of Mathematics, Payame Noor University, Tehran 19395–4697, Iran

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Abstract

The approximate equation for the forced fractional oscillator is obtained by approximation of the Riemann- Liouville fractional derivatives. And the approximate symmetries and conservation laws of the forced fractional oscillator are derived when the system is in resonance.

Keywords: forced fractional oscillator, approximate symmetry, resonance, approximate conservation law
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1 Introduction

A wide class of fractional-order control systems can be described by ordinary fractional differential equations e.g., [1, 17]. Also fractional calculus is widely used in formulation of constitutive relations for viscoelastic materials e.g., [7, 13, 18]. Such constitutive relations have many advantages over classical ones. Problems of vibrations of the continuous structures (like beams, bars etc.), when one use fractional constitutive relations, leads to fractional differential equations similar to the equation of forced, harmonic, damped oscillator e.g., [8, 9, 14]. We will call such models: forced fractional oscillators. We consider here, Scott-Blair model which is the simplest model of this kind e.g., [3, 16]. Scott-Blair model is a generalization of Hook's law for perfectly elastic material and Newton law (stress proportional to the rate of strain) for perfectly viscous material; for$\alpha = 0$ one get Hook's law, while for $\alpha = 1$ Newton's law.

In this paper we approximate the fractional Scott-Blair oscillator by an perturbed integer order differential equation and then in resonance case, we get approximate symmetries and approximate conservation laws of the system.

2 Approximation of Fractional-Order Operator

Riemann-Liouville and Caputo fractional derivatives of $\alpha-$th order ($0 < \alpha < 1$) of function $x(t)$, denoted respectively by $D^{\alpha}_{RL}x(t)$ and $D^{\alpha}_{C}x(t)$, are defined below:

$$D^{\alpha}_{RL}x(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\Big(\int_0^t (t-\tau)^{-\alpha}x(\tau)d\tau\Big) \tag{2.1}$$

*Corresponding author
Email addresses: m_nadjafikhah@iust.ac.ir (Mehdi Nadjafikhah), m.mirala@yahoo.com (Mansoureh Mirala), chaichi@pnu.ac.ir (Mohamad Chaichi)

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$$D^{\alpha}_{C}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{dx(\tau)}{d\tau}d\tau$$

where Γ denote gamma function. These definitions are not equivalent. Condition under which Riemann-Liouville and Caputo fractional derivatives give the same result for a given function x(t) is: x(0)=0. We will consider only functions x(t) which fulfill this requirement e.g.,[10], so we will denote fractional derivative omitting index RL or C and write simply $D^{\alpha}x(t)$.

Assuming that the order of fractional differentiation α in (2.1) close to integer number n, we can write $\alpha = n\pm\epsilon$ where $\epsilon > 0$ is a small parameter (i.e. $\epsilon << 1$). We will have e.g.,[2] the following first order approximation in ϵ for the left-sided Riemann-Liouville fractional derivative:

$$_0D_t^{n\pm\epsilon}x \approx x^{(n)}(t)\pm\epsilon\{[\psi(n+1)-\ln t]x^{(n)}(t)-\sum_{k=0,k\neq n}^{\infty}\frac{(-1)^{k-n}n!}{(k-n)k!}t^{k-n}x^{(k)}(t)\} \tag{2.2}$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$ is the digamma function. A similar approximation can be derived for the right-sided Riemann-Liouville fractional derivative.

The approximation (2.2) makes possible to approximate a FDE by an integer-order differential equation with a small parameter. For example, an approximate equation for the FDE

$$F(t,x(t),\dot{x}(t),...,x^{(k)}(t),D^{\alpha}x(t))=0$$

has the form

$$F_{(0)}(t,x,\dot{x},...,x^{(l)})+\epsilon F_{(1)}(t,x,\dot{x},\ddot{x},...)\approx 0 \tag{2.3}$$

where $l=\max\{k,n\}$. Note that (2.3) can be considered as a specific perturbation of the integer-order differential equation $F_{(0)}(z)=0$, where $z=(t,x,\dot{x},...,x^{(l)})$, in which the function $F_{(1)}$ depends on all integer-order derivatives of function $x(t)$.

3 Approximate Fractional Forced Oscillator

Application of the Scott-Blair model of viscoelastic material to the vibration problems of the continuous structures like bars, beams etc., after using Rayleigh-Ritz method, leads to the following equation of motion e.g.,[8, 9, 14] for selected mode of free vibrations of given structure:

$$\ddot{x}(t)+\nu(\alpha)D^{\alpha}x(t)=f(t) \tag{3.1}$$

We will call model represented by the equation (3.1) the Scott-Blair oscillator. Zero initial conditions are assumed for oscillator: $x(0)=0,\dot{x}(0)=0$, which means that continuous structure is initially at rest. Unknown function $x(t)$ plays role of the free vibration displacement for the considered continuous structure and $f(t)$ being oscillating excitation (external loading) is a given known function:

$$f(t)=A_0\sin(\Omega_0 t)$$

where A_0 and Ω_0 are respectively amplitude and frequency of external loading. We impose the following conditions on function $\nu(\alpha)$:

$$\nu(0)=\omega_0^2,\nu(1)=2\omega_0$$

where ω_0 is undamped angular frequency. For Scott-Blair oscillator, for $\alpha=0$ we get classical undamped forced harmonic oscillator:

$$\ddot{x}(t)+\omega_0^2x(t)=f(t)$$

and for $\alpha=1$ we get equation of motion for a mass connected to the damper and loaded by a given external force:

$$\ddot{x}(t)+2\omega_0\dot{x}(t)=f(t).$$

If α is close to 0 then we get $\alpha=\epsilon$ and (2.2) takes the form

$$D^{\epsilon}x\approx x+\epsilon\{[\psi(1)-\ln t]x-\sum_{k=1}^{\infty}\frac{(-1)^k}{k!k}t^kx^{(k)}\}.$$

