Int. J. Nonlinear Anal. Appl. 14 (2023) 4, 111–124
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.29383.4149

Local well-posedness and blow-up of solution for a higher-order wave equation with viscoelastic term and variable-exponent

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(Communicated by Haydar Akca)

Abstract

We investigate in this paper a value problem related to the following nonlinear higher-order wave equation

$$\eta_{tt} + (-\Delta)^m \eta - \int_0^t g(t-s)(-\Delta)^m \eta(s)\, ds + \eta_t = |\eta|^{p(x)-2}\eta.$$

Firstly, we prove the existence and uniqueness of the local solution under suitable conditions for the relaxation function g and viable-exponent $p(.)$, using a method, which is a mixture of the Faedo-Galarkin and Banach fixed point theorem, and prove also the solution blows up in finite time. Finally, we give a two-dimensional numerical example to illustrate the blow-up result.

Keywords: Higher-order equation, wave equation, variable-exponent, local solution, blow up
2020 MSC: Primary 35B40; Secondary 35L25

1 Introduction

We consider the following boundary value problem:

$$\begin{cases} \eta_{tt} + (-\Delta)^m \eta - \int_0^t g(t-s)(-\Delta)^m \eta(s)\, ds + \eta_t = |\eta|^{p(x)-2}\eta, & \text{in } \Omega_t, \\ \eta(x,t) = 0,\ \frac{\partial^i \eta}{\partial v^i} = 0,\ i = 1,\ 2, ..., m-1, & \text{on } \Gamma_t \\ \eta(x,0) = \eta_0(x),\ \eta_t(x,0) = \eta_1(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $m \geq 1$ is a natural number, Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\partial\Omega$ is smooth boundary of Ω, $\Omega_t = \Omega \times \mathbb{R}^+$, $\Gamma_t = \partial\Omega \times \mathbb{R}^+$, g is the relaxation function satisfying some condition to be specified later. $p(.)$ is given measurable functions on Ω, satisfying

$$\begin{cases} 2 < p_1 \leq p(x) \leq p_2 \leq p^* \text{ for } n \leq 2m, \\ 2 < p_1 \leq p(x) \leq p_2 \leq p^* \text{ for } n > 2m. \end{cases} \tag{1.2}$$

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Received: December 2022 *Accepted:* February 2023

with

$$p^{*}=\left\{\begin{array}{c}\infty,\text{ if }\ n\leq 2m,\\ \frac{2n}{n-2m},\text{ if }n>2m,\end{array}\right.$$

where

$$p_{1}:=\ ess_{x\in\Omega}\inf\left(p\left(x\right)\right),\ p_{2}:=ess_{x\in\Omega}\sup\left(p\left(x\right)\right)$$

We also assume that $p\left(x\right)$ satisfy the following condition:

$$\left|\xi\left(x_{1}\right)-\xi\left(x_{2}\right)\right|\leq-\frac{R}{\log\left|x_{1}-x_{2}\right|},\ \text{for a.e. }x_{1},\,x_{2}\in\Omega,\ \text{with }\ \left|x_{1}-x_{2}\right|<\mu,\tag{1.3}$$

$R>0,\ 0<\mu<1.$

The exponents of nonlinearity are given constants:

Many authors looked into the following equation with memory and source terms

$$\eta_{tt}+\left(-\Delta\right)^{m}\eta-\int_{0}^{t}g\left(t-s\right)\left(-\Delta\right)^{m}\eta\left(x,s\right)ds=\left|\eta\right|^{p-2}\eta.\tag{1.4}$$

When $m=2$, Tahamatani et al [20] in the both instance of nonpositive initial energy and positive initial energy demostreted the existence of weak solution and proved that solution blow-up in finite time and gave the lifespan estimates of solutions. In the case of $m\geq 1$ and by used the Galerkin's method Yaojun Ye [23], studied weak global solution, under appropriate conditions on the relaxtion function g and the positive initial energy as well as the non-positive initial energy proved that solution blow up in finite time. The higher-order in case when $m\geq 1$, equation (1.4) with damping term and without the viscoelastic term becomes

$$\eta_{tt}+\left(-\Delta\right)^{m}\eta+a\eta_{t}\left|\eta_{t}\right|^{\gamma-1}=b\left|\eta\right|^{m-2}\eta.\tag{1.5}$$

Brenner et al. [3] proved the existence and uniqueness of classical solutions in the Hilbert space. Pecher in [16] by used the potential well method, investigated the existence and uniqueness of the Cauchy problem for the equation (1.5) .

For more attached results concerning the existence and asymptotic properties of solutions of (1.1), can also referred to [2, 4, 6, 11, 12, 13, 14, 21, 24].

The exponents of nonlinearity are given functions:

Messaoudi et al. In [9] considered the following equation:

$$\eta_{tt}-\Delta\eta+a\left|\eta_{t}\right|^{m(x)-2}\eta_{t}=b\left|\eta\right|^{p(x)-2}\eta,\tag{1.6}$$

and used the Faedo-Galerkin method to establish the existence of a unique weak local solution. They also proved with negative initial that the solution blow up in finite time. Messaoudi and Talahmeh [7], considered the following equation:

$$\eta_{tt}-div\left(\left|\nabla\eta\right|^{r(x)-2}\nabla\eta\right)+a\left|\eta_{t}\right|^{m(x)-2}\eta_{t}=b\left|\eta\right|^{p(x)-2}\eta,\tag{1.7}$$

where a, b and are all positive constants. They proved a finite-time blow-up result for the solution with negative initial energy as well as for certain solutions with positive initial energy; in the case where $m\left(x\right)=2$ and under suitable conditions on the exponents, they established a blow-up result for solutions with arbitrary positive initial energy.

In [17] Pişkin studied the global nonexistence of solutions for the Klien-Gorden equation

$$\eta_{tt}-\Delta\eta+m^{2}u+\left|\eta_{t}\right|^{m(x)-2}\eta_{t}=\left|\eta\right|^{p(x)-2}\eta,\tag{1.8}$$

Antontsev et al. In [2] considered the strong damping Petrovsky equation with variable exponents

$$\eta_{tt}+\Delta^{2}\eta-\Delta\eta_{t}+\left|\eta_{t}\right|^{m(x)-1}\eta_{t}=\left|\eta\right|^{p(x)-1}\eta,\tag{1.9}$$

they proved the local existence and blow up of solution.

Sun-Hye Park in [15] studied a blow up result for the following viscoelastic wave equation with varible exponents

$$\eta_{tt}-\Delta\eta+\int_{0}^{t}g(t-s)\Delta\eta(s)ds+a|\eta|^{m(x)-2}\eta_{t}=b|\eta|^{p(x)-2}\eta.\tag{1.10}$$

Our objective in this paper is to study, in section two, we state some notations and assumptions are introduced, in section third, we prove the existence of local solution, in section foor, we show that the solution with the negative initial energy blow up in the finite time. In section five, we give a two-dimension numerical example to illustrate the blow up result.

