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Some contraction fixed point theorems in partially ordered modular metric spaces

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Abstract

In this paper, we prove some fixed point theorems for modular metric spaces endowed with partial order sets by using the mixed monotone mapping property which is a generalization of the definitions and results of T. Gnana Bhaskar and V. Lakshmikantham.

Keywords: Coupled fixed point, Modular metric spaces, Partially ordered sets
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1 Introduction

The theory of modular spaces was initiated by Nakano [15] in 1950, and generalized and redefined by Musielak and Orlicz [14] in 1959.

In 2008, Chistyakov [5] introduced the notation of modular metric spaces generated by F-modular and developed the theory of modular spaces. By the same idea he defined the modular metric spaces on an arbitrary set which is a new generalization of metric spaces [6], [7]. The field of the metric fixed point theory and its applications [10], [12] are far reaching developments of Banach's Contraction Principle [3], and the first fixed point results in modular function space were given by Khamsi [12].

Recently some authors have introduced and have established some notions and fixed point results in modular metric spaces (c.f.[4, 8, 13]). Many authors investigated on the existence of the fixed points for contraction type mapping in partially ordered metric spaces [1, 2, 11].

In this paper we state and prove some coupled fixed point theorems for partially ordered modular metric spaces. These results are extensions of the results obtained by T. Gnana Bhaskar and Lakshmikantham [9].

Definition 1.1. Let X be an arbitrary set. A function $\omega : (0,\infty) \times X \times X \longrightarrow [0,\infty]$ that will be written as $\omega_\lambda(x,y) = \omega(\lambda,x,y)$ for all $x,y \in X$ and for all $\lambda > 0$, is said to be a modular metric on X (or simply a modular if no ambiguity arises) if it satisfies the following three conditions:
(i) given $x,y \in X$, $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$ iff $x = y$;
(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$, for all $\lambda > 0$ and $x,y \in X$;
(iii) $\omega_{\lambda+\mu}(x,y) \leq \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda,\mu > 0$ and $x,y,z \in X$.

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If instead of (i), we have only the condition:
(i_1) $\omega_\lambda(x,x) = 0$ for all $\lambda > 0$ and $x \in X$, then ω is said to be a (metric) pseudomodular on X and if ω satisfies (i_1) and
(i_2) given $x, y \in X$, if there exists $\lambda > 0$, possibly depending on x and y, such that $\omega_\lambda(x,y) = 0$ implies that $x = y$, then ω is called a *strict modular* on X.

Definition 1.2. [6] Given a modular ω on X, the sets

$$X_\omega \equiv X_\omega(x_\circ) = \{x \in X : \omega_\lambda(x, x_\circ) \to 0 \; as \; \lambda \to \infty\}$$

and

$$X^*_\omega \equiv X^*_\omega(x_\circ) = \{x \in X : \omega_\lambda(x, x_\circ) < \infty \;\; for\ some \;\; \lambda > 0\}$$

are said to be modular spaces (around $x_\circ$). Also the modular spaces X_ω and X^*_ω can be equipped with metrics d_ω and d^*_ω, generated by ω and given by

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le \lambda\}, \quad x, y \in X_\omega$$

and

$$d^*_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le 1\}, \quad x, y \in X^*_\omega$$

If ω is a convex modular on X, then according to [6, Theorem 3.6] the two modular spaces coincide, $X_\omega = X^*_\omega$.

Definition 1.3. Given a modular metric space X_ω, a sequence of elements $\{x_n\}_{n=1}^\infty$ from X_ω is said to be modular convergent ($\omega-$convergent) to an element $x \in X$ if there exists a number $\lambda > 0$, possibly depending on $\{x_n\}$ and x such that $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$. This will be written briefly as $x_n \overset{\omega}{\to} x$, as $n \to \infty$.

Definition 1.4. [8] A sequence $\{x_n\} \subset X_\omega$ is said to be ω-*Cauchy* if there exists a number $\lambda = \lambda(\{x_n\}) > 0$ such that $\lim_{m,n\to\infty} \omega_\lambda(x_n, x_m) = 0$, i.e.,

$$\forall \varepsilon > 0 \;\; \exists \;\; n_\circ(\varepsilon) \in \mathbb{N} \;\; such \;\; that \;\; \forall n, m \ge n_\circ(\varepsilon) \;\; : \;\; \omega_\lambda(x_n, x_m) \le \varepsilon.$$

Modular metric space X_ω is said to be ω-*complete* if each ω-Cauchy sequence from X_ω is modular convergent to an $x \in X_\omega$.

Remark 1.5. A modular $\omega = \omega_\lambda$ on a set X, for given $x, y \in X$, is non-increasing on λ. Indeed if $0 < \lambda < \mu$, then we have

$$\omega_\mu(x,y) \le \omega_{\mu-\lambda}(x,x) + \omega_\lambda(x,y) = \omega_\lambda(x,y)$$

for all $x, y \in X$.