In this case from $\epsilon\ddot{x} \approx \epsilon(-\nu x + f(t)) \approx \epsilon(-\omega_0^2 x + f(t))$ we will have

$$\ddot{x}(t) + \nu x(t) - f(t) + \epsilon\nu\{[\psi(1) - \ln t - p(t)]x(t) - q(t)\dot{x}(t) - r(t)\} = 0 \tag{3.2}$$

such that

$$p(t) = \int \frac{\cos(\omega_0 t)}{t} dt - \ln t + \frac{\omega_0^2}{4} t^2 - c$$

$$q(t) = \frac{-1}{\omega_0} \int \frac{\sin(\omega_0 t)}{t} dt + t - c'$$

$$r(t) = \frac{A_0}{\omega_0^2} \sum_{k=1}^{\infty} \{\cos(\Omega_0 t)(\frac{\Omega_0}{\omega_0})^{2k-1}[\int \frac{\sin(\omega_0 t)}{t} dt + \sum_{n=1}^{k} (-1)^n \frac{(\omega_0 t)^{2n-1}}{(2n-1)!(2n-1)}]$$

$$- \sin(\Omega_0 t)(\frac{\Omega_0}{\omega_0})^{2k-2}[\int \frac{\cos(\omega_0 t)}{t} dt - ln(\omega_0 t) - \sum_{n=1}^{k-1} (-1)^n \frac{(\omega_0 t)^{2n}}{(2n)!(2n)}]\}$$

and c, c' are arbitrary constants. If α is close to 1 then $\alpha = 1 - \epsilon$ and in view of (2.2), we have

$$D^{1-\epsilon} x \approx \dot{x} - \epsilon\{[\psi(2) - \ln t]\dot{x} - \frac{x}{t} - \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k!(k-1)} t^{k-1} x^{(k)}\}.$$

It is follows from(3.1) that $\epsilon\ddot{x} \approx \epsilon(-\nu\dot{x} + f(t)) \approx \epsilon(-2\omega_0\dot{x} + f(t))$. Thus, we get, the approximate differential equation for (3.1) as

$$\ddot{x}(t) + \nu\dot{x}(t) - f(t) - \epsilon\nu\{[\psi(2) - \ln t - g(t)]\dot{x}(t) - \frac{1}{t} x(t) - h(t)\} = 0 \tag{3.3}$$

such that for arbitrary constant c''

$$g(t) = \frac{1}{2\omega_0} \int \frac{e^{2\omega_0 t}}{t^2} dt + \frac{1}{2\omega_0 t} - \ln t - \omega_0 t - c''$$

$$h(t) = \frac{A_0}{2\omega_0} \sum_{k=0}^{\infty} (-1)^k (\frac{\Omega_0}{2\omega_0})^{2k} \{\cos(\Omega_0 t)(\frac{\Omega_0}{2\omega_0})[\int \frac{e^{2\omega_0 t}}{2\omega_0 t^2} dt - \ln t - \sum_{n=0, n\neq 1}^{2k+3} \frac{(2\omega_0 t)^{n-1}}{n!(n-1)}]$$

$$- \sin(\Omega_0 t)[\int \frac{e^{2\omega_0 t}}{2\omega_0 t^2} dt - \ln t - \sum_{n=0, n\neq 1}^{2k+2} \frac{(2\omega_0 t)^{n-1}}{n!(n-1)}]\}.$$

3.1 Approximate Symmetries

Let $t = (t^1, ..., t^n)$ is a vector of independent variables and $x = (x^1, ..., x^m)$ is a vector of dependent variables. A set of approximate point transformations

$$\begin{aligned} \bar{t}^i &\approx f^i_{(0)}(t, x, a) + f^i_{(1)}(t, x, a) \qquad i = 1, .., n \\ \bar{x}^j &\approx g^j_{(0)}(t, x, a) + g^j_{(1)}(t, x, a) \qquad j = 1, .., m \end{aligned} \tag{3.4}$$

satisfying the conditions

$$\bar{t}^i|_{a=0} \approx t^i \qquad \bar{x}^j|_{a=0} \approx x^j$$

are called a one-parameter approximate transformation group if the group property is satisfied with the accuracy $o(\epsilon)$. The generator of an approximate transformation group (3.4) has the form

$$V \approx V_{(0)} + \epsilon V_{(1)} \equiv (\zeta^i_{(0)}(t, x) + \epsilon\zeta^i_{(1)}(t, x))\frac{\partial}{\partial t^i} + (\theta^j_{(0)}(t, x) + \epsilon\theta^j_{(1)}(t, x))\frac{\partial}{\partial x^j} \tag{3.5}$$

where

$$\zeta^i_{(0)}(t, x) = \frac{\partial f^i_{(0)}(t, x, a)}{\partial a}|_{a=0} \qquad \zeta^i_{(1)}(t, x) = \frac{\partial f^i_{(1)}(t, x, a)}{\partial a}|_{a=0}$$

$$\theta^j_{(0)}(t, x) = \frac{\partial g^j_{(0)}(t, x, a)}{\partial a}|_{a=0} \qquad \theta^j_{(1)}(t, x) = \frac{\partial g^j_{(1)}(t, x, a)}{\partial a}|_{a=0}$$

An approximate equation

$$F(t,x,x_{(1)},...,x_{(l)},\epsilon)=F_{(0)}(t,x,x_{(1)},...,x_{(l)})+\epsilon F_{(1)}(t,x,x_{(1)},...,x_{(l)},D_{t_1}^{l+1},D_{t_1}^{l+2},\cdots)\approx 0 \quad (3.6)$$

(such that $x_{(k)}=\{\frac{\partial^k x^j}{\partial t^{i_1}...\partial t^{i_k}}\}$) is said to be approximately invariant with respect to approximate transformation group (3.4) if and only if e.g.,[6]

$$V^{(k)}F|_{F\approx 0}=o(\epsilon)$$

or

$$[V_{(0)}^{(k)}F_{(0)}+\epsilon(V_{(1)}^{(k)}F_{(0)}+V_{(0)}^{(k)}F_{(1)})]=o(\epsilon). \quad (3.7)$$

In which k is order of equation and $V^{(k)}$ is k-th order prolongation of V. The operator(3.5) satisfying equation (3.7) is called an infinitesimal approximate symmetry of Eq.(3.6). If (3.6) admits an approximate transformation group with the generator (3.5), where $V_{(0)}\neq 0$, then the operator $V_{(0)}$ is an exact symmetry of the unperturbed equation and it is called a stable symmetry of this equation.