2 Preliminaries

We begin this section with some notations and definitions. Let $H^m(\Omega)$ be the Sobolev space. $H_0^m(\Omega)$ denotes the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. we denote the norm $\|D^m.\|_2$ instead of $H_0^m(\Omega)$ norm $\|.\|_{H_0^m(\Omega)}$, where D denotes the gradient operator, that is $D^m. = \nabla. = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right)$. Moreover $D^m. = \Delta^j.$ if $m = 2j$ and $D^m. = D\Delta^j.$ if $m = 2j+1$.

Let $\xi : \Omega \to [1, +\infty]$ be a measurable function. The Lebesque space with variale exponent $\xi(.)$ by:

$$L^{\xi(.)}(\Omega) := \left\{v : \Omega \to \mathbb{R} : \text{ measurable in } \Omega,\ \varrho_{\xi(.)}(\lambda v) < +\infty, \text{ for some } \lambda > 0\right\},$$

where $\varrho_{\xi(.)}(v) = \int_\Omega |v(x)|^{\xi(x)}\, dx$.

The set $L^{\xi(.)}(\Omega)$ equipped with the Luxemburg's norm

$$\|v\|_{q(.)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{v(x)}{\lambda}\right|^{\xi(x)} dx \le 1\right\},$$

$L^{\xi(.)}(\Omega)$ is a Banach space [5]. The Sobolev space with variable-exponent $W^{1,\xi(.)}(\Omega)$ is:

$$W^{1,\xi(.)}(\Omega) := \left\{v \in L^{\xi(.)}(\Omega) \text{ such that } \nabla v \text{ exists and } |\nabla v| \in L^{\xi(.)}(\Omega)\right\}.$$

This is a Banach space with respect to the norm $\|v\|_{W^{1,\xi(.)}(\Omega)} = \|v\|_{\xi(.)} + \|\nabla v\|_{\xi(.)}$. Furthermore, we set $W_0^{1,\xi(.)}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in the space $W^{1,\xi(.)}(\Omega)$.

Lemma 2.1. [5] If

$$1 \le \xi_1 := ess\inf_{x\in\Omega} \xi(x) \le \xi(x) \le \xi_2 := ess\sup_{x\in\Omega} \xi(x) < \infty,$$

then we have

$$\min\left\{\|\eta\|_{\xi(.)}^{\xi_1},\ \|\eta\|_{\xi(.)}^{\xi_2}\right\} \le \varrho_{\xi(.)}(\eta) \le \max\left\{\|\eta\|_{\xi(.)}^{\xi_1},\ \|\eta\|_{\xi(.)}^{\xi_2}\right\},$$

for any $\eta \in L^{q(.)}(\Omega)$.

For that purpose, we assume that

(H) $g \in \mathbb{C}^1([0, +\infty))$ is non-negative function satisfaying

$$1 - \int_0^t g(s)\, ds = \beta > 0,\ g'(t) \le 0 \text{ for } t \ge 0. \tag{2.1}$$

In the proof of our main result, we shall make use of the following Lemma.

Lemma 2.2. [25] Assume $\rho(t)$ is a twice continuously differentiable satisfying

$$\begin{cases} \rho''(t) + \rho'(t) \ge C\rho^{1+\alpha}(t), & t, C, \alpha > 0 \\ \rho(0) > 0,\ \rho'(0) \ge 0. \end{cases} \tag{2.2}$$

Then, $\rho(t)$ blows up in finite time.

Furthermore, the energy of problem (1.1) is

$$\begin{aligned} E(t) = &\frac{1}{2}\|\eta_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right)\|D^m\eta(t)\|_2^2 + \frac{1}{2}(g \circ D^m\eta)(t) \\ &- \int_\Omega \frac{1}{p(x)}|\eta(t)|^{p(x)}\, dx, \end{aligned} \tag{2.3}$$

where $(g \circ D^m v)(t) = \int_0^t g(t-s)\|D^m v(t) - D^m v(s)\|^2\, ds$.

Lemma 2.3. Suppose (H)and (1.2) hold. Then $E(t)$ decreases, which

$$E^{'}(t) = \frac{1}{2}\left(g' \circ D^m \eta\right)(t) - \frac{1}{2} g(t) \left\|D^m \eta(t)\right\|_2^2 - \left\|\eta_t\right\|_2^2 \le 0,$$

furthermore,

$$E(t) - E(0) \le 0, \quad t \ge 0. \tag{2.4}$$

Proof . Multiplying the first equation in (1.1) by η_t and integrating over Ω, we get

$$\begin{aligned}&\int_\Omega \eta_t \eta_{tt} dx + \int_\Omega \eta_t (-\Delta)^m \eta dx - \int_\Omega \eta_t \int_0^t g(t-s)(-\Delta)^m \eta(x,s)\, dxds + \int_\Omega \eta_t^2 dx \\ &= \int_\Omega \eta_t \eta |\eta|^{p(x)-2} dx,\end{aligned}$$

then use integration par parts, yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\eta_t|^2 dx + \int_\Omega |D^m \eta|^2 dx\right) - \int_0^t g(t-s) \int_\Omega D^m \eta_t(t)\, D^m \eta(s)\, dxds + \int_\Omega \eta_t^2 dx \\ &= \frac{d}{dt}\left(\int_\Omega \frac{1}{p(x)} |\eta|^{p(x)} dx\right).\end{aligned} \tag{2.5}$$

We take account $\int_0^t g(t-s)\, ds = \int_0^t g(s)\, ds$, the third term in (2.5) can be estimated as:

$$\begin{aligned}&\int_0^t g(t-s) \int_\Omega D^m \eta_t(t)\, D^m \eta(s)\, dxds \\ &= \int_0^t g(t-s) \int_\Omega D^m \eta_t(t) . \left(D^m \eta(s) - D^m \eta(t)\right) dxds + \int_0^t g(t-s) \int_\Omega D^m \eta_t(t) . D^m \eta(t)\, dxds \\ &= -\frac{1}{2} \int_0^t g(t-s) \frac{d}{dt} \int_\Omega \left|D^m \eta(s) - D^m \eta(t)\right|^2 dxds + \int_0^t g(t-s) \left(\frac{d}{dt}\frac{1}{2} \int_\Omega \left|D^m \eta(t)\right|^2 dx\right) ds \\ &= -\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-s) \int_\Omega \left|D^m \eta(s) - D^m \eta(t)\right|^2 dxds\right] \\ &+ \frac{1}{2} \int_0^t g'(t-s) \int_\Omega \left|D^m \eta(s) - D^m \eta(t)\right|^2 dxds + \frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)\, ds \int_\Omega \left|D^m \eta(t)\right|^2 dx\right] - \frac{1}{2} g(t) \int_\Omega \left|D^m \eta(t)\right|^2 dx.\end{aligned} \tag{2.6}$$