Let ω be a modular on X such that X_ω is a ω-complete modular metric space and $(X_\omega, \preceq)$ is a partially ordered set. Further, we endow the product space $X_\omega \times X_\omega$ with the following partial order:

$$for \quad (x,y), (u,v) \in X_\omega \times X_\omega, \;\; (u,v) \preceq (x,y) \Leftrightarrow x \succeq u \;,\; y \preceq v.$$

Definition 1.6. [17, 18] Let X_ω be a modular metric space , we say that $T : X_\omega \to X_\omega$ is modular continuous (ω-continuous) if

$$x_n \overset{\omega}{\to} x \Rightarrow Tx_n \overset{\omega}{\to} Tx$$

for each $\{x_n\} \in X_\omega$ as $n \to \infty$.

Definition 1.7. Let $(X_\omega, \preceq)$ be a partially ordered set and $T : X_\omega \times X_\omega \to X_\omega$. We say that T has the mixed monotone property if $T(x, y)$ is monotone non-decreasing with respect to x and is monotone non-increasing on y, that is, for any $x, y \in X_\omega$,

$$x_1, x_2 \in X_\omega, x_1 \preceq x_2 \to T(x_1, y) \preceq T(x_2, y)$$

and

$$y_1, y_2 \in X_\omega, y_1 \preceq y_2 \to T(x, y_1) \succeq T(x, y_2).$$

Definition 1.8. We call an element $(x, y) \in X_\omega \times X_\omega$ a *coupled fixed* point of the mapping $T : X_\omega \times X_\omega \longrightarrow X_\omega$ if

$$T(x, y) = x, \quad T(y, x) = y.$$

Remark 1.9. For each $(x, y), (u, v) \in X_\omega \times X_\omega$ we set,

$$\omega_\lambda((x, y), (u, v)) = \omega_\lambda(x, u) + \omega_\lambda(y, v).$$

It's obvious that ω_λ is (metric) modular on $X_\omega \times X_\omega$.

Lemma 1.10. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in modular metric space X_ω. Then $\{x_n\}$ and $\{y_n\}$ are ω-convergent to x and y (respectively) iff coupled sequence $\{(x_n, y_n)\}$ is ω-convergent to (x, y).

Proof . Let $\{x_n\}$ and $\{y_n\}$ be two ω-convergent sequences such that $x_n \overset{\omega}{\to} x$ and $y_n \overset{\omega}{\to} y$ as $n \to \infty$. By definition of the modular convergence, there exist $\lambda = \lambda(\{x_n\}, x) > 0$ and $\mu = \mu(\{y_n\}, y) > 0$ such that, $\omega_\lambda(x_n, x) \longrightarrow 0$ and $\omega_\mu(y_n, y) \longrightarrow 0$, as $n \to \infty$. Let $\zeta \geq \max\{\lambda, \mu\}$, it follows from Remark 1.5 that

$$\omega_\zeta(x_n, x) \leq \omega_\lambda(x_n, x) \longrightarrow 0 \quad as \;\; n \longrightarrow \infty$$

and

$$\omega_\zeta(y_n, y) \leq \omega_\mu(y_n, y) \longrightarrow 0 \quad as \;\; n \longrightarrow \infty.$$

So we have

$$\omega_\zeta((x_n, y_n), (x, y)) = \omega_\zeta(x_n, x) + \omega_\zeta(y_n, y) \longrightarrow 0 \quad as \;\; n \longrightarrow \infty.$$

It follows that $\{(x_n, y_n)\}$ is modular convergent, i.e., $(x_n, y_n) \overset{\omega}{\to} (x, y)$ as $n \to \infty$. Conversely, suppose that $(x_n, y_n) \overset{\omega}{\to} (x, y)$ as $n \to \infty$, then there exists $\lambda = \lambda(\{(x_n, y_n)\}, (x, y))$ such that

$$\omega_\lambda((x_n, y_n), (x, y)) \longrightarrow 0, \text{as} \;\; n \longrightarrow \infty.$$

By Remark 1.9 we have

$$0 \leq \omega_\lambda((x_n, y_n), (x, y)) = \omega_\lambda(x_n, x) + \omega_\lambda(y_n, y) \longrightarrow 0.$$

Therefore $x_n \overset{\omega}{\to} x$ and $y_n \overset{\omega}{\to} y$ as $n \to \infty$. □

2 Preliminaries

In this section, we prove the fixed point theorem for a function T on the partial ordered product space $X_\omega \times X_\omega$ which is a generalization of T. Gnana Bhaskar and V. Lakshmikantham [9].