Suppose that the system is on resonance, that is, the angular frequency and the loading frequency are equal. In other words, Ω_0 is resonant frequency i.e in equations(3.2),(3.3) $\Omega_0=\omega_0$ e.g.,[11] and the amplitude of the oscillator increases. Then we get the approximate symmetries of the Eq.(3.1) in both cases when α is close to 0 and α is close to 1.

Case1: The exact symmetries of unperturbed equation corresponding to Eq.(3.2) are

$$\mathbf{v}_{(0)}^1=\frac{\partial}{\partial t}+\frac{A_0}{2}t\sin(\omega_0 t)\frac{\partial}{\partial x},\quad \mathbf{v}_{(0)}^2=\cos(\omega_0 t)\frac{\partial}{\partial x},\quad \mathbf{v}_{(0)}^3=\sin(\omega_0 t)\frac{\partial}{\partial x}$$

So the approximate symmetries are

$$\begin{aligned}
\mathbf{v}_1=&\sin(\omega_0 t)\frac{\partial}{\partial x}+\epsilon[((\frac{\psi(1)}{2}+\frac{c}{2}+\frac{5}{16})\omega_0 t-\frac{1}{24}\omega_0^3t^3)\cos(\omega_0 t)+(-\frac{c'}{2}t+\frac{5}{16}t^2)\omega_0^2\sin(\omega_0 t)\\
&-\frac{1}{2}\omega_0 t\cos(\omega_0 t)Ci(\omega_0 t)-\frac{1}{2}\omega_0 t\sin(\omega_0 t)Si(\omega_0 t)+\frac{1}{2}\sin(\omega_0 t)Ci(\omega_0 t)-\frac{1}{2}\cos(\omega_0 t)Si(\omega_0 t)]\frac{\partial}{\partial x},\\
\mathbf{v}_2=&\cos(\omega_0 t)\frac{\partial}{\partial x}+\epsilon[(-\frac{c'}{2}t+\frac{5}{16}t^2)\omega_0^2\cos(\omega_0 t)-((\frac{\psi(1)}{2}+\frac{c}{2}+\frac{5}{16})\omega_0 t-\frac{1}{24}\omega_0^3t^3)\sin(\omega_0 t)\\
&+\frac{1}{2}\omega_0 t\sin(\omega_0 t)Ci(\omega_0 t)-\frac{1}{2}\omega_0 t\cos(\omega_0 t)Si(\omega_0 t)-\frac{1}{2}]\frac{\partial}{\partial x},\\
\mathbf{v}_3=&\epsilon(\frac{\partial}{\partial t}+\frac{A_0}{2}t\sin(\omega_0 t)\frac{\partial}{\partial x}),\\
\mathbf{v}_4=&\epsilon\cos(\omega_0 t)\frac{\partial}{\partial x},\\
\mathbf{v}_5=&\epsilon\sin(\omega_0 t)\frac{\partial}{\partial x},
\end{aligned}$$

where $Ci(t)=\gamma+\ln t-\int_0^t\frac{1-\cos x}{x}dx$ and $Si(t)=\int_0^t\frac{\sin x}{x}dx$. An optimal system of one dimensional approximate Lie algebras of the equation is provided by

$$\mathbf{v}_1+\alpha\mathbf{v}_5 \qquad \mathbf{v}_2+\alpha\mathbf{v}_1+\beta\mathbf{v}_4$$

$$\mathbf{v}_3+\alpha\mathbf{v}_2+\beta\mathbf{v}_1 \qquad \mathbf{v}_4+\alpha\mathbf{v}_5 \qquad \mathbf{v}_5$$

Case2: The exact symmetries of unperturbed equation corresponding to Eq.(3.3) are

$$\mathbf{v}_{(0)}^1=\frac{\partial}{\partial x} \qquad \mathbf{v}_{(0)}^2=e^{-2\omega t}\frac{\partial}{\partial x}$$

$$\mathbf{v}_{(0)}^3=e^{\omega_0 t}\frac{\partial}{\partial t}+\frac{A_0}{10\omega_0}e^{\omega_0 t}(\cos(\omega_0 t)+2\sin(\omega_0 t))\frac{\partial}{\partial x}$$

$$\mathbf{v}_{(0)}^4=\frac{\partial}{\partial t}+\frac{A_0}{5\omega_0}(-\cos(\omega_0 t)+2\sin(\omega_0 t))\frac{\partial}{\partial x}$$

So the approximate symmetries are

$$\mathbf{v}_1 = \epsilon\frac{\partial}{\partial x} \qquad \mathbf{v}_2 = \epsilon e^{-2\omega_0 t}\frac{\partial}{\partial x}$$

$$\mathbf{v}_3 = \epsilon e^{2\omega_0 t}(\frac{1}{\omega_0}\frac{\partial}{\partial t} + \frac{A_0}{\omega_0^2}(\frac{2}{5}\sin(\omega_0 t) - \frac{1}{5}\cos(\omega_0 t))\frac{\partial}{\partial x})$$

$$\mathbf{v}_4 = \epsilon(\frac{\partial}{\partial t} + \frac{A_0}{\omega_0}(\frac{2}{5}\sin(\omega_0 t) - \frac{1}{5}\cos(\omega_0 t))\frac{\partial}{\partial x})$$

$$\mathbf{v}_5 = (1+e^{-2\omega_0 t})\frac{\partial}{\partial x} + \epsilon[((2\psi(2)+2c''+1)\omega_0 t + \omega_0^2 t^2)e^{-2\omega_0 t} - e^{-2\omega_0 t}(2\omega_0 t Ei(2\omega_0 t) - Ei(2\omega_0 t)) - \ln t + 1]\frac{\partial}{\partial x}$$

where $Ei(t)$ is the exponential integral function. An optimal system of one dimensional approximate Lie algebras of the equation is provided by

$$\mathbf{v}_1 \qquad \mathbf{v}_2 + \alpha\mathbf{v}_1 \qquad \mathbf{v}_3 + \alpha\mathbf{v}_2$$

$$\mathbf{v}_4 + \alpha\mathbf{v}_3 + \beta\mathbf{v}_1 \qquad \mathbf{v}_5 + \alpha\mathbf{v}_4 + \beta\mathbf{v}_3$$