Insert (2.6) in (2.5) to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega |\eta_t|^2 dx + \int_\Omega |D^m \eta|^2 dx\right\} + \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(t-s) \int_\Omega \left|D^m \eta(s) - D^m \eta(t)\right|^2 dxds\right) \\ &- \frac{1}{2} \int_0^t g'(t-s) \int_\Omega \left|D^m \eta(s) - D^m \eta(t)\right|^2 dxds - \frac{1}{2}\frac{d}{dt} \int_0^t g(s)\, ds \int_\Omega \left|D^m \eta(t)\right|^2 dx + \frac{1}{2} g(t) \int_\Omega \left|D^m \eta(t)\right|^2 dx + \int_\Omega \eta_t^2 dx \\ &= \frac{d}{dt}\left(\int_\Omega \frac{1}{p(x)} |\eta|^{p(x)} dx\right).\end{aligned}$$

We deduce that

$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\Omega}|\eta_t|^2\,dx+\frac{1}{2}\int_{\Omega}|D^m\eta|^2\,dx-\int_{\Omega}\frac{1}{p(x)}|\eta|^{p(x)}\,dx\right.$$
$$\left.+\frac{1}{2}\int_0^t g(t-s)\int_{\Omega}|D^m\eta(s)-D^m\eta(t)|^2\,dxds-\frac{1}{2}\int_0^t g(s)\,ds\int_{\Omega}|D^m\eta(t)|^2\,dx\right\}$$
$$=\frac{1}{2}\int_0^t g'(t-s)\int_{\Omega}|D^m\eta(s)-D^m\eta(t)|^2\,dxds-\frac{1}{2}g(t)\int_{\Omega}|D^m\eta(t)|^2\,dx-\int_{\Omega}\eta_t^2dx.$$

Using the equality $(g\circ D^m\eta)(t)=\int_0^t g(t-s)\,\|D^m\eta(t)-D^m\eta(s)\|_2^2\,ds$, we get

$$\frac{d}{dt}\left\{\frac{1}{2}\|\eta_t(t)\|_2^2+\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|D^m\eta(t)\|_2^2+\frac{1}{2}(g\circ D^m\eta)(t)-\int_{\Omega}\frac{1}{p(x)}|\eta(t)|^{p(x)}\,dx\right\}$$
$$=\frac{1}{2}\int_0^t g'(t-s)\int_{\Omega}|D^m\eta(s)-D^m\eta(t)|^2\,dxds-\frac{1}{2}g(t)\int_{\Omega}|D^m\eta(t)|^2\,dx-\int_{\Omega}\eta_t^2dx,$$

hence, using (2.3), we obtain

$$E'(t)=\frac{1}{2}(g'\circ D^m\eta)-\frac{1}{2}g(t)\,\|D^m\eta\|_2^2-\|\eta_t\|^2\leq 0.$$

Using integration of last inequality, we get

$$E(t)\leq E(0) \tag{2.7}$$

□

Lemma 2.4. [8] Suppose that(1.2), (1.3) hold and $E(0)<0$.Then the solution of (1.1) satisfies

$$\int_{\Omega}|\eta|^{p(x)}\,dx\geq c\,\|\eta\|_{p_1}^{p_1}. \tag{2.8}$$

3 Local existence

In this section, we will prove the nonexistence global solution of (1.1), we state the following lemma witch can be obtained by using the Faedo-Galerkin method by combining the argument of [9, 15, 22, 23].

Lemma 3.1. Assume that (1.2) and (1.3) hold and $(\eta_0,\eta_1)\in\left(H_0^m(\Omega),\,L^2(\Omega)\right)$ and give $f(t,\ x)$ a fixed function on $\Omega\times(0,t)$.Then there existe a unique local solution η of

$$\begin{cases}\eta_{tt}+(-\Delta)^m\eta-\int_0^t g(t-s)(-\Delta)^m\eta(s)\,ds+\eta_t=f(t,x)\,, & \text{in }\Omega_t,\\ \eta(x,t)=0,\ \frac{\partial^i\eta}{\partial v^i}=0,\ i=1,\ 2,...,m-1, & \text{on }\Gamma_t\\ \eta(x,0)=\eta_0(x)\,,\ \eta_t(x,0)=\eta_1(x)\,, & \text{in }\Omega,\end{cases} \tag{3.1}$$

Satisfying $\eta\in L^\infty((0,\ T),\ H_0^m(\Omega))\,,\eta_t\in L^\infty((0,\ T)\,,L^2(\Omega))\cap L^2(\Omega\times(0,\ T))$, where $f\in L^2(\Omega\times(0,\ T))$. Now, we prove the local existence of (1.1) by using the method of Banach fixed point theorem.

Theorem 3.2. Suppose that (1.2) holds. Suppose further that

$$2\leq p_1\leq p(x)\leq p_2\leq\frac{2(n-m)}{n-2m},\ \ (n>2m) \tag{3.2}$$

and $(\eta_0,\eta_1)\in\left(H_0^m(\Omega),\,L^2(\Omega)\right)$. Then there exists $T>0$, such that (1.1) has unique local solution

$$\eta\in L^\infty((0,\ T),\ H_0^m(\Omega))\,,\eta_t\in L^\infty((0,\ T),\ L^2(\Omega))\cap L^2(\Omega\times(0,\ T)).$$

Proof . Let $v\in L^{\infty}\left((0,T),\ H_0^m\left(\Omega\right)\right)$ and $f\left(v\right)=\left|v\right|^{p(x)-2}v$, we have

$$\left\|f\left(v\right)\right\|^2=\int\limits_{\Omega}\left|v\right|^{2p(x)-2}dx\leq\int\limits_{\Omega}\left|v\right|^{2p_2-2}dx+\int\limits_{\Omega}\left|v\right|^{2p_1-2}dx<\infty,$$

Since

$$2p_1-2\leq 2p_2-2\leq\frac{2n}{n-2m},\ \ (n>2m).$$

We have

$$f\left(v\right)\in L^{\infty}\left((0,t),H_0^m\left(\Omega\right)\right)\subset L^2\left(\Omega\times(0,\ T)\right).$$