Theorem 2.1. Let ω be a strict modular on X such that X_ω is a $\omega-$complete modular metric space and $T : X_\omega \times X_\omega \longrightarrow X_\omega$ is a ω-continuous mapping having the mixed monotone property on X_ω. Assume that there exists a $k \in [0, 1)$ with

$$\omega_\lambda(T(x, y), T(u, v)) \leq \frac{k}{2} \left[\omega_\lambda(x, u) + \omega_\lambda(y, v)\right], \quad (x \succeq u \;,\; y \preceq v). \tag{2.1}$$

If there exist $x_0, y_0 \in X_\omega$ such that

$$x_0 \preceq T(x_0, y_0) \quad and \quad y_0 \succeq T(y_0, x_0),$$

then there exist $x, y \in X_\omega$ such that

$$x = T(x, y) \quad and \quad y = T(y, x).$$

Proof . Taking $T(x_0, y_0) = x_1, T(y_0, x_0) = y_1$, $x_2 = T(x_1, y_1)$ and $y_2 = T(y_1, x_1)$, we obtain

$$T^2(x_0, y_0) = T(T(x_0, y_0), T(y_0, x_0)) = T(x_1, y_1) = x_2$$

and

$$T^2(y_0, x_0) = T(T(y_0, x_0), T(x_0, y_0)) = T(y_1, x_1) = y_2.$$

By mixed monotone property of T we get,

$$x_2 = T^2(x_0, y_0) = T(x_1, y_1) \succeq T(x_0, y_0) = x_1$$

and

$$y_2 = T^2(y_0, x_0) = T(y_1, x_1) \preceq T(y_0, x_0) = y_1.$$

We construct sequences $\{x_n\}$ and $\{y_n\}$ as following:

$$x_{n+1} = T^{n+1}(x_0, y_0) = T(T^n(x_0, y_0), T^n(y_0, x_0))$$

and

$$y_{n+1} = T^{n+1}(y_0, x_0) = T(T^n(y_0, x_0), T^n(x_0, y_0)).$$

By induction and the mixed monotone property of T, we get the following two relation:

$$x_0 \preceq T(x_0, y_0) = x_1 \preceq T^2(x_0, y_0) = x_2 \preceq ... \preceq T^{n+1}(x_0, y_0) = x_{n+1} \preceq ...$$

and

$$y_0 \succeq T(y_0, x_0) = y_1 \succeq T^2(y_0, x_0) = y_2 \succeq ... \succeq T^{n+1}(y_0, x_0) = y_{n+1} \succeq ... .$$

Now we show that for $n \in \mathbb{N}$,

$$\omega_\lambda(T^{n+1}(x_0, y_0), T^n(x_0, y_0)) \leq \frac{k^n}{2}[\omega_\lambda(T(x_0, y_0), x_0) + \omega_\lambda(T(y_0, x_0), y_0)], \tag{2.2}$$

and

$$\omega_\lambda(T^{n+1}(y_0, x_0), T^n(y_0, x_0)) \leq \frac{k^n}{2}[\omega_\lambda(T(y_0, x_0), y_0) + \omega_\lambda(T(x_0, y_0), x_0)]. \tag{2.3}$$

For $n = 1$, using (2.1) we get

$$\begin{aligned}\omega_\lambda(T^2(x_0, y_0), T(x_0, y_0)) &= \omega_\lambda(T(T(x_0, y_0), T(y_0, x_0)), T(x_0, y_0)) \\ &\leq \frac{k}{2}[\omega_\lambda(T(x_0, y_0), x_0) + \omega_\lambda(T(y_0, x_0), y_0)].\end{aligned}$$

Similarly,

$$\begin{aligned}\omega_\lambda(T^2(y_0, x_0), T(y_0, x_0)) &= \omega_\lambda(T(T(y_0, x_0), T(x_0, y_0)), T(y_0, x_0)) \\ &\leq \frac{k}{2}[\omega_\lambda(T(y_0, x_0), y_0) + \omega_\lambda(T(x_0, y_0), x_0)].\end{aligned}$$

Now, assume that (2.2) and (2.3) hold. Using

$$T^{n+1}(x_0, y_0) \succeq T^n(x_0, y_0) \ \ and \ \ T^{n+1}(y_0, x_0) \preceq T^n(y_0, x_0),$$

we get

$$\begin{aligned}
&\omega_\lambda(T^{n+2}(x_0, y_0), T^{n+1}(x_0, y_0)) \\
&\quad = \omega_\lambda(T(T^{n+1}(x_0, y_0), T^{n+1}(y_0, x_0)), T(T^n(x_0, y_0), T^n(y_0, x_0))) \\
&\quad \leq \frac{k}{2}[\omega_\lambda(T^{n+1}(x_0, y_0), T^n(x_0, y_0)) + \omega_\lambda(T^{n+1}(y_0, x_0), T^n(y_0, x_0))] \\
&\quad \leq \frac{k^{n+1}}{2}[\omega_\lambda(T(x_0, y_0), x_0) + \omega_\lambda(T(y_0, x_0), y_0)].
\end{aligned}$$

Similarly, one can show that

$$\omega_\lambda(T^{n+2}(y_0, x_0), T^{n+1}(y_0, x_0)) \leq \frac{k^{n+1}}{2}[\omega_\lambda(T(y_0, x_0), y_0) + \omega_\lambda(T(x_0, y_0), x_0)].$$

