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Numerical solution of Lane-Emden type equations using Flatlet oblique multiwavelets collocation approach

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Abstract

The presented paper examines a numerical method for solving Lane-Emden type equations based on Flatlet oblique multiwavelet properties. In this paper, using the Flatlet multiwavelet features, an operator matrix is created and then the Lane-Emden equation reduces to a set of algebraic equations. Also, comparing the results presented in previous articles, it is observed that this wavelet due to having different high ranks, has the ability to solve this problem more accurately than other methods.

Keywords: Flatlet oblique multiwavelet, Lane-Emden equation, Operational matrix of derivative, Operational matrix of integration, Collocation method, Biorthogonal system
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1 Introduction

One of the most widely used equations in physics and engineering is linear or nonlinear differential equations. In particular, the Lane-Emden type of these equations increases its applicability in various sciences. Jonathan Homer Lane and Robert Emden were the first who studied a special singular differential equation that later called Lane-Emden differential equation. They studied the thermal behavior of a spherical cloud. A spherical cloud is a gas that behaves under the mutual attraction of its molecules, following the classical laws of thermodynamics [24, 25]. The general form of this equation is as follows,

$$y''(x) + \frac{k}{x}y'(x) + f(x)g(y(x)) = h(x), \quad k > 0 \quad , \quad x > 0, \tag{1.1}$$

with initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta,$$

where k, α and β are constants, $f(x)$, $g(y(x))$ and $h(x)$ are given continuous functions and $y(x)$ is the unknown function.

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It should be noted that several studies on various methods of solving Lane-Emden type equations have been done so far as AdomianPade approximation method [12], Quasi-Newton's method [31], Adomian decomposition method [25, 26], Legender spectral method [1] and Jacobi-Gauss collocation method [20]. Also, Chebyshev collocation method [6], Bessel collocation method [29], Hermite function collocation method (HFCM) [28], and Laguerre collocation method [30] are the other techniques used in studying Lane-Emden type equations.

In this paper, we use an operator matrix of integral constructed using Flatlet oblique multiwavelets in order to solve Lane-Emden type equations.

2 Flatlet Multiwavelet System and the Duale Functions

The basis for the construction of Flatlet oblique multiwavelets are a set of $(m+1)$ unit constant functions $\phi_0(x), ..., \phi_m(x)$ that defined by

$$\phi_i(x) = \begin{cases} 1, & \frac{i}{m+1} \leq x < \frac{i+1}{m+1}, \\ 0, & \text{otherwise,} \end{cases} \quad , i = 0, 1, ..., m. \tag{2.1}$$

The simplest member of this family is the Haar wavelet which happens for the case $m = 0$. Each of these unit constant functions is called a scale function. The flatlet mother wavelets $\psi_0(x), ..., \psi_m(x)$, corresponding to flatlet scaling functions are constructed using two-scale relation which will be discussed in the following.

First, consider two vector functions

$$\Phi(x) = \begin{bmatrix} \phi_0(x) \\ \vdots \\ \phi_i(x) \\ \vdots \\ \phi_m(x) \end{bmatrix}, \Psi(x) = \begin{bmatrix} \psi_0(x) \\ \vdots \\ \psi_i(x) \\ \vdots \\ \psi_m(x) \end{bmatrix}, \tag{2.2}$$

whose components are flatlet scaling functions and mother wavelets, respectively.

The two-scale relations for the flatlet multiwavelet system are expressed as

$$\Phi(x) = \mathbf{R} \begin{bmatrix} \Phi(2x) \\ \Phi(2x-1) \end{bmatrix}, \Psi(x) = \mathbf{S} \begin{bmatrix} \Phi(2x) \\ \Phi(2x-1) \end{bmatrix}, \tag{2.3}$$

where $\mathbf{R}$ and $\mathbf{S}$ are $(m+1) \times 2(m+1)$ matrices. The matrix form of two-scale relations (2.3) are as follows

$$\begin{bmatrix} \Phi(x) \\ \Psi(x) \end{bmatrix} = \begin{bmatrix} \mathbf{R} \\ \mathbf{S} \end{bmatrix} \begin{bmatrix} \Phi(2x) \\ \Phi(2x-1) \end{bmatrix}, \tag{2.4}$$

and the coefficient matrix in (2.4) is called reconstruction matrix that is invertible [9]. Also, equation (2.4) can be extended as follows

$$\Phi_{i-1}(x) = \sum_{j=1}^{m+1} R_{i,j}\phi_{j-1}(2x) + \sum_{j=m+2}^{2m+2} R_{i,j}\phi_{j-m-2}(2x-1),$$

$$\Psi_{i-1}(x) = \sum_{j=1}^{m+1} S_{i,j}\phi_{j-1}(2x) + \sum_{j=m+2}^{2m+2} S_{i,j}\phi_{j-m-2}(2x-1), \tag{2.5}$$

which is called reconstruction relations.