Therefore, for each $v\in L^{\infty}\left((0,\ T),\ H_0^m\left(\Omega\right)\right)$, there existe a unique

$$\eta\in L^{\infty}\left((0,\ T),\ H_0^m\left(\Omega\right)\right),\ \eta_t\in L^{\infty}((0,T),\ L^2\left(\Omega\right))\cap L^2\left(\Omega\times(0,\ T)\right)$$

Satisfying the following problem

$$\begin{cases}\eta_{tt}+\left(-\Delta\right)^m\eta-\int\limits_0^t g\left(t-s\right)\left(-\Delta\right)^m\eta\left(s\right)ds+\eta_t=f\left(v\right), & \text{in }\Omega_t,\\ \eta\left(x,\ t\right)=0,\ \frac{\partial^i\eta}{\partial v^i}=0,\ i=1,\ 2,...,m-1, & \text{on }\Gamma_t\\ \eta\left(x,\ 0\right)=\eta_0\left(x\right),\ \eta_t\left(x,\ 0\right)=\eta_1\left(x\right), & \text{in }\Omega,\end{cases}\tag{3.3}$$

Let a map $G:X_T\longrightarrow X_T$ by $G\left(v\right)=u$, where

$$X_T=\left\{w\in L^{\infty}\left((0,T),\ H_0^m\left(\Omega\right)\right),w_t\in L^{\infty}((0,T),\ L^2\left(\Omega\right))\right\}.$$

where, X_T is Banach space with respect to the norm

$$\left\|w\right\|_{X_T}=\frac{1}{2}\sup_{(0,T)}\left\|w_t\right\|_2^2+\frac{1}{2}l\sup_{(0,T)}\left\|D^mw\right\|_2^2.$$

Multiplying the equation (3.3) by η_t and integrating over $\Omega\times(0,t)$, to get

$$\begin{aligned}&\frac{1}{2}\left\|\eta_t\left(t\right)\right\|_2^2+\frac{1}{2}\left(1-\int_0^t g\left(s\right)ds\right)\left\|D^m\eta\left(t\right)\right\|_2^2+\frac{1}{2}\left(g\circ D^m\eta\right)\left(t\right)\\&-\int_0^t\left[\frac{1}{2}\left(g'\circ D^m\eta\right)\left(s\right)-\frac{1}{2}g\left(s\right)\left\|D^m\eta\left(s\right)\right\|_2^2\right]ds+\int_0^t\int_{\Omega}\eta_t^2dxds\frac{1}{2}\left\|\eta_1\right\|_2^2+\frac{1}{2}\left\|D^m\eta_0\right\|_2^2+\int_0^t\int_{\Omega}\left|v\right|^{2p(x)-2}v\eta_tdxds.\end{aligned}\tag{3.4}$$

Using the Young and the Sobolev-Poincare inequalities, we obtain

$$\begin{aligned}\left|\int_{\Omega}\left|v\right|^{p(x)-2}v\eta_tdx\right|&\leq\frac{\delta}{4}\left\|\eta_t\left(t\right)\right\|_2^2+\frac{4}{\delta}\int_{\Omega}\left|v\right|^{2p(x)-2}dx\\&\leq\frac{\delta}{4}\left\|\eta_t\left(t\right)\right\|_2^2+\frac{4}{\delta}\left(\int_{\Omega}\left|v\right|^{2p_2-2}dx+\int_{\Omega}\left|v\right|^{2p_1-2}dx\right)\\&\leq\frac{\delta}{4}\left\|\eta_t\left(t\right)\right\|_2^2+\frac{4c_*}{\delta}\left(\left\|D^mv\right\|_2^{2p_2-2}+\left\|D^mv\right\|_2^{2p_1-2}\right)\end{aligned}\tag{3.5}$$

Thus by (3.4) and (3.5), we get

$$\frac{1}{2}\left\|\eta_t\left(t\right)\right\|_2^2+\frac{1}{2}l\left\|D^m\eta\left(t\right)\right\|_2^2\leq k_0+\frac{\delta T}{4}\sup_{(0,\ T)}\left\|\eta_t\left(t\right)\right\|_2^2+\frac{4c_*}{\delta}\left(\int_0^T\left\|D^mv\right\|_2^{2p_2-2}+\int_0^T\left\|D^mv\right\|_2^{2p_1-2}\right)ds.$$

Then we have

$$\frac{1}{2}\sup_{(0,\ T)}\left\|\eta_t\left(t\right)\right\|_2^2+\frac{1}{2}l\sup_{(0,\ T)}\left\|D^m\eta\left(t\right)\right\|_2^2\leq k_0+\frac{\delta T}{4}\sup_{(0,\ T)}\left\|\eta_t\left(t\right)\right\|_2^2+\frac{4c_*T}{\delta l^{2p_2-2}}\left[\int_0^T\|v\|_{X_T}^{p_2-1}+\int_0^T\|v\|_{X_T}^{p_1-1}\right],$$

where $k_0=\frac{1}{2}\left\|\eta_1\right\|_2^2+\frac{1}{2}\left\|D^m\eta_0\right\|_2^2$ and c_* is the embedding constant. Taking $\delta T=1$, we get

$$\frac{1}{2}\sup_{(0,\ T)}\left\|\eta_t\left(t\right)\right\|_2^2+\frac{1}{2}l\sup_{(0,\ T)}\left\|D^m\eta\left(t\right)\right\|_2^2\leq 2k_0+\frac{8c_*T}{\delta l^{2p_2-2}}\left[\|v\|_{X_T}^{p_2-1}+\|v\|_{X_T}^{p_1-1}\right].$$

Then

$$\left\|\eta\right\|_{X_T}\leq K+T\alpha\left[\|v\|_{X_T}^{p_2-1}+\|v\|_{X_T}^{p_1-1}\right].$$

Choosing M_0 large enough and T Sufficiently small such that

$$\left\|\eta\right\|_{X_T}\leq K+2T\alpha M_0^{2p_2-2}\leq M_0.$$

If $K\leq M_0^2$ and $T\leq T_0<\frac{M_0^2-K}{2\alpha M_0^{2P_2-2}}$. Thus, we have $G:\Lambda\longrightarrow\Lambda$, where

$$\Lambda=\left\{w\in X_T,\ \left\|w\right\|_{X_T}\leq M_0^2\right\}.$$

Next, we show that G is contraction. For this purpose, let $\eta_1=G\left(v_1\right)$ and $\eta_2=G\left(v_2\right)$ and set $\eta=\eta_1-\eta_2$ satisfyies

$$\begin{cases}\eta_{tt}+\left(-\Delta\right)^m\eta-\int\limits_0^t g\left(t-s\right)\left(-\Delta\right)^m\eta\left(s\right)ds+\left(\eta_{1t}-\eta_{2t}\right) & \\ \qquad =\left|v_1\right|^{p(x)-2}v_1-\left|v_2\right|^{p(x)-2}v_2 & \text{in } \Omega\times(0,t) \\ \eta\left(x,t\right)=0,\ \frac{\partial^i\eta}{\partial v^i}=0,\ i=1,\ 2,...,m-1, & \text{on } \Gamma_t \\ \eta\left(x,\ 0\right)=\eta_t\left(x,\ 0\right)=0, & \text{in } \Omega,\end{cases}\tag{3.6}$$