This implies that $\{T^n(x_0, y_0)\}$ and $\{T^n(y_0, x_0)\}$ are ω-Cauchy sequences in X_ω. In fact, for $m > n$,

$$\begin{aligned}
\omega_\lambda(T^m(x_0, y_0), T^n(x_0, y_0)) &= \omega_{\frac{\lambda(m-n)}{m-n}}(T^m(x_0, y_0), T^n(x_0, y_0)) \\
&\leq \omega_{\frac{\lambda}{m-n}}(T^n(x_0, y_0), T^{n+1}(x_0, y_0)) + ... + \omega_{\frac{\lambda}{m-n}}(T^{m-1}(x_0, y_0), T^m(x_0, y_0)) \\
&\leq \frac{(k^{m-1} + ... + k^n)}{2}[\omega_{\frac{\lambda}{m-n}}(T(x_0, y_0), x_0) + \omega_{\frac{\lambda}{m-n}}(T(y_0, x_0), y_0)] \\
&= \frac{(k^n - k^m)}{2(1-k)}[\omega_{\frac{\lambda}{m-n}}(T(x_0, y_0), x_0) + \omega_{\frac{\lambda}{m-n}}(T(y_0, x_0), y_0)] \\
&< \frac{k^n}{2(1-k)}[\omega_{\frac{\lambda}{m-n}}(T(x_0, y_0), x_0) + \omega_{\frac{\lambda}{m-n}}(T(y_0, x_0), y_0)].
\end{aligned}$$

Similarly, we can verify that $\{T^n(y_0, x_0)\}$ is also a ω-Cauchy sequence. Since X_ω is a ω-complete modular metric space, there exist $x, y \in X_\omega$ such that

$$T^n(x_0, y_0) \stackrel{\omega}{\to} x \ \ ; \ \ T^n(y_0, x_0) \stackrel{\omega}{\to} y.$$

Now we claim that $T(x, y) = x$ and $T(y, x) = y$. Because $T^n(x_0, y_0) \stackrel{\omega}{\to} x$ and $T^n(y_0, x_0) \stackrel{\omega}{\to} y$ as $n \to \infty$, so there exist $\lambda_1 = \lambda_1(\{T^n(x_0, y_0)\}, x) > 0$ and $\lambda_2 = \lambda_2(\{T^n(y_0, x_0)\}, y) > 0$ such that,

$$\omega_{\lambda_1}(T^n(x_0, y_0), x) \longrightarrow 0 \ \ ; \ \ \omega_{\lambda_2}(T^n(y_0, x_0), y) \longrightarrow 0.$$

The mapping $T : X_\omega \times X_\omega \longrightarrow X_\omega$ is ω-continuous. So if $(x_n, y_n) \stackrel{\omega}{\to} (x, y)$, then there exists $\lambda_3 = \lambda_3(\{T(x_n, y_n)\}, T(x, y))$ such that

$$\omega_{\lambda_3}(T(x_n, y_n), T(x, y)) \longrightarrow 0.$$

This implies that

$$\omega_{\lambda_3}(T^{n+1}(x_0, y_0), T(x, y)) = \omega_{\lambda_3}(T(T^n(x_0, y_0), T^n(y_0, x_0)), T(x, y)) \longrightarrow 0.$$

Therefore for $\lambda \geq \lambda_1 + \lambda_3$ we have,

$$\begin{aligned}
\omega_\lambda(T(x, y), x) &\leq \omega_{\lambda_1+\lambda_3}(T(x, y), x) \\
&\leq \omega_{\lambda_1}(T^{n+1}(x_0, y_0), x) + \omega_{\lambda_3}(T(x, y), T^{n+1}(x_0, y_0)) \longrightarrow 0,
\end{aligned}$$

as $n \longrightarrow \infty$. So by strictness of ω, $T(x, y) = x$. Similarly $T(y, x) = y$. $\square$

Theorem 2.2. Let ω be a strict modular on X such that X_ω is a $\omega-$complete modular metric space and $(X_\omega, \preceq)$ is a partially ordered set. Suppose that X_ω has the following properties:
(i) if a nondecreasing sequence $x_n \overset{\omega}{\to} x$, then $x_n \preceq x$, for all n;
(ii) if a non-increasing sequence $y_n \overset{\omega}{\to} y$, then $y \preceq y_n$, for all n.

Let $T : X_\omega \times X_\omega \longrightarrow X_\omega$ be a mapping having the mixed monotone property on X_ω. Assume that there exists a $k \in [0, 1)$ with

$$\omega_\lambda(T(x, y), T(u, v)) \leq \frac{k}{2}[\omega_\lambda(x, u) + \omega_\lambda(y, v)],\ (x \succeq u,\ y \preceq v).$$

If there exist $x_0, y_0 \in X_\omega$ such that $x_0 \preceq T(x_0, y_0)$ and $y_0 \succeq T(y_0, x_0)$, then there exist $x, y \in X_\omega$ such that $x = T(x, y)$ and $y = T(y, x)$.