3.2 Approximate conservation laws

If approximate symmetries of Eq.(3.6) are known, then corresponding approximate conservation laws can be constructed using the concept of nonlinear self-adjointness e.g.,[5, 12]. This concept is applicable for integer-order differential equations with a small parameter e.g.,[6]. Let $\mathcal{L}$ be the formal Lagrange of equation(3.6):

$$\mathcal{L} \approx \mathcal{L}_{(0)} + \epsilon\mathcal{L}_{(1)} \equiv yF_{(0)} + \epsilon yF_{(1)}$$

hence, the adjoint equations of Eq.(3.6) are defined as

$$\frac{\delta\mathcal{L}}{\delta x} \approx F^*_{(0)}(t,x,y,\dot{x},\dot{y},\cdots,x^{(l)},y^{(l)}) + \epsilon F^*_{(1)}(t,x,y,\dot{x},\dot{y},\cdots,x^{(l)},y^{(l)},D^{l+1}_{t_1}x,D^{l+1}_{t_1}y,D^{l+2}_{t_1}x,D^{l+2}_{t_1}y,...) \approx 0 \tag{3.8}$$

where $\frac{\delta\mathcal{L}}{\delta x}$ is the variational derivative written in terms of the total derivative operator D_i:

$$\frac{\delta\mathcal{L}}{\delta x} = \frac{\partial}{\partial x} + \sum_{k=1}^{\infty}(-1)^k D_{i_1}...Di_k\frac{\partial^k}{\partial x_{i_1...i_k}}.$$

If we consider $y \approx \phi_{(0)}(t,x) + \epsilon\phi_{(1)}(t,x) \neq 0$, we have

$$\mathcal{L} \approx \phi_{(0)}F_{(0)} + \epsilon(\phi_{(0)}F_{(1)} + \phi_{(1)}F_{(0)})$$

and if it satisfies the nonlinear self adjoint condition:

$$F^*_{(0)}|_{y\approx\phi_{(0)}+\epsilon\phi_{(1)}} + \epsilon F^*_{(1)}|_{y\approx\phi_{(0)}} \approx \lambda_{(0)}F_{(0)} + \epsilon(\lambda_{(0)}F_{(1)} + \lambda_{(1)}F_{(0)}). \tag{3.9}$$

In which $\lambda_{(0)}$ and $\lambda_{(1)}$ are to be determined coefficients. Any approximate symmetry equation (3.5) of Eq.(3.6) leads to a conservation law

$$D_i(C^i) \approx 0, \qquad C^i \approx C^i_{(0)} + \epsilon C^i_{(1)}$$

where the components C^i are obtained by

$$\begin{aligned} C^i_{(0)} &= W_{(0)}(\tfrac{\partial\mathcal{L}_{(0)}}{\partial x_i} + \textstyle\sum_{k=1}^{l-1}(-1)^k D_{i_1}...Di_k\frac{\partial^k\mathcal{L}_{(0)}}{\partial x_{ii_1...i_k}}) \\ &+ \textstyle\sum_{s=1}^{l-1}D_{j_1}...D_{j_s}(W_{(0)})(\frac{\partial\mathcal{L}_{(0)}}{\partial x_{ij_1...j_s}} + \sum_{k=1}^{l-s-1}(-1)^k D_{i_1}...Di_k\frac{\partial^k\mathcal{L}_{(0)}}{\partial x_{ii_1...i_kj_1...j_s}}) \end{aligned} \tag{3.10}$$

$$\begin{aligned} C^i_{(1)} =& W_{(1)}(\frac{\partial\mathcal{L}_{(0)}}{\partial x_i} + \sum_{k=1}^{l-1}(-1)^k D_{i_1}...Di_k\frac{\partial^k\mathcal{L}_{(0)}}{\partial x_{ii_1...i_k}}) + \sum_{s=1}^{l-1}D_{j_1}...D_{j_s}(W_{(1)})(\frac{\partial\mathcal{L}_{(0)}}{\partial x_{ij_1...j_s}} + \sum_{k=1}^{l-s-1}(-1)^k D_{i_1}...Di_k\frac{\partial^k\mathcal{L}_{(0)}}{\partial x_{ii_1...i_kj_1...j_s}}) \\ &+ W_{(0)}(\frac{\partial\mathcal{L}_{(1)}}{\partial x_i} + \sum_{k=1}^{l-1}(-1)^k D_{i_1}...Di_k\frac{\partial^k\mathcal{L}_{(0)}}{\partial x_{ii_1...i_k}}) \\ &+ \sum_{s=1}^{l-1}D_{j_1}...D_{j_s}(W_{(0)})(\frac{\partial\mathcal{L}_{(1)}}{\partial x_{ij_1...j_s}} + \sum_{k=1}^{l-s-1}(-1)^k D_{i_1}...Di_k\frac{\partial^k\mathcal{L}_{(0)}}{\partial x_{ii_1...i_kj_1...j_s}}) \end{aligned} \tag{3.11}$$

in which $W_{(0)}=\theta_{(0)}+\epsilon\zeta^i_{(0)}x_i$, $W_{(1)}=\theta_{(1)}+\epsilon\zeta^i_{(1)}x_i$. An approximate conservation law is called a trivial approximate conservation law if

$$D_i(C^i_{(0)})\equiv 0, \qquad D_i(C^i_{(1)})\equiv 0.$$

Case1: By choosing approximate formal Lagrange

$$\mathcal{L}\approx(\phi_{(0)}+\epsilon\phi_{(1)})[\ddot{x}+\nu x-f(t)+\epsilon\nu\{(\psi(1)-\ln t-p(t))x-q(t)\dot{x}-r(t)\}]$$

we obtain adjoint equation using Eq.(3.8) as:

$$F^*\approx\ddot{y}+\omega_0^2 y+\epsilon\omega_0^2[q(t)\dot{y}+(\psi(1)-\ln t-p(t)+q'(t))y].$$