Multiplying (3.6) by $\eta_t=\eta_{1t}-\eta_{2t}$ and integrate on $\Omega\times(0,t)$, we get

$$\frac{1}{2}\left\|\eta_t\left(t\right)\right\|_2^2+\frac{1}{2}\left(1-\int_0^t g\left(s\right)ds\right)\left\|D^m\eta\left(t\right)\right\|_2^2+\frac{1}{2}\left(g\circ D^m\eta\right)\left(t\right)$$
$$-\int_0^t\left[\frac{1}{2}\left(g'\circ D^m\eta\right)\left(s\right)-\frac{1}{2}g\left(s\right)\left\|D^m\eta\left(s\right)\right\|_2^2\right]ds+\int_0^t\left\|\eta_{1t}-\eta_{2t}\right\|^2dx+\int_0^t\int_\Omega\left(\left|v_1\right|^{p(x)-2}v_1-\left|v_2\right|^{p(x)-2}v_2\right)\eta_t ds.$$

Then, we have

$$\frac{1}{2}\left\|\eta_t\left(t\right)\right\|_2^2+\frac{1}{2}l\left\|D^m\eta\left(t\right)\right\|_2^2\leq\int_0^t\int_\Omega\left(h\left(v_1\right)-h\left(v_2\right)\right)\eta_t dxds,\tag{3.7}$$

where $h\left(s\right)=\left|s\right|^{p(x)-2}s$. Now, we estimate $I=\int_0^t\int_\Omega\left(h\left(v_1\right)-h\left(v_2\right)\right)\eta_t dxds$.We get

$$I\leq\left|\int_0^t\int_\Omega\left(h\left(v_1\right)-h\left(v_2\right)\right)\eta_t dxds\right|\leq\int_\Omega\left|h'\left(\xi\right)\right|\left|v\right|\left\|\eta_t\right\|dx,$$

where $v=v_1-v_2$ and $\xi=\alpha v_1+\left(1-\alpha\right)v_2,\ \ 1\geq\alpha\geq 0$. By the Young inequality implies

$$\begin{aligned}I&\leq\frac{\delta}{2}\left\|\eta_t\left(t\right)\right\|_2^2+\frac{2}{\delta}\int_\Omega\left|h'\left(\xi\right)\right|^2\left|v\right|^2dx\\&\leq\frac{\delta}{2}\left\|\eta_t\left(t\right)\right\|_2^2+\frac{2\left(p_2-1\right)^2}{\delta}\int_\Omega\left|\alpha v_1+\left(1-\alpha\right)v_2\right|^{2(p(x)-2)}\left|v\right|^2dx\\&\leq\frac{\delta}{2}\left\|\eta_t\left(t\right)\right\|_2^2+\frac{2\left(p_2-1\right)}{\delta}\left(\int_\Omega\left|v\right|^{\frac{2n}{n-2m}}dx\right)^{\frac{n-2m}{n}}\left[\int_\Omega\left|\alpha v_1+\left(1-\alpha\right)v_2\right|^{\frac{n}{m}(p(x)-2)}\right]^{\frac{2m}{n}}\\&\leq\frac{\delta}{2}\left\|\eta_t\left(t\right)\right\|_2^2+c_\delta\left(\int_\Omega\left|v\right|^{\frac{2n}{n-2m}}dx\right)^{\frac{n-2m}{n}}\left[\int_\Omega\left|\alpha v_1+\left(1-\alpha\right)v_2\right|^{\frac{n}{m}(p_2-2)}dx+\int_\Omega\left|\alpha v_1+\left(1-\alpha\right)v_2\right|^{\frac{n}{m}(p_2-2)}dx\right]^{\frac{2m}{n}}.\end{aligned}\tag{3.8}$$

Since $2\leq p_1\leq p(x)\leq p_2\leq \frac{2(n-m)}{n-2m}$, $(n>2)$, we get

$$\begin{aligned} I &\leq \frac{\delta}{2}\|\eta_t(t)\|_2^2 + c_\delta c_* \|D^m v\|_2^2 \left(\|D^m v_1\|_2^{2(p_2-2)} + \|D^m v_1\|_2^{2(p_1-2)} + \|D^m v_2\|_2^{2(p_2-2)} + \|D^m v_2\|_2^{2(p_1-2)}\right) \\ &\leq \frac{\delta}{2}\|\eta_t(t)\|_2^2 + 4c_{\delta,l} c_* M_0^{2(p_2-2)} \|D^m v\|_2^2 . \end{aligned}$$

Therefore, (3.7) takes the form

$$\frac{1}{2}\sup_{(0,\,T)}\|\eta_t(t)\|_2^2 + \frac{1}{2} l \sup_{(0,T)} \|D^m \eta(t)\|_2^2 \leq \frac{\delta}{2} T_0 \frac{1}{2}\sup_{(0,\,T)}\|\eta_t(t)\|_2^2 + 4c_{\delta,l} c_* T_0 M_0^{2(p_2-2)} \sup_{(0,\,T)} \|D^m v\|_2^2 .$$

Then, we arrive at

$$\|\eta\|_{X_T} \leq \delta T_0 \|\eta\|_{X_T} + 8c_{\delta,l} c_* T_0 M_0^{2(p_2-2)} \|v\|_{X_T} .$$

We take δ small sufficient, we obtain

$$\|\eta\|_{X_T} \leq \lambda T_0 \|v\|_{X_T} .$$

There exists T_0 small for that $\lambda T_0 < 1$, then we get

$$\|\eta\|_{X_T} \leq \theta \|v\|_{X_T}, \quad 0<\theta<1.$$

Then, G is a contraction maping. Thus, implies that the unique solution $\eta \in \Lambda$ satisfied $G(\eta)=\eta$. Thus, η is nonglobal solution of (1.1). □

4 Blow up of solution

Now we state our main result

Theorem 4.1. Suppose that (H) holds. Assume further

$$\int_0^t g(s)\,ds < \frac{p_1(p_1-2)}{(p_1-1)^2},\ \forall t\geq 0, \tag{4.1}$$

and the initial condition

$$(\eta_0,\eta_1)\in H_0^m(\Omega)\times L^2(\Omega),$$

satisfying

$$E(0)<0 \text{ and } \eta_0\eta_1>0.$$

Then the solution of (1.1) blows up in finite time.