As is clear, the flatlet scaling functions have a simple form so the matrix $\mathbf{R}$ can be calculated as

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & & & & 0 \\ & & 1 & 1 & & \\ & & & \ddots & & \\ 0 & & & & 1 & 1 \end{bmatrix}.$$

In order to compute $2(m+1)^2$ entries of matrix $\mathbf{S}$, $2(m+1)^2$ independent conditions are needed. For this purpose, $\frac{(m+1)(m+2)}{2}$ orthonormality conditions

$$\int_0^1 \psi_i(x)\psi_j(x)dt = \delta_{i,j}, \qquad i,j = 0,1,...,m, \tag{2.6}$$

and $\frac{(m+1)(3m+2)}{2}$ vanishing moment conditions

$$\int_{-\infty}^{\infty} \psi_i(x)x^j dx = 0, \quad i = 0,1,...,m, \quad j = 0,1,...,m+i, \tag{2.7}$$

come to our aid. By using (2.1) and (2.5), equation (2.7) can be written as a system of linear equations

$$\sum_{l=0}^{2(m+1)} \{(l+1)^{j+1} - (l)^{j+1}\} s_{j,l} = 0, \qquad j = 0,...,m+i. \tag{2.8}$$

Now, the unknown matrix $\mathbf{S}$ and so $\mathbf{\Psi}(x)$ are obtained by solving (2.6) – (2.8). As an example, for the first order flatlet basis functions we have

$$\phi_0(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ 0, & \text{otherwise}, \end{cases} \quad , \quad \phi_1(x) = \begin{cases} 1, & \frac{1}{2} \le x < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.9}$$

and the matrix $\mathbf{S}$ is computed as

$$\mathbf{S} = \pm \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{bmatrix}. \tag{2.10}$$

This computation implies that the associated multiwavelets are not unique. A simple form of mother wavelets for the above example may be given as

$$\psi_0(x) = \sqrt{2} \begin{cases} \frac{1}{2}, & 0 \le x < \frac{1}{4}, \\ -\frac{1}{2}, & \frac{1}{4} \le x < \frac{3}{4}, \\ \frac{1}{2}, & \frac{3}{4} \le x < 1, \\ 0, & \text{otherwise}, \end{cases} \quad , \quad \psi_1(x) = \sqrt{10} \begin{cases} \frac{1}{10}, & 0 \le x < \frac{1}{4}, \\ -\frac{3}{10}, & \frac{1}{4} \le x < \frac{1}{2}, \\ \frac{3}{10}, & \frac{1}{2} \le x < \frac{3}{4}, \\ -\frac{1}{10}, & \frac{3}{4} \le x < 1, \\ 0, & \text{otherwise}. \end{cases} \tag{2.11}$$

Also the second order flatlet multiwavelet system can be represented as

$$\phi_i(x) = \begin{cases} 1, & \frac{i}{3} \le x < \frac{i+1}{3}, \\ 0, & \text{otherwise}, \end{cases} , i = 0,1,2,$$