It is easy to achieve an approximate formal Lagrange by solving characteristic equation of the Eq.(3.9):

$$\begin{aligned} y=&c_0\cos(\omega_0 t)+\epsilon[(\frac{-3}{16}\omega_0^2 c_0 t^2+\frac{1}{2}\omega_0^2 c' c_0 t+l_1)\cos(\omega_0 t)\\ &+\{\frac{1}{24}\omega_0^3 c_0 t^3-\frac{1}{2}\omega_0 c_0(\psi(1)+\frac{5}{8}+c)t+l_2\}\sin(\omega_0 t)+\frac{1}{2}c_0\cos(2\omega_0 t)\\ &+\frac{1}{2}\omega_0 c_0 t\sin(\omega_0 t)Ci(\omega_0 t)+\frac{1}{2}\omega_0 c_0 t\cos(\omega_0 t)Si(\omega_0 t)]\end{aligned}$$

and $\mathcal{L}\approx\mathcal{L}_{(0)}+\epsilon\mathcal{L}_{(1)}$ where

$$\mathcal{L}_{(0)}=c_0\cos(\omega_0 t)(\ddot{x}+\omega_0^2 x-A_0\sin(\omega_0 t))$$

$$\mathcal{L}_{(1)}=\phi_{(1)}(\ddot{x}+\omega_0^2 x-A_0\sin(\omega_0 t))+c_0\omega_0^2\cos(\omega_0 t)[-q(t)\dot{x}+(\psi(1)-\ln t-p(t))x-r(t)]$$

such that c_0, l_1, l_2 are arbitrary constants. Applying the formula equations(3.10) and (3.11), we perform all computations to approximate conservation laws. Finally, we obtain

$$\begin{aligned} C^t_{(0)}=&W_{(0)}c_0\omega_0\sin(\omega_0 t)+D_t(W_{(0)})c_0\cos(\omega_0 t)\\ C^t_{(1)}=&W_{(1)}c_0\omega_0\sin(\omega_0 t)+D_t(W_{(1)})c_0\cos(\omega_0 t)+W_{(0)}(-c_0\omega_0^2\cos(\omega_0 t)q(t)-D_t\phi_{(1)})+D_t(W_{(0)})\phi_{(1)}.\end{aligned}$$

The only nontrivial approximate conservation law of the Eq.(3.2) corresponding to approximate symmetry $\mathbf{v}_3$ is

$$\mathbf{c}=\epsilon(-c_0\cos(\omega_0 t)\ddot{x}-c_0\omega_0\sin(\omega_0 t)\dot{x}+\frac{A_0}{2}c_0\omega_0 t+\frac{A_0}{4}c_0\sin(2\omega_0 t)).$$

Case2: By approximate equation(3.3):

$$\mathcal{L}\approx(\phi_{(0)}+\epsilon\phi_{(1)})[\ddot{x}+\nu\dot{x}-f(t)-\epsilon\nu\{(\psi(2)-\ln t-g(t))\dot{x}-\frac{1}{t}x-h(t)\}]$$

in this case the adjoint equation is

$$F^*\approx\ddot{y}-2\omega_0\dot{y}+2\epsilon\omega_0[(\psi(2)-\ln t-g(t))\dot{y}-g'(t)y]$$

therefore

$$\begin{aligned} y=&c_0'+c_0'e^{2\omega_0 t}+\epsilon[-c_0'\ln t+l_0'-c_0'\omega_0 t+c_0'e^{4\omega_0 t}\\ &+\{l'+c_0'Ei(2\omega_0 t)-2c_0'\omega_0(Ei(2\omega_0 t)+\psi(2)+c''+\frac{1}{2})t+c_0'\omega_0^2 t^2\}e^{2\omega_0 t}]\end{aligned}$$

and $\mathcal{L}_{(0)}=c_0'(1+e^{2\omega_0 t})(\ddot{x}+2\omega_0\dot{x}-A_0\sin(\omega_0 t))$

$$\mathcal{L}_{(1)}=\phi_{(1)}(\ddot{x}+2\omega_0\dot{x}-A_0\sin(\omega_0 t))-2\omega_0 c_0'(1+e^{2\omega_0 t})[(\psi(2)-\ln t-g(t))\dot{x}-\frac{x}{t}-h(t)].$$

Here c_0', l_0', l' are arbitrary constants. Then

$$\begin{aligned} C^t_{(0)}=&2W_{(0)}c_0'\omega_0+D_t(W_{(0)})(c_0'+c_0'e^{2\omega_0 t})\\ C^t_{(1)}=&2W_{(1)}c_0'\omega_0+D_t(W_{(1)})(c_0'+c_0'e^{2\omega_0 t})+W_{(0)}[2\omega_0\phi_{(1)}-2\omega_0(c_0'+c_0'e^{2\omega_0 t})(\psi(2)-\ln t-g(t))-D_t\phi_{(1)}]\\ &+D_t(W_{(0)})\phi_{(1)}.\end{aligned}$$

By computing the components of approximate conserved vectors we find

$$\mathbf{c}_1 = \epsilon \frac{c'_0}{\omega_0} e^{2\omega_0 t}[-(1+e^{2\omega_0 t})\ddot{x} - 2\omega_0(2+e^{2\omega_0 t})\dot{x} + A_0(-\frac{2}{5}\cos(\omega_0 t) + (\frac{9}{5} + e^{2\omega_0 t})\sin(\omega_0 t))]$$

$$\mathbf{c}_2 = \epsilon c'_0[-(1+e^{2\omega_0 t})\ddot{x} - 2\omega_0\dot{x} + A_0\sin(\omega_0 t) + \frac{A_0}{5}e^{2\omega_0 t}(2\cos(\omega_0 t) + \sin(\omega_0 t))]$$

which are the non-trivial conservation laws corresponding to $\mathbf{v}_3, \mathbf{v}_4$ respectively.

4 Conclusion

We obtained the approximate equation of the fractional oscillator for nearly integer orders 0 and 1, which is applicable to calculating approximate symmetries, approximate conservation laws and approximate solutions. In this work, in special case, when the system is on resonance, we compuetd the approximate symmetries and conservation laws.

Appendix A appendix:calculation of approximate equations

We want to demonstrate the calculation steps of the equations(3.2) and (3.3).