Proof . To apply the Lemma 2.2, the following is defined:

$$\rho(t) = \frac{1}{2}\int_\Omega |\eta(x,t)|^2\,dx.$$

Therefore,

$$\rho'(t) = \int_\Omega \eta\eta_t dx,\ \rho''(t) = \int_\Omega \left(\eta\eta_{tt} + |\eta_t|^2\right) dx. \tag{4.2}$$

By using the first equation of (1.1), the second equation of (4.2) becomes

$$\begin{aligned}\rho''(t) &= \int_\Omega \left(\eta\eta_{tt} + |\eta_t|^2\right) dx = \int_\Omega \eta\eta_{tt} dx + \int_\Omega |\eta_t|^2 dx \\ &= \int_\Omega |\eta_t|^2 dx + \int_\Omega \eta(t) \left(-(-\Delta)^m \eta(t) + \int_0^t g(t-s)(-\Delta)^m \eta(s)\, ds - \eta_t + |\eta(t)|^{p(x)-2}\eta(t) \right) dx \\ &= \int_\Omega |\eta_t|^2 dx - \int_\Omega |D^m\eta(t)|^2 dx - \int_\Omega \eta(t)\eta_t dx + \int_\Omega |\eta(t)|^{p(x)} dx \\ &+ \int_0^t g(t-s) \int_\Omega D^m\eta(t) . D^m\eta(s)\, dxds.\end{aligned}$$

We add and subtract the term $\int_0^t g(t-s) \int_\Omega D^m\eta(t) . D^m\eta(t)\, dxds$, and we take account that $\int_0^t g(t-s) \int_\Omega D^m\eta(t) . D^m\eta(t)\, dxds = \int_0^t g(t-s)\, ds \int_\Omega D^m\eta(t) . D^m\eta(t)\, dx$, we obtain

$$\begin{aligned}\rho''(t) = \int_\Omega |\eta_t|^2 dx - \int_\Omega |D^m\eta|^2 dx - \int_\Omega \eta\eta_t dx + \int_\Omega |\eta|^{p(x)} dx + \int_0^t g(t-s) \int_\Omega D^m\eta(t) . D^m\eta(t)\, dxds \\ - \int_0^t g(t-s) \int_\Omega D^m\eta(t) . (D^m\eta(t) - D^m\eta(s))\, dxds.\end{aligned}$$

We recall that $\int_0^t g(t-s)\, ds = \int_0^t g(s)\, ds$, then

$$\begin{aligned}\rho''(t) = \int_\Omega |\eta_t|^2 dx - \int_\Omega |D^m\eta|^2 dx - \int_\Omega \eta\eta_t dx + \int_\Omega |\eta|^{p(x)} dx + \int_0^t g(s)\, ds \int_\Omega D^m\eta(t) . D^m\eta(t)\, dx \\ - \int_0^t g(t-s) \int_\Omega D^m\eta(t) . (D^m\eta(t) - D^m\eta(s))\, dxds\end{aligned}$$

So that,

$$\begin{aligned}\rho''(t) = -\left(1 - \int_0^t g(s)\, ds\right) \int_\Omega |D^m\eta(t)|^2 dx - \int_0^t g(t-s) \int_\Omega D^m\eta(t) . (D^m\eta(t) - D^m\eta(s))\, dxds \\ + \int_\Omega |\eta|^{p(x)} dx + \int_\Omega |\eta_t|^2 dx - \int_\Omega \eta\eta_t dx. \qquad (4.3)\end{aligned}$$

Using the following Young inequality

$$ab \le \delta a^2 + \frac{1}{4\delta} b^2,$$

for a, $b \in \mathbb{R}$, and $\delta > 0$, we estimate

$$\begin{aligned}&\int_0^t g(t-s) \int_\Omega D^m\eta(t) . (D^m\eta(t) - D^m\eta(s))\, dxds = \int_\Omega D^m\eta(t) \int_0^t g(t-s)(D^m\eta(t) - D^m\eta(s))\, dsdx \\ &\le \int_\Omega \delta |D^m\eta(t)|^2 dx + \int_\Omega \frac{1}{4\delta} \left(\int_0^t g(t-s)(D^m\eta(t) - D^m\eta(s))\, ds \right)^2 dx\end{aligned}$$

Using the Hölder inequality, we get

$$\begin{aligned}
&\int_0^t g(t-s)\int_\Omega D^m\eta(t).(D^m\eta(t)-D^m\eta(s))\,dxds\\
&\le \delta\int_\Omega |D^m\eta(t)|^2\,dx+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-s)\,ds\right)\left(\int_0^t g(t-s)\,|D^m\eta(t)-D^m\eta(s)|^2\,ds\right)dx\\
&=\delta\int_\Omega |D^m\eta(t)|^2\,dx+\frac{1}{4\delta}\left(\int_0^t g(t-s)\,ds\right)\int_0^t g(t-s)\,\|D^m\eta(t)-D^m\eta(s)\|_2^2\,ds\\
&=\delta\int_\Omega |D^m\eta(t)|^2\,dx+\frac{1}{4\delta}\left(\int_0^t g(t-s)\,ds\right)(g\circ D^m\eta)(t)\\
&=\delta\int_\Omega |D^m\eta(t)|^2\,dx+\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)(g\circ D^m\eta)(t).
\end{aligned}$$

We deduce that

$$-\int_0^t g(t-s)\int_\Omega D^m\eta(t).(D^m\eta(t)-D^m\eta(s))\,dxds\ge-\delta\int_\Omega |D^m\eta(t)|^2\,dx-\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)(g\circ D^m\eta)(t).\tag{4.4}$$

By combining (4.3) and (4.4), we get

$$\rho''(t)\ge-\left(1+\delta-\int_0^t g(s)\,ds\right)\|D^m\eta\|_2^2-\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)(g\circ D^m\eta)(t)+\int_\Omega|\eta|^{p(x)}\,dx+\int_\Omega|\eta_t|^2\,dx-\int_\Omega\eta\eta_t dx.$$

Now, we exploit (2.3) to substitute for $\|D^m\eta\|_2^2$. Therefore,

$$\begin{aligned}
\rho''(t)+\rho'(t)\ge&-\frac{2}{\beta}\left(1+\delta-\int_0^t g(s)\,ds\right)E(t)+\left(1+\frac{1+\delta-\int_0^t g(s)\,ds}{\beta}\right)\int_\Omega|\eta_t|^2\,dx\\
&+\left(\frac{1+\delta-\int_0^t g(s)\,ds}{\beta}-\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)\right)(g\circ D^m\eta)+\left(1-2\frac{1+\delta-\int_0^t g(s)\,ds}{\beta p_1}\right)\int_\Omega|\eta|^{p(x)}\,dx.
\end{aligned}$$