$$\psi_0(x) = \sqrt{10} \begin{cases} \frac{1}{6}, & 0 \le x < \frac{1}{6}, \\ -\frac{7}{30}, & \frac{1}{6} \le x < \frac{1}{3}, \\ -\frac{2}{15}, & \frac{1}{3} \le x < \frac{1}{2}, \\ \frac{2}{15}, & \frac{1}{2} \le x < \frac{2}{3}, \\ \frac{7}{30}, & \frac{2}{3} \le x < \frac{5}{6}, \\ -\frac{1}{6}, & \frac{5}{6} \le x < 1, \\ 0, & \text{otherwise}, \end{cases} \quad , \quad \psi_1(x) = \sqrt{14} \begin{cases} \frac{1}{14}, & 0 \le x < \frac{1}{6}, \\ -\frac{3}{14}, & \frac{1}{6} \le x < \frac{1}{3}, \\ \frac{1}{7}, & \frac{1}{3} \le x < \frac{2}{3}, \\ -\frac{3}{14}, & \frac{2}{3} \le x < \frac{5}{6}, \\ \frac{1}{14}, & \frac{5}{6} \le x < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\psi_2(x) = \sqrt{14} \begin{cases} -\frac{1}{42}, & 0 \le x < \frac{1}{6}, \\ \frac{5}{42}, & \frac{1}{6} \le x < \frac{1}{3}, \\ -\frac{5}{21}, & \frac{1}{3} \le x < \frac{1}{2}, \\ \frac{5}{21}, & \frac{1}{2} \le x < \frac{2}{3}, \\ -\frac{5}{42}, & \frac{2}{3} \le x < \frac{5}{6}, \\ \frac{1}{42}, & \frac{5}{6} \le x < 1, \\ 0, & \text{otherwise}. \end{cases} . \tag{2.12}$$

2.1 Biorthogonal Flatlet Multiwavelet System (BFMS)

In this section, we introduce the dual scaling and wavelet vector functions in biorthogonal flatlet multiwavelet system (BFMS) respectively by $\tilde{\Phi}(x)$ and $\tilde{\Psi}(x)$ that are represented as

$$\tilde{\Phi}(x)=\begin{bmatrix}\tilde{\phi_0}(x)\\ \vdots\\ \tilde{\phi_i}(x)\\ \vdots\\ \tilde{\phi_m}(x)\end{bmatrix},\ \tilde{\Psi}(x)=\begin{bmatrix}\tilde{\psi_0}(x)\\ \vdots\\ \tilde{\psi_i}(x)\\ \vdots\\ \tilde{\psi_m}(x)\end{bmatrix}. \tag{2.13}$$

According to the biorthogonality conditions we must have

$$\langle\tilde{\phi_i},\phi_j\rangle=\int_0^1\tilde{\phi_i}(x)\phi_j(x)dx=\delta_{i,j}, \tag{2.14}$$

$$\langle\tilde{\psi_i},\psi_j\rangle=\int_0^1\tilde{\psi_i}(x)\psi_j(x)dx=\delta_{i,j}, \tag{2.15}$$

$$\langle\tilde{\psi_i},\phi_j\rangle=\int_0^1\tilde{\psi_i}(x)\phi_j(x)dx=0, \tag{2.16}$$

$$i,j=0,1,...,m.$$

Now we can introduce the $\tilde{\phi_i}(x)$ and $\tilde{\psi_i}(x)$ as polynomials and piecewise polynomials of degree m respectively, by

$$\tilde{\phi_i}(x)=\begin{cases}a_{i1}+a_{i2}x+...+a_{i,m+1}x^m, & 0\le x<1,\\ 0, & \text{otherwise},\end{cases} \tag{2.17}$$

$$\tilde{\psi_i}(x)=\begin{cases}b^1_{i1}+b^1_{i2}x+...+b^1_{i,m+1}x^m, & 0\le x<\frac{1}{2},\\ b^2_{i1}+b^2_{i2}x+...+b^2_{i,m+1}x^m, & \frac{1}{2}\le x<1,\\ 0, & \text{otherwise}.\end{cases} \tag{2.18}$$

Based on biorthogonal conditions (2.14) − (2.16), we can show that coefficients $a_{i,j}$, $b^1_{i,j}$ and $b^2_{i,j}, i=0,...,m$ and $j=1,...,m+1$, in (2.17) and (2.18) are uniquely determined (see[9]). For example, we compute the dual multiwavelets corresponding to (2.9) and (2.11) as

$$\tilde{\phi_0}(x)=\begin{cases}3-4x, & 0\le x<1,\\ 0, & \text{otherwise},\end{cases}\quad,\quad \tilde{\phi_1}(x)=\begin{cases}-1+4x, & 0\le x<1,\\ 0, & \text{otherwise},\end{cases}$$
$$\tilde{\psi_0}(x)=\begin{cases}2\sqrt{2}(1-4x), & 0\le x<\frac{1}{2},\\ -2\sqrt{2}(3-4x), & \frac{1}{2}\le x<1,\\ 0, & \text{otherwise},\end{cases}\quad,\quad \tilde{\psi_1}(x)=\begin{cases}\sqrt{10}(1-4x), & 0\le x<\frac{1}{2},\\ \sqrt{10}(3-4x), & \frac{1}{2}\le x<1,\\ 0, & \text{otherwise}.\end{cases} \tag{2.19}$$