Case1: When $\alpha = \epsilon$, the fractional derivative takes the form

$$D^{\epsilon}x \approx x + \epsilon\{[\psi(1) - \ln t]x - \sum_{k=1}^{\infty} \frac{(-1)^k}{k!k} t^k x^{(k)}\}. \tag{A.1}$$

In this case, we have $\epsilon\ddot{x} \approx \epsilon(-\omega_0^2 x + f(t))$ in the same way

$$\begin{aligned}
\epsilon x^{(3)} &\approx \epsilon(-\omega_0^2\dot{x} + f'(t)) \\
\epsilon x^{(4)} &\approx \epsilon(\omega_0^4 x - \omega_0^2 f(t) + f''(t)) \\
\epsilon x^{(5)} &\approx \epsilon(\omega_0^4\dot{x} - \omega_0^2 f'(t) + f'''(t)) \\
\epsilon x^{(6)} &\approx \epsilon(-\omega_0^6 x + \omega_0^4 f(t) - \omega_0^2 f''(t) + f^{(4)}(t)) \\
&\vdots
\end{aligned}$$

Using these, we can obtain representations for $\epsilon x^{(k)}(k = 3, 4, ...)$ as functions of ϵx and $\epsilon\dot{x}$ and by substituting in the infinite series in Eq.(A.1), we can derive the coefficient of x as $p(t) = \sum_{k=1}^{\infty} \frac{(-1)^k(\omega_0 t)^{2k}}{(2k)!(2k)}$. So

$$tp'(t) = \sum_{k=2}^{\infty} \frac{(-1)^k(\omega_0 t)^{2k}}{(2k)!} = \cos(\omega_0 t) - 1 + \frac{\omega_0^2}{2}t^2$$

$$p(t) = \int \frac{\cos(\omega_0 t)}{t}dt - \ln t + \frac{\omega_0^2}{4}t^2 - c$$

such that c is an arbitrary constant. Similarly, the coefficient of $\dot{x}$ is $q(t) = -\sum_{k=0}^{\infty} \frac{(-1)^k\omega_0^{2k}t^{2k+1}}{(2k+1)!(2k+1)}$ then

$$-t\omega_0 q'(t) = \sum_{k=1}^{\infty} \frac{(-1)^k(\omega_0 t)^{2k+1}}{(2k+1)!} = \sin(\omega_0 t) - \omega_0 t$$

$$q(t) = \frac{-1}{\omega_0}\int \frac{\sin(\omega_0 t)}{t}dt + t - c'$$

such that c' is constant. Finally, we get the remaining of the series as

$$\sum_{k=1}^{\infty}(-\omega_0^2)^{k-1}\frac{t^{2k}}{(2k)!(2k)}f(t) + \sum_{k=1}^{\infty}(-\omega_0^2)^{k-1}\frac{t^{2k+1}}{(2k+1)!(2k+1)}f'(t) +$$
$$\sum_{k=2}^{\infty}(-\omega_0^2)^{k-2}\frac{t^{2k}}{(2k)!(2k)}f''(t) + \sum_{k=2}^{\infty}(-\omega_0^2)^{k-2}\frac{t^{2k+1}}{(2k+1)!(2k+1)}f'''(t) + \cdots$$

Since $f(t) = A_0 \sin(\Omega_0 t)$ then by summation of the coefficients of $A_0 \sin(\Omega_0 t)$ and $A_0 \cos(\Omega_0 t)$ we obtain

$$\begin{aligned}P(t) =& \sum_{k=1}^{\infty}(-1)^{k-1}\Omega_0^{2k-2}\sum_{n=k}^{\infty}(-\omega_0^2)^{n-k}\frac{t^{2n}}{(2n)!(2n)}\\ =&\frac{-1}{\omega_0^2}\sum_{k=1}^{\infty}(\frac{\Omega_0}{\omega_0})^{2k-2}\sum_{n=k}^{\infty}(-1)^n\frac{(\omega_0 t)^{2n}}{(2n)!(2n)}\\ =&-\frac{1}{\omega_0^2}\sum_{k=1}^{\infty}(\frac{\Omega_0}{\omega_0})^{2k-2}[\int\frac{\cos(\omega_0 t)}{t}dt - ln(\omega_0 t) - \sum_{n=1}^{k-1}(-1)^n\frac{(\omega_0 t)^{2n}}{(2n)!(2n)}]\end{aligned}$$

as the coefficient of $A_0 \sin(\Omega_0 t)$ and

$$\begin{aligned}Q(t) =& \sum_{k=1}^{\infty}(-1)^{k}\Omega_0^{2k-1}\sum_{n=k}^{\infty}(-\omega_0^2)^{n-k}\frac{t^{2n+1}}{(2n+1)!(2n+1)}\\ =&\frac{-1}{\omega_0^2}\sum_{k=1}^{\infty}(\frac{\Omega_0}{\omega_0})^{2k-1}\sum_{n=k}^{\infty}(-1)^n\frac{(\omega_0 t)^{2n+1}}{(2n+)!(2n+1)}\\ =&\frac{1}{\omega_0^2}\sum_{k=1}^{\infty}(\frac{\Omega_0}{\omega_0})^{2k-1}[\int\frac{\sin(\omega_0 t)}{t}dt + \sum_{n=1}^{k}(-1)^n\frac{(\omega_0 t)^{2n-1}}{(2n-1)!(2n-1)}]\end{aligned}$$

as the coefficient of $A_0 \cos(\Omega_0 t)$.

Case2: When $\alpha = 1-\epsilon$, the fractional derivative takes the form

$$D^{1-\epsilon}x \approx \dot{x} - \epsilon\left\{[\psi(2) - \ln t]\dot{x} - \frac{x}{t} - \sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{k!(k-1)}t^{k-1}x^{(k)}\right\}. \tag{A.2}$$

In this case, we have $\epsilon\ddot{x} \approx \epsilon(-2\omega_0\dot{x} + f(t))$ in the same way

$$\begin{aligned}\epsilon x^{(3)} &\approx \epsilon(4\omega_0^2\dot{x} - 2\omega_0 f(t) + f'(t))\\ \epsilon x^{(4)} &\approx \epsilon(-8\omega_0^3\dot{x} + 4\omega_0^2 f(t) - 2\omega_0 f'(t) + f''(t))\\ \epsilon x^{(5)} &\approx \epsilon(16\omega_0^4\dot{x} - 8\omega_0^3 f(t) + 4\omega_0^2 f'(t) - 2\omega_0 f''(t) + f'''(t))\\ &\vdots\end{aligned}$$