Proof . Following the proof of Theorem 2.1, we only have to show that $T(x, y) = x$ and $T(y, x) = y$. Because $T^n(x_0, y_0) \overset{\omega}{\to} x$ and $T^n(y_0, x_0) \overset{\omega}{\to} y$, there exist $\lambda_1 = \lambda_1(T^n(x_0, y_0), x) > 0$ and $\lambda_2 = \lambda_2(T^n(y_0, x_0), y) > 0$ such that

$$\omega_{\lambda_1}(T^n(x_0, y_0), x) \longrightarrow 0, \omega_{\lambda_2}(T^n(y_0, x_0), y) \longrightarrow 0,\ as\ n \longrightarrow +\infty.$$

Now for $\lambda \geq 2\max\{\lambda_1, \lambda_2\}$ we obtain:

$$\begin{aligned}
\omega_\lambda(T(x, y), x) &\leq \omega_{\frac{\lambda}{2}}(T(x, y), T^{n+1}(x_0, y_0)) + \omega_{\frac{\lambda}{2}}(T^{n+1}(x_0, y_0), x)\\
&= \omega_{\frac{\lambda}{2}}(T(x, y), T(T^n(x_0, y_0), T^n(y_0, x_0))) + \omega_{\frac{\lambda}{2}}(T^{n+1}(x_0, y_0), x)\\
&\leq \frac{k}{2}[\omega_{\frac{\lambda}{2}}(x, T^n(x_0, y_0)) + \omega_{\frac{\lambda}{2}}(y, T^n(y_0, x_0))] + \omega_{\frac{\lambda}{2}}(T^{n+1}(x_0, y_0), x)\\
&\leq [\omega_{\lambda_1}(T^n(x_0, y_0), x) + \omega_{\lambda_2}(T^n(y_0, x_0), y)] + \omega_{\lambda_1}(T^{n+1}(x_0, y_0), x) \longrightarrow 0,\\
&as\ n \longrightarrow \infty.
\end{aligned}$$

Therefore by strictness of ω we have $T(x, y) = x$. Similarly one can show that $T(y, x) = y$. $\square$ From [16] we have that for each $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x^* \\ y^* \end{pmatrix} \in X_\omega \times X_\omega$ there exists a $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in X_\omega \times X_\omega$ which is comparable to $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x^* \\ y^* \end{pmatrix}$. In other words, each pair of elements in the product space has an upper bound or a lower bound.

Remark 2.3. The contractivity assumption is taken on the comparable elements in $X_\omega \times X_\omega$, and so Theorem 2.1 doesn't imply the uniqueness of the fixed point. In the next theorem we add the comparability condition to the hypothesis of Theorem 2.1 and conclude the uniqueness of the coupled fixed point of T.

Theorem 2.4. Adding comparability condition to the hypothesis of Theorem 2.1, we obtain the uniqueness of the coupled fixed point of T.

Proof . If $\begin{pmatrix} x^* \\ y^* \end{pmatrix} \in X_\omega \times X_\omega$ is another coupled fixed point of T, then we show that

$$\omega_\lambda(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x^* \\ y^* \end{pmatrix}) = 0,$$

for some $\lambda > 0$, where

$$\lim_{n\to\infty} T^n(x_0, y_0) = x\ \ ;\ \ \lim_{n\to\infty} T^n(y_0, x_0) = y.$$

We consider two cases:
Case 1: If $\begin{pmatrix} x \\ y \end{pmatrix}$ is comparable to $\begin{pmatrix} x^* \\ y^* \end{pmatrix}$ with respect to the ordering in $X_\omega \times X_\omega$, then for all $n \in \mathbb{N} \cup \{0\}$, $\begin{pmatrix} T^n(x, y) \\ T^n(y, x) \end{pmatrix} =$

$\begin{pmatrix} x \\ y \end{pmatrix}$ is comparable to $\begin{pmatrix} T^n(x^*,y^*) \\ T^n(y^*,x^*) \end{pmatrix} = \begin{pmatrix} x^* \\ y^* \end{pmatrix}$, also

$$\begin{aligned}
\omega_\lambda(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x^* \\ y^* \end{pmatrix}) &= \omega_\lambda(x,x^*) + \omega_\lambda(y,y^*) \\
&= \omega_\lambda(T^n(x,y), T^n(x^*,y^*)) + \omega_\lambda(T^n(y,x), T^n(y^*,x^*)) \\
&= \omega_\lambda(T(T^{n-1}(x,y), T^{n-1}(y,x)), T(T^{n-1}(x^*,y^*), T^{n-1}(y^*,x^*))) \\
&+ \omega_\lambda(T(T^{n-1}(y,x), T^{n-1}(x,y)), T(T^{n-1}(y^*,x^*), T^{n-1}(x^*,y^*))) \\
&\leq \frac{k}{2}[\omega_\lambda(T^{n-1}(x,y), T^{n-1}(x^*,y^*)) + \omega_\lambda(T^{n-1}(y,x), T^{n-1}(y^*,x^*))] \\
&\leq k^n[\omega_\lambda(x,x^*) + \omega_\lambda(y,y^*)] \\
&= k^n\omega_\lambda(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x^* \\ y^* \end{pmatrix}) \xrightarrow{\omega} 0, \ as\ n \longrightarrow \infty.
\end{aligned}$$

So $\omega_\lambda(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x^* \\ y^* \end{pmatrix}) = 0$ for some $\lambda > 0$.