Also computation of dual multiwavelets corresponding to (2.12), yields

$$\tilde{\phi_0}(x) = \begin{cases} \frac{11}{2} - 18x + \frac{27}{2}x^2, & 0 \le x < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tilde{\phi_1}(x) = \begin{cases} \frac{-7}{2} - 27x + 27x^2, & 0 \le x < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tilde{\phi_2}(x) = \begin{cases} 1 - 9x + \frac{27}{2}x^2, & 0 \le x < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tilde{\psi_0}(x) = \begin{cases} \sqrt{10}(\frac{7}{4} - \frac{33}{2}x + 27x^2), & 0 \le x < \frac{1}{2}, \\ -\sqrt{10}(\frac{49}{4} - \frac{75}{2}x + 27x^2), & \frac{1}{2} \le x < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tilde{\psi_1}(x) = \begin{cases} \sqrt{14}(\frac{9}{4} - \frac{45}{2}x + \frac{81}{2}x^2), & 0 \le x < \frac{1}{2}, \\ \sqrt{14}(\frac{81}{4} - \frac{117}{2}x + \frac{81}{2}x^2), & \frac{1}{2} \le x < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tilde{\psi_3}(x) = \begin{cases} -\sqrt{14}(1 - 12x + 27x^2), & 0 \le x < \frac{1}{2}, \\ \sqrt{14}(16 - 42x + 27x^2), & \frac{1}{2} \le x < 1, \\ 0, & \text{otherwise}. \end{cases} \tag{2.20}$$

Theorem 2.1. Let $A = [a_{i,j}]_{m\times n}$ be a square matrix with $a_{i,j} = p_{i-1}(j)$, a polynomial of exact degree $i-1$, then A is invertible.

Proof . See [9] □

Theorem 2.2. For oblique flatlet multiwavelets, the dual functions defined in (2.17) and (2.18) are uniquely determined.

Proof . See [9] □

Suppose $\Lambda(x)$ and $\tilde{\Lambda}(x)$ are two vector functions as

$$\Lambda(x) = \begin{bmatrix} \phi_0(x) \\ \vdots \\ \phi_m(x) \\ \psi_0(x) \\ \vdots \\ \psi_i(2^l x - k) \\ \vdots \\ \psi_m(2^J x - 2^J + 1) \end{bmatrix}, \quad \tilde{\Lambda}(x) = \begin{bmatrix} \tilde{\phi_0}(x) \\ \vdots \\ \tilde{\phi_m}(x) \\ \tilde{\psi_0}(x) \\ \vdots \\ \tilde{\psi_i}(2^l x - k) \\ \vdots \\ \tilde{\psi_m}(2^J x - 2^J + 1) \end{bmatrix}. \tag{2.21}$$

Now we can approximate a function $f(x)$ defined on $[0, 1]$ by the flatlet multiwavelets [13] as

$$f(x) \simeq \Lambda^T(x).\mathbf{C}, \tag{2.22}$$

or

$$f(x) \simeq \tilde{\Lambda}^T(x).\tilde{\mathbf{C}}, \tag{2.23}$$

where $\mathbf{C}, \tilde{\mathbf{C}}$ are N-vectors as

$$\begin{aligned} \mathbf{C} &= [c_0, ..., c_m, d_{0,0,0}, ..., d_{i,l,k}, ..., d_{m,J,2^J-1}], \\ \tilde{\mathbf{C}} &= [\tilde{c}_0, ..., \tilde{c}_m, \tilde{d}_{0,0,0}, ..., \tilde{d}_{i,l,k}, ..., \tilde{d}_{m,J,2^J-1}], \end{aligned}$$

in which $N = 2^J(m+1)$.