Using these, we can obtain representations for $\epsilon x^{(k)}(k = 3, 4, ...)$ as function of $\epsilon\dot{x}$ and by substituting in the infinite series in Eq.(A.2), we get the coefficient of $\dot{x}$ as $g(t) = \sum_{k=2}^{\infty}\frac{(2\omega_0 t)^{k-1}}{k!(k-1)}$ then

$$\begin{aligned}2\omega_0 t^2 g'(t) &= \sum_{k=3}^{\infty}\frac{(2\omega_0 t)^k}{k!} = e^{2\omega_0 t} - 1 - 2\omega_0 t - \frac{(2\omega_0 t)^2}{2!}\\ g(t) &= \frac{1}{2\omega_0}\int\frac{e^{2\omega_0 t}}{t^2}dt + \frac{1}{2\omega_0 t} - \ln t - \omega_0 t - c''\end{aligned}$$

such that c'' is constant. Also the coefficient of $A_0 \sin(\Omega_0 t)$ is

$$\begin{aligned}I(t) =&\frac{1}{2\omega_0}\sum_{n=0}^{\infty}(-1)^{n+1}(\frac{\Omega_0}{2\omega_0})^{2n}\sum_{k=2n+2}^{\infty}\frac{(2\omega_0 t)^{k-1}}{k!(k-1)} = \sum_{n=0}^{\infty}(-1)^{n+1}(\frac{\Omega_0}{2\omega_0})^{2n}\left[\int\frac{e^{2\omega_o t}}{(2\omega_0 t)^2}dt - \sum_{k=0}^{2n+2}\int\frac{(2\omega_0 t)^k}{k!(2\omega_0 t)^2}dt\right]\\ =&\sum_{n=0}^{\infty}(-1)^{n+1}(\frac{\Omega_0}{2\omega_0})^{2n}\left[\int\frac{e^{2\omega_o t}}{(2\omega_0 t)^2}dt - \frac{1}{2\omega_0}(lnt + \sum_{k=0,k\neq 1}^{2n+2}\frac{(2\omega_0 t)^{k-1}}{k!(k-1)})\right]\end{aligned}$$

and the coefficient of $A_0\cos(\Omega_0 t)$ is

$$
\begin{aligned}
J(t) =&\frac{1}{2\omega_0}\sum_{n=0}^{\infty}(-1)^n(\frac{\Omega_0}{2\omega_0})^{2n+1}\sum_{k=2n+3}^{\infty}\frac{(2\omega_0 t)^{k-1}}{k!(k-1)} = \sum_{n=0}^{\infty}(-1)^n(\frac{\Omega_0}{2\omega_0})^{2n+1}\left[\int\frac{e^{2\omega_o t}}{(2\omega_0 t)^2}dt - \sum_{k=0}^{2n+3}\int\frac{(2\omega_0 t)^k}{k!(2\omega_0 t)^2}dt\right]\\
=&\sum_{n=0}^{\infty}(-1)^n(\frac{\Omega_0}{2\omega_0})^{2n+1}\left[\int\frac{e^{2\omega_o t}}{(2\omega_0 t)^2}dt - \frac{1}{2\omega_0}(lnt + \sum_{k=0,k\neq 1}^{2n+3}\frac{(2\omega_0 t)^{k-1}}{k!(k-1)})\right].
\end{aligned}
$$

Appendix B appendix:calculation of approximate symmetries and conservation laws

We consider the resonance state, $\Omega_0 = \omega_0$.

Case1: Let us consider the approximate group generators(3.5). The exact symmetry of unperturbed equation $\ddot{x}(t) + \omega_0^2 x(t) = A_0\sin(\Omega_0 t)$ is

$$
V_{(0)} = c_1\frac{\partial}{\partial t} + [(\hat{c} + \frac{c_1}{2}A_0 t)\sin(\omega_0 t) + c_2\cos(\omega_0 t)]\frac{\partial}{\partial x} \tag{B.1}
$$

s.t $\hat{c}, c_1, c_2$ are constant. We need to determine the auxiliary function H by virtue of Eq.(3.7) e.g.,[15] , i.e by the equation

$$
H = \frac{1}{\epsilon}[V_{(0)}(F_{(0)} + \epsilon F_{(1)})|_{F_{(0)}+\epsilon F_{(1)}=0}].
$$

Substituting the expresion (B.1) of the generator into above equation, we obtain the auxiliary function

$$
\begin{aligned}
H =& -c_1\omega_0^2[(\frac{-\sin(\omega_0 t)}{\omega_0 t} + 1)\dot{x} + (\frac{\cos(\omega_0 t)}{t} + \frac{\omega_0^2}{2}t)x + (P'(t) - \omega_0 Q(t))A_0\sin(\omega_0 t) + (Q'(t)+\\
&\omega_0 P(t))A_0\cos(\omega_0 t)] + \omega_0^2(\psi(1) - \int\frac{\cos(\omega_0 t)}{t}dt - \frac{\omega_0^2}{4}t^2)[(\hat{c} + \frac{c_1}{2}A_0 t)\sin(\omega_0 t)+\\
&c_2\cos(\omega_0 t)] + \omega_0^2[(\frac{c_1}{2}A_0 - c_2\omega_0)\sin(\omega_0 t) + (\hat{c} + \frac{c_1}{2}A_0 t)\omega_0\cos(\omega_0 t)](\int\frac{\sin(\omega_0 t)}{\omega_0 t}dt - t)
\end{aligned}
$$

Now calculate the operators $V_{(1)}$ by solving the determining equation for deformations

$$
V_1^{(k)}F_0(z)|_{F_0(z)} + H = 0.
$$

Finally the generator of approximate transformation group is

$$
\begin{aligned}
&[\hat{c}\sin(\omega_0 t) + c_2\cos(\omega_0 t)]\frac{\partial}{\partial x} + \epsilon[\bar{c}\frac{\partial}{\partial t} + (\{(\frac{\psi(1)}{2}\hat{c} + \frac{c}{2}\hat{c} + \frac{5\hat{c}}{16} - \frac{\omega_0}{2}c_2 c' - \frac{\hat{c}}{2}Ci(\omega_0 t) - \frac{c_2}{2}Si(\omega_0 t))\omega_0 t + c_2\frac{5}{16}\omega_0^2 t^2\\
&-\frac{\hat{c}}{24}\omega_0^3 t^3 - \frac{\hat{c}}{2}Si(\omega_0 t) + k_0\}\cos(\omega_0 t) + \{(-\frac{\psi(1)}{2}c_2 - \frac{c}{2}c_2 - \frac{5c_2}{16} - \frac{\omega_0}{2}\hat{c}c' + \bar{c}\frac{A_0}{2\omega_0} - \frac{\hat{c}}{2}Si(\omega_0 t) + \frac{c_2}{2}Ci(\omega_0 t))\omega_0 t\\
&+\hat{c}\frac{5}{16}\omega_0^2 t^2 + \frac{c_2}{24}\omega_0^3 t^3 + \frac{\hat{c}}{2}Ci(\omega_0 t) + k_0'\}\sin(\omega_0 t) - \frac{c_2}{2})\frac{\partial}{\partial x}]
\end{aligned}
$$