Case 2: Suppose that $\hat{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ is not comparable to $\hat{x^*} = \begin{pmatrix} x^* \\ y^* \end{pmatrix}$. We know that there exists an upper bound or a lower bound $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in X_\omega \times X_\omega$ of $\hat{x}, \hat{x^*}$. So, for all $n = 0, 1, 2, ...,$ $z = \begin{pmatrix} T^n(z_1,z_2) \\ T^n(z_2,z_1) \end{pmatrix}$ is comparable to $\begin{pmatrix} T^n(x,y) \\ T^n(y,x) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} T^n(x^*,y^*) \\ T^n(y^*,x^*) \end{pmatrix} = \begin{pmatrix} x^* \\ y^* \end{pmatrix}$, so

$$\begin{aligned}
\omega_\lambda(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x^* \\ y^* \end{pmatrix}) &= \omega_\lambda(\begin{pmatrix} T^n(x,y) \\ T^n(y,x) \end{pmatrix}, \begin{pmatrix} T^n(x^*,y^*) \\ T^n(y^*,x^*) \end{pmatrix}) \\
&\leq \omega_{\frac{\lambda}{2}}(\begin{pmatrix} T^n(x,y) \\ T^n(y,x) \end{pmatrix}, \begin{pmatrix} T^n(z_1,z_2) \\ T^n(z_2,z_1) \end{pmatrix}) + \omega_{\frac{\lambda}{2}}(\begin{pmatrix} T^n(z_1,z_2) \\ T^n(z_2,z_1) \end{pmatrix}, \begin{pmatrix} T^n(x^*,y^*) \\ T^n(y^*,x^*) \end{pmatrix}) \\
&= \omega_{\frac{\lambda}{2}}(T^n(x,y), T^n(z_1,z_2)) + \omega_{\frac{\lambda}{2}}(T^n(y,x), T^n(z_2,z_1)) \\
&+ \omega_{\frac{\lambda}{2}}(T^n(z_1,z_2), T^n(x^*,y^*)) + \omega_{\frac{\lambda}{2}}(T^n(z_2,z_1), T^n(y^*,x^*)) \\
&< k^n\{[\omega_{\frac{\lambda}{2}}(x,z_1) + \omega_{\frac{\lambda}{2}}(y,z_2)] + [\omega_{\frac{\lambda}{2}}(x^*,z_1) + \omega_{\frac{\lambda}{2}}(y^*,z_2)]\} \xrightarrow{\omega} 0,
\end{aligned}$$

$as\ n \to \infty.$

Thus, $\omega_\lambda(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x^* \\ y^* \end{pmatrix}) = 0$ for some $\lambda > 0$, and by the strictness of ω, we get $\hat{x^*} = \hat{x}$. □

Theorem 2.5. In addition to the hypothesis of Theorem 2.1, suppose that each pair of elements in X_ω has an upper bound or a lower bound in X_ω. Then there exist $x, y \in X_\omega$, such that $x = y$.

Proof . Case 1: Suppose that x is not comparable to y. By the assumption there exists a $z \in X_\omega$ comparable to x and y such that $x \preceq z$ and $y \preceq z$. Then we have,

$$T(x,y) \preceq T(z,y) \ \ and \ \ T(x,y) \succeq T(x,z)$$

and

$$T(y,x) \preceq T(z,x) \ \ and \ \ T(y,x) \succeq T(y,z).$$

The mixed monotone property of T yields:

(1) $T^2(x,y) = T(T(x,y), T(y,x)) \preceq T(T(z,y), T(y,z)) = T^2(z,y)$. This implies that $T^2(x,y) \preceq T^2(z,y)$;

(2) $T^2(y,x) = T(T(y,x), T(x,y)) \preceq T(T(z,x), T(x,z)) = T^2(z,x)$. This implies that $T^2(y,x) \preceq T^2(z,x)$;

(3) $T^2(x,y) = T(T(x,y), T(y,x)) \succeq T(T(x,z), T(z,x)) = T^2(x,z)$, which is equivalent to $T^2(x,y) \succeq T^2(x,z)$;

(4) $T^2(y,x) = T(T(y,x), T(x,y)) \succeq T(T(y,z), T(z,y)) = T^2(y,z)$, or equivalently $T^2(y,x) \succeq T^2(y,z)$.