So we can rewrite (2.22) and (2.23) respectively as

$$f(x) \simeq \sum_{i=0}^{m} c_i \phi_i(x) + \sum_{i=0}^{m} \sum_{l=0}^{J} \sum_{k=0}^{2^j-1} d_{i,l,k} \psi_{i,l,k}(x), \tag{2.24}$$

and

$$f(x) \simeq \sum_{i=0}^{m} \tilde{c}_i \tilde{\phi}_i(x) + \sum_{i=0}^{m} \sum_{l=0}^{J} \sum_{k=0}^{2^J-1} \tilde{d}_{i,l,k} \tilde{\psi}_{i,l,k}(x), \tag{2.25}$$

where $\psi_{i,l,k}(x) = \psi_i(2^l x - k)$ and $\tilde{\psi}_{i,l,k}(x) = \tilde{\psi}_i(2^l x - k)$. The vectors of $\mathbf{C}$ and $\tilde{\mathbf{C}}$ can be obtained respectively as

$$\mathbf{C} = \int_0^1 f(x).\tilde{\Lambda}^T(x)dt, \tag{2.26}$$

and

$$\tilde{\mathbf{C}} = \int_0^1 f(x).\Lambda^T(x)dt. \tag{2.27}$$

In other words

$$c_i = \int_0^1 f(x).\tilde{\phi}_i(x) \quad , \quad d_{i,l,k} = \int_0^1 f(x).\tilde{\psi}_{i,j,k}(x), \tag{2.28}$$

and

$$\tilde{c}_i = \int_0^1 f(x).\phi_i(x) \quad , \quad \tilde{d}_{i,l,k} = \int_0^1 f(x).\psi_{i,j,k}(x). \tag{2.29}$$

It is useful to note that the dual flatlet multiwavelets are defined based on polynomials and have more flexibility in approximating functions so we use the last relation and equation (2.23) because of the higher order of accuracy.

3 Description of Flatlet Oblique Multiwavelet Method for Solving Lane-Emden Equation

In this section we proposed a numerical method for solving Lane-Emden differential equation

$$y''(x) + \frac{2}{x}y'(x) + g(x, y(x)) = f(x), \tag{3.1}$$

with initial conditions $y(0) = \alpha$ and $y'(0) = 0$. For this purpose, first we approximate the second-order derivative of the unknown function $y(x)$ by scaling and mother wavelet functions of dual flatlet multiwavelets as

$$y''(x) = \mathbf{C}^T \tilde{\Lambda}(x), \tag{3.2}$$

or

$$y''(x) = \sum_{i=1}^{(m+1)2^J} c_i \tilde{\Lambda}_i(x),$$

where $\mathbf{C}$ is an unknown vector of order $(m+1)2^J$ as follows

$$\mathbf{C} = [c_1, c_2, ..., c_{(m+1)2^J}]^T.$$

By integrating both sides of equation (3.2) from 0 to x the relation

$$y'(x) = \mathbf{C}^T.P.\tilde{\Lambda}(x), \tag{3.3}$$

is obtained in which P is integral operational matrix that is calculated from the flatlet multiwavelet properties described in (2.14), (2.15) and (2.16) as

$$P = \int_0^1 \left(\int_0^x \tilde{\Lambda}(t)dt\right).\Lambda^T(x)dx, \tag{3.4}$$

To solve the problem of singularity of Lane-Emden equation, we define a square matrix like T of order $(m+1)2^J$ for which the following relation holds,

$$x.\tilde{\Lambda}(x) = T.\tilde{\Lambda}(x), \tag{3.5}$$

and similar to (3.4), it can be obtained as

$$T = \int_0^1 \left(x.\tilde{\Lambda}(x)\right)\Lambda^T(x)dx, \tag{3.6}$$

and

$$T_{i,j} = \int_0^1 x.\tilde{\Lambda}_i(x).\Lambda_j(x)dx. \tag{3.7}$$

So we can write

$$y'(x) = \mathbf{C}^T.P.T^{-1}.x.\tilde{\Lambda}(x). \tag{3.8}$$

Also, integrating both sides of equation (3.3) from 0 to x and using equation (3.4) we get

$$y(x) = \mathbf{C}^T.P^2.\tilde{\Lambda}(x) + \alpha. \tag{3.9}$$

Finally, we need to extend the nonlinear function $g(x, y(x))$ and the function $f(x)$ by flatlet