Using Lemma 2.2, we get

$$\begin{aligned}
\rho''(t)+\rho'(t)\ge&-\frac{2}{\beta}\left(1+\delta-\int_0^t g(s)\,ds\right)E(t)+\left(1+\frac{1+\delta-\int_0^t g(s)\,ds}{\beta}\right)\int_\Omega|\eta_t|^2\,dx\\
&+\left(\frac{1+\delta-\int_0^t g(s)\,ds}{\beta}-\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)\right)(g\circ D^m\eta)+c\left(1-2\frac{1+\delta-\int_0^t g(s)\,ds}{\beta p_1}\right)\|\eta\|_{p_1}^{p_1}.
\end{aligned}\tag{4.5}$$

At this point $\delta>0$ is chosen so that:

$$\frac{1+\delta-\int_0^t g(s)\,ds}{\beta}-\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)\ge 0$$

$$c\left(1-2\frac{1+\delta-\int\limits_0^t g(s)\,ds}{\beta p_1}\right)>0$$

This is, of course, possible by (4.1).Thus by using (2.4) and the negative initial energy, (4.5) becomes:

$$\rho''(t)+\rho'(t)\geq\gamma\,\|\eta\|_{p1}^{p_1} \tag{4.6}$$

where $\gamma=c\left(1-2\frac{1+\delta-\int\limits_0^t g(s)ds}{\beta p_1}\right)$. Now, we use Hölder's inequality to estimate

$$\int_\Omega|\eta|^2\,dx\leq\left(\int_\Omega|\eta|^{p1}\,dx\right)^{\frac{2}{p_1}}\left(\int_\Omega 1dx\right)^{\frac{p_1-2}{p1}}.$$

where $|\Omega|$ is measure of the doman Ω, then

$$\left(\int_\Omega|\eta|^{p_1}\,dx\right)^{\frac{2}{p_1}}\geq\left(\int_\Omega|\eta|^2\,dx\right)|\Omega|^{\frac{2-p_1}{p_1}}.$$

So,

$$\int_\Omega|\eta|^{p_1}\,dx\geq\left(\int_\Omega|\eta|^2\,dx\right)^{\frac{p_1}{2}}|\Omega|^{\frac{2-p_1}{2}}. \tag{4.7}$$

From the expression of $\rho(t)=\frac{1}{2}\int_\Omega|\eta(x,t)|^2\,dx$,we get

$$2\rho(t)=\int_\Omega|\eta(x,t)|^2\,dx.$$

Then

$$(2\rho(t))^{\frac{p_1}{2}}=\left(\int_\Omega|\eta(x,t)|^2\,dx\right)^{\frac{p_1}{2}}. \tag{4.8}$$

Combining (4.7), (4.8),and (4.6) yield

$$\rho''(t)+\rho'(t)\geq 2^{\frac{p_1}{2}}\gamma(\rho(t))^{\frac{p_1}{2}}|\Omega|^{\frac{2-p_1}{2}}$$

We simplify the last inequality, we arrive at

$$\rho''(t)+\rho'(t)\geq\varpi\rho^{1+\alpha}(t) \tag{4.9}$$

where

$$\varpi=2^{\frac{p_1}{2}}\gamma\,|\Omega|^{\frac{2-p_1}{2}}>0,\qquad\alpha=\frac{p_1-2}{2}$$

Therefore $\rho(t)$ blows up in the finite time. □

5 Numerical example

Now, we present an example to illustrate numerically the result of Theorem 4.1. For solve problem (1.1), we consider $m=1$, $n=2$ where the domain is taken to be $\Omega=[-1,\ 1]^2$. We chosen $g(t)=\lambda e^{-t}$, $(0<\lambda<1)$, $\eta_0(x_1,x_2)=\eta_1(x_1,x_2)=3(2-x_1^2+x_2^2)$, such that $0>E(0)$, $\eta_0\eta_1>0$, and we take $p(x_1,x_2)=4.8$, which satisfy condition (1.2).

5.1 Numerical method

We first choose a suitable numerical scheme to discretize (1.1) using finite differences for the time variable t and the space variable $x=(x_1,x_2)$. Comprehensive details about the finite difference methods, see in [18, 19]. We subdivide the time interval $[0,\ T]$ into N equal subintervals $[t_{n-1},\ t_n]$, $\quad t_n=n\ \delta t,\quad n=1,\,2,...,N+1$, where δt is the time step.

Let $\eta^n\ (x)=\eta\,(x_1,x_2,\ t_n)$, and use the finite-difference formulas:

$$\partial_t\eta^n\,(x)=\frac{\eta^n\,(x)-\eta^{n-1}\,(x)}{\delta t}.$$

and

$$\partial_{tt}\eta^n\,(x)=\frac{\eta^{n+1}\,(x)-2\eta^n\,(x)+\eta^{n-1}\,(x)}{(\delta t)^2}.$$

Then the discrete problem of (1.1) reads: Let η_0 and η_1, calculate $\left\{\eta^2,\ \eta^3,...,\eta^{n+1}\right\}$ such that

$$\begin{cases} \frac{\eta^{n+1}}{(\delta t)^2}-\Delta\eta^{n+1}=\frac{2\eta^n-\eta^{n-1}}{(\delta t)^2}-\frac{\eta^n-\eta^{n-1}}{\delta t} & \\ \quad -\int_0^{t_{n+1}} g\,(t_{n+1}-s)\,\Delta\eta^n\,(s)\,ds+|\eta^n|^{p(x_1,x_2)-2}\,\eta^n, & \text{in } \Omega \\ \eta^{n+1}=0, & \text{on } \partial\Omega \\ \eta^0=\eta_0,\quad \eta^1=\eta^0+(\delta t)\,\eta_1, & \text{in } \Omega \end{cases}. \tag{5.1}$$

Problem (5.1) is solved iteratively by using the history data η^n and η^{n-1} in the second side of the equation, satisfies the boundary-value problem:

$$\begin{cases} \frac{\eta^{n+1}}{(\delta t)^2}-\Delta\eta^{n+1}=F\left(\eta^n,\ \eta^{n-1}\right), & \text{in } \Omega_h \\ \eta^{n+1}=0, & \text{on } \partial\Omega_h \end{cases} \tag{5.2}$$

where $F\left(\eta^n,\ \eta^{n-1}\right)=\frac{2\eta^n-\eta^{n-1}}{(\delta t)^2}-\frac{\eta^n-\eta^{n-1}}{\delta t}-\int_0^{t_{n+1}} g\,(t_{n+1}-s)\,\Delta\eta^n\,(s)\,ds+|\eta^n|^{p(x_1,x_2)-2}\,\eta^n$.