By induction it can be shown that similar relations hold for all $n > 2$ too. Now, consider

$$\begin{aligned}\omega_\lambda(x,y) &= \omega_\lambda(T^{n+1}(x,y), T^{n+1}(y,x))\\ &= \omega_\lambda(T(T^n(x,y),T^n(y,x)), T(T^n(y,x),T^n(x,y))\\ &\le \omega_{\frac{\lambda}{3}}(T(T^n(x,y),T^n(y,x)), T(T^n(x,z),T^n(z,x))\\ &+ \omega_{\frac{2\lambda}{3}}(T(T^n(x,z),T^n(z,x)), T(T^n(y,x),T^n(x,y))\\ &\le \omega_{\frac{\lambda}{3}}(T(T^n(x,y),T^n(y,x)), T(T^n(x,z),T^n(z,x))\\ &+ \omega_{\frac{\lambda}{3}}(T(T^n(x,z),T^n(z,x)), T(T^n(z,x),T^n(x,z))\\ &+ \omega_{\frac{\lambda}{3}}(T(T^n(z,x),T^n(x,z)), T(T^n(y,x),T^n(x,y))\\ &\le \frac{k}{2}[\omega_{\frac{\lambda}{3}}(T^n(x,y),T^n(x,z)) + \omega_{\frac{\lambda}{3}}(T^n(y,x),T^n(z,x))]\\ &+ \frac{k}{2}[\omega_{\frac{\lambda}{3}}(T^n(x,z),T^n(z,x)) + \omega_{\frac{\lambda}{3}}(T^n(z,x),T^n(x,z))]\\ &+ \frac{k}{2}[\omega_{\frac{\lambda}{3}}(T^n(z,x),T^n(y,x)) + \omega_{\frac{\lambda}{3}}(T^n(x,z),T^n(x,y))]\\ &= k[\omega_{\frac{\lambda}{3}}(T^n(x,y),T^n(x,z)) + \omega_{\frac{\lambda}{3}}(T^n(x,z),T^n(z,x))\\ &+ \omega_{\frac{\lambda}{3}}(T^n(z,x),T^n(y,x))].\end{aligned}$$

Proceeding, then we obtain $\omega_\lambda(x,y) \le k^{n+1}[\omega_{\frac{\lambda}{3}}(x,z) + \omega_{\frac{\lambda}{3}}(z,y)] \xrightarrow{\omega} 0$ as $n \to \infty$, so $\omega_\lambda(x,y) = 0$, and by the strictness of ω, we get $x = y$.
Case 2: If x is comparable to y, then $x = T(x,y)$ is comparable to $y = T(y,x)$, and we obtain

$$\omega_\lambda(x,y) = \omega_\lambda(T(x,y),T(y,x)) \le k\omega_\lambda(x,y).$$

Because $0 \le k < 1$ we conclude that $\omega_\lambda(x,y) = 0$, so $x = y$. □

As an application of Theorem 2.1, we consider the following example.

Example 2.6. Let the triple $(\mathbb{R}, d, +)$ be a metric semigroup, i.e., the pair $(\mathbb{R}, d)$ is an Abelian semigroup with respect to addition, $+$, and d is translation invariant in the sense that $d(p+r, q+r) = d(p,q)$ for all $p,q,r \in \mathbb{R}$. Let X be the set of all real valued functions x on the closed interval $[a,b] \subset \mathbb{R}$ with $a < b$ such that $x(a) = x_\circ$. The function $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a φ-function, i.e., a continuous nondecreasing unbounded function vanishing only at zero. Suppose that φ is a convex φ-function on $\mathbb{R}^+$. We define the function $\omega : (0,\infty) \times X \times X \longrightarrow [0,\infty]$ for all $\lambda > 0$ and $x,y \in X$

$$\omega_\lambda(x,y) = \sup_\tau \sum_{i=1}^m \varphi\left(\frac{d(x(t_i)+y(t_{i-1}), x(t_{i-1})+y(t_i))}{\lambda(t_i - t_{i-1})}\right)(t_i - t_{i-1})$$

where the supremum is taken over all partitions $\tau = \{t_i\}_{i=0}^m$ of the interval $[a,b]$. It was shown in [8] that ω is a strict convex modular on X, and $X^*_\omega(x_\circ) = X^*_\omega$, (here $x_\circ$ denotes the constant mapping $x_\circ(t) = x_\circ$ for all $t \in [a,b]$). For more details see [10]. Fix an $x_\circ \in \mathbb{R}$,it is easy to show that

$$X^*_\omega = \{x : [a,b] \to \mathbb{R} \mid x(a) = x_\circ \; and \; \omega_\lambda(x,x_\circ) < \infty \; for \; some \; \lambda > 0\}$$

where

$$\omega_\lambda(x,x_\circ) = \sup_\tau \sum_{i=1}^m \varphi\left(\frac{d(x(t_i), x(t_{i-1}))}{\lambda \cdot (t_i - t_{i-1})}\right)(t_i - t_{i-1}).$$

Note that by the last relation $\omega_\lambda(x,x_\circ)$ is independent from $x_\circ \in \mathbb{R}$. This value is called the generalized φ_λ-variation of x, where $\varphi_\lambda(u) = \varphi(u/\lambda), u \in \mathbb{R}^+$. Recall that $\mathbb{R}$ with $d(x,y) = |x-y|$ is a complete metric space so the modular space $X^*_\omega = X_\omega$ is ω-complete, (see [10]).