such that k_0, k' are arbitrary constants. To determine the conservation laws, first we need the adjoint equation F^* of Eq.(3.2). By (3.8) we have

$$
\begin{aligned}
F_{(0)}^* + \epsilon F_{(1)}^* \approx& (\frac{\partial}{\partial x} - D_t\frac{\partial}{\partial\dot{x}} + D_{tt}\frac{\partial}{\partial\ddot{x}})y(t,x)(\ddot{x}(t) + \omega_0^2 x(t) - f(t) + \epsilon\omega_0^2\{[\psi(1) - \ln t - p(t)]x(t)\\
&- q(t)\dot{x}(t) - r(t)\}) = \ddot{y} + \omega_0^2 y + \epsilon\omega_0^2[q(t)\dot{y} + (\psi(1) - lnt - p(t) + q'(t))y]
\end{aligned}
$$

s.t $y \approx \phi_{(0)} + \epsilon\phi_{(1)}$. Then selfadjointness condition (3.9) leads to equations

$$
\begin{aligned}
&\ddot{\phi}_{(0)} + \omega_0^2\phi_{(0)} = 0\\
&\ddot{\phi}_{(1)} + \omega_0^2\phi_{(1)} + \omega_0^2[q(t)\dot{\phi}_{(0)} + (\psi(1) - lnt - p(t) + q'(t))\phi_{(0)}] = 0
\end{aligned} \tag{B.2}
$$

By solving the system of differential equations(B.2), we can get y and $\mathcal{L}$ as mentioned in paper. Finally by (3.10), (3.11), the conserved vector can be obtained.

Case2: The exact symmetry of unperturbed equation $\ddot{x}(t)+2\omega_0\dot{x}(t)=A_0\sin(\Omega_0 t)$ is

$$V_{(0)}=(s_1+s_2e^{\omega_0 t})\frac{\partial}{\partial t}+[(k_1+k_2e^{-2\omega_0 t}+\frac{s_1}{5\omega_0}A_0(2\sin(\omega_0 t)-\cos(\omega_0 t))+\frac{s_2}{10\omega_0}A_0e^{\omega_0 t}(2\sin(\omega_0 t)+\cos(\omega_0 t))]\frac{\partial}{\partial x}$$

such that s_1, s_2, k_1, k_2 are constant, and the auxiliary function H is

$$\begin{aligned}H=&-2\omega_0(s_1+s_2e^{\omega_0 t})[(-\frac{1}{t}-g'(t))\dot{x}+\frac{1}{t^2}x+(-I'(t)+\omega_0 J(t))A_0\sin(\omega_0 t)-(J'(t)+\omega_0 I(t))\\ &A_0\cos(\omega_0 t)]-2\omega_0(\psi(2)-lnt-g(t))[-2k_2\omega_0e^{-2\omega_0 t}+\frac{s_1}{5}A_0(\sin(\omega_0 t)+2\cos(\omega_0 t))+\frac{s_2}{10}A_0e^{\omega_0 t}\\ &(\sin(\omega_0 t)+3\cos(\omega_0 t))]+\frac{2}{t}[\omega_0(k_1+k_2e^{-2\omega_0 t})+\frac{s_1}{5}A_0(2\sin(\omega_0 t)-\cos(\omega_0 t))+\frac{s_2}{10}A_0e^{\omega_0 t}(2\sin(\omega_0 t)+\cos(\omega_0 t))].\end{aligned}$$

The generator of approximate transformation group is

$$\begin{aligned}&(k_1+k_1e^{-2\omega_0 t)}\frac{\partial}{\partial x}+\epsilon[(\frac{c_3}{\omega_0}e^{2\omega_0 t}+c_4)\frac{\partial}{\partial t}+\{\bar{l_1}+(\bar{l_2}+k_1\omega_0 t(2\psi(2)+2c''+1)+k_1\omega^2t^2)e^{-2\omega_0 t}+\\ &\frac{A_0}{\omega_0}(c_4+\frac{1}{\omega_0}c_3e^{2\omega_0 t)})(\frac{2}{5}\sin(\omega_0)-\frac{1}{5}\cos(\omega_0))-k_1e^{-2\omega_0 t}Ei(2\omega_0 t)(2\omega_0 t-1)-k_1\ln t+k_1\}\frac{\partial}{\partial x}]\end{aligned}$$

where $c_3, c_4, \bar{l_1}, \bar{l_2}$ are arbitrary constants. The adjoint equation F^* of Eq.(3.3) is

$$\begin{aligned}F^*_{(0)}+\epsilon F^*_{(1)}\approx&(\frac{\partial}{\partial x}-D_t\frac{\partial}{\partial \dot{x}}+D_{tt}\frac{\partial}{\partial \ddot{x}})y(t,x)(\ddot{x}(t)+2\omega_0\dot{x}(t)-f(t)-2\epsilon\omega_0\{[\psi(2)-\ln t-g(t)]\dot{x}(t)\\ &-\frac{1}{t}x(t)-h(t)\}=\ddot{y}-2\omega_0\dot{y}+2\epsilon\omega_0[(\psi(2)-lnt-g(t))\dot{y}-g'(t)y]\end{aligned}$$

By solving the system of differential equations

$$\begin{aligned}\ddot{\phi}_{(0)}-2\omega_0\dot{\phi}_{(0)}=0\\ \ddot{\phi}_{(1)}-2\omega_0\dot{\phi}_{(1)}+2\omega_0[(\psi(2)-lnt-g(t))\dot{\phi}_{(0)}-g'(t)\phi_{(0)}]=0\end{aligned}$$

we can get y and $\mathcal{L}$ as mentioned.

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