5.2 Numerical results

Now, we present the results of the numerical scheme (5.1). The numerical results are obtained using the Matlab codes.

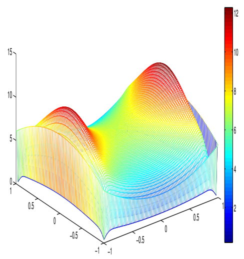

Figure 1: U^{16}

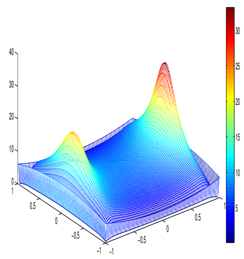

Figure 2: U^{26}

The parameters that have been set up for numerical experiments are:

- Number of discretisation points is: 100×100;
- Time step is: $\delta t=0.01$;
- The spatial discretisation step $h\simeq 0.01$;
- $\lambda=10^{-3}$.

Figures. 1, 2, 3 and 4 present η^n for iterations $n=16$ $(t=0.16)$, $n=26$ $(t=0.26)$, $n=29$ $(t=0.29)$ and $n=30$ $(t=0.30)$ respectively.

Figure. 4 present η^n for iteration $n=30$ $(t=0.30)$, which the blowup.

In conclusion, the previous numerical example verifies and agrees with the results of Theorem 4.1.

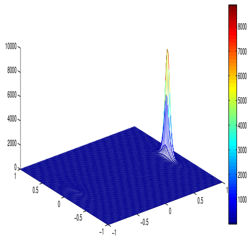

Figure 3: U^{29}

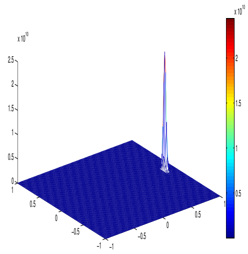

Figure 4: U^{30}

References

[1] D. Andrade, L.H. Fatori and J.E.M. Rivera, *Nonlinear transmission problem with a dissipative boundary condition of memory type*, Electron J. Differ. Equ. **2006** (2006), no. 53, 1–16.

[2] S. Antontsev, J. Ferreira and E. Pişkin, *Existence and blow up of petrovsky equation solutions with strong damping and variable exponents*, Electron. J. Differ. Equ. **2021** (2021), 1–18.

[3] P. Bernner and W. von Whal, *Global classical solutions of nonlinear wave equations*, Math. Z. **176** (1981), 87–221.

[4] M.M. Cavalcanti, V.N. Domingos Cavalcanti, T.F. Ma and J.A. Soriano, *Global existence and asymptotic stability for viscoelastic problem*, Differ. Integral Equ. **15** (2002), 731–748.

[5] D. Lars, P. Harjulehto, P. Hasto and M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, 2017.

[6] S.A. Messaoudi and N.-e. Tatar, *Exponential and polynomial decay for a quasilinear viscoelastic equation*, Nolinear Anal. **68** (2008), 785-793.

[7] S.A. Messaoudi and A. Talahmeh, *A blow-up result for a nonlinear wave equation with variable-exponent nonlinearities*, Appl. Anal. **96** (2017), 71509–1515.

[8] S.A. Messaoudi and A. Talahmeh, *Blowup in solutions of a quasilinear wave equation with variable-exponent nonlinearities*, Math. Meth. Appl. Sci. **40** (2017), no. 18, 6976–6986.

[9] S.A. Messaoudi, A.A. Talahmeh and J.H. Al-Shail, *Nonlinear damped wave equation: Existence and blow-up* , Comput. Math. Appl. **74** (2017), no. 12, 3024–3041.

[10] J.E. Munoz Rivera, E.C. Lapa and R. Baretto, *Decay rates for viscolastic plates with memory*, J. Elastic. **40** (1996), 61–87.

[11] A. Ouaoua and M. Maouni, *Blow-up, exponential growth of solution for a nonlinear parabolic equation with $p(x)$-Laplacian*, Int. J. Anal. Appl. **17** (2019), no. 4, 620–629.

[12] A. Ouaoua and M. Maouni, *Exponential growth of positive initial energy solutions for coupled nonlinear Klein-Gordon equations with degenerate damping and source terms*, Bol. Soc. Paran. Mat. **40** (2022), 1–9.

[13] A. Ouaoua, A. Khaldi and M. Maouni, *Existence and stabiliy results of a nonlinear Timoshenko equation with damping and source terms*, Theor. Appl. Mech. **48** (2021), no. 1, 53–66.

[14] A. Ouaoua, M. Maouni and A. Khaldi, *Exponential decay of solutions with Lp -norm for a class to semilinear wave equation with damping and source terms*, Open J. Math. Anal. **4** (2020), no. 2, 123–131.

[15] S.H. Park and J.R. Kang, *Blow-up of solutions for a viscoelastic wave equation with variable exponents*, Math. Methods Appl. Sci. **42** (2019), no. 6, 2083–2097.

[16] H. Pecher, *Die existenz regulaer Losungen fur Cauchy-und anfangs-randwertproble-me michtlinear wellengleichungen*, Math. Z. **140** (1974), 263–679.

[17] E. Pişkin, *Global nonexistence of solutions for a nonlinear Klein-Gordon equation with variable exponents*, Appl. Math. E-Notes. **19** (2019), 315–323.

[18] A. Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations*, SCM Series, vol. 23 ,

Springer-Verlag, Heidelberg, 1994.

[19] A. Quarteroni, R. Sacconand, and F. Saleri, *Numerical Mathematics*, second ed., TAM Series, vol. 37 , Springer-Verlag, New-York, 2007.

[20] F. Tahamatani and M. Shahrouzi, *General existence and blow up of solutions to a Petrovsky equation with memory and nonlinear source*, Bound. Value Probl. **2012** (2012), no. 50, 1–15.

[21] Y. Wang, *A global nonexistence theorem for viscoelastic equation with arbitrary positive initial energy*, Appl. Math. Lett. **22** (2009), 1394–1400.

[22] S.T. Wu and L.Y. Tsai, *Existence and nonexistence of global solutions for a nonlinear wave equation*, Taiwanese J. Math. **13** (2009), no. 6B, 2069–2091.

[23] Y. Ye, *Global existence and blow-up of solutions for higher-order viscoelastic wave equation with a nonlinear source term*, Nonlinear Anal. **112** (2015), 129–146.

[24] K. Zennir, *Exponential growth of solutions with Lp- norm of a nonlinear viscoelastic hyperbolic equation*, J. Nonlinear Sci. Appl. **6** (2013), 252–262.

[25] Y. Zhou, *Global existence and nonexistence for a nonlinear wave equation with damping and source terms*, Math. Nachr. **278** (2005), no. 11, 1341–1358.