Now we set $[a,b] = [-1,1]$, $x_\circ = 0$ and $\varphi(t) = \sqrt{t}$ for $t \in \mathbb{R}^+$. With this modification we have

$$X = \{x : [-1,1] \to \mathbb{R} \mid x(-1) = 0\}$$

and

$$X^*_\omega = \{x : [-1,1] \to \mathbb{R} \mid x(-1) = 0 \; and \; \omega_\lambda(x,0) < \infty \; for \; some \; \lambda > 0\}.$$

Let the binary relation $\preceq$ on X_ω be the ordinary relation that define for all $x, y \in X$ as follows:

$$x \preceq y \text{ iff } x(t) \leq y(t) \; for \; all \; t \in [-1,1].$$

Define $T : X_\omega \times X_\omega \longrightarrow X_\omega$ by $T(x,y) = \frac{1}{5}\,(x(t) - y(t))$. We show that T has the properties of Theorem 2.1 Indeed,

$$\begin{aligned}
\omega_\lambda(T(x,y), x_\circ) &= \omega_\lambda\left(\frac{1}{5}(x-y), 0\right)\\
&= \sup_\tau \sum_{i=1}^{m} \varphi\left(\frac{|\frac{1}{5}(x(t_i) - x(t_{i-1})) - \frac{1}{5}(y(t_i) + y(t_{i-1}))|}{\lambda(t_i - t_{i-1})}\right)(t_i - t_{i-1})\\
&\leq \frac{1}{\sqrt{5}} \sup_\tau \sum_{i=1}^{m} \varphi\left(\frac{|(x(t_i) - x(t_{i-1}))| + |(y(t_i) - y(t_{i-1}))|}{\lambda(t_i - t_{i-1})}\right)(t_i - t_{i-1})\\
&\leq \frac{1}{\sqrt{5}}\left(\omega_{\lambda_1}(x,0) + \omega_{\lambda_2}(y,0)\right)\\
&< \infty
\end{aligned}$$

where, $\lambda \geq \max\{\lambda_1, \lambda_2\}$. On the other hand $(T(x,y))(-1) = \left(\frac{1}{5}(x-y)\right)(-1) = 0$. This implies that $T(x,y) \in X^*_\omega = X_\omega$.

T has mixed monotone property, because if $x_1(t) \leq x_2(t)$ for all $t \in [-1,1]$ then $T(x_1,y) \leq T(x_2,y)$, similarly if $y_1(t) \leq y_2(t)$ for all $t \in [-1,1]$ then $T(x,y_1) \geq T(x,y_2)$.

Now let $x_\circ(t) = -1$ and $y_\circ(t) = 1$ be constant functions on the closed interval $[-1,1]$, then $T(x_\circ, y_\circ) = -\frac{2}{5} = x_1(t)$ is constant function that implies $x_\circ \leq x_1$. Similarly $y_\circ(t) = 1 \geq y_1(t) = T(y_0, x_0) = \frac{2}{5}$. For inequality (2.1) we have;

$$\begin{aligned}
&\omega_\lambda(T(x,y), T(u,v)) = \omega_\lambda(\frac{1}{5}(x-y), \frac{1}{5}(u-v))\\
&= \sup_\tau \sum_{i=1}^{m} \varphi(\frac{\frac{1}{5}|(x(t_i) - y(t_i)) + (u(t_{i-1}) - v(t_{i-1})) - (x(t_{i-1}) - y(t_{i-1})) - (u(t_i) - v(t_i))|}{\lambda(t_i - t_{i-1})})(t_i - t_{i-1})\\
&\leq \frac{1}{\sqrt{5}} \sup_\tau \sum_{i=1}^{m} \varphi(\frac{|(x(t_i) + u(t_{i-1}) - x(t_{i-1}) - u(t_i))| + |(y(t_i) + v(t_{i-1}) - y(t_{i-1}) - v(t_i))|}{\lambda(t_i - t_{i-1})})(t_i - t_{i-1})\\
&\leq \frac{1}{\sqrt{5}}(\omega_\lambda(x,u) + \omega_\lambda(y,v)).
\end{aligned}$$

Therefore $k = \frac{2}{\sqrt{5}} < 1$, and by Theorem 2.1 there exit $x, y \in X_\omega$ such that $x = T(x,y)$ and $y = T(y,x)$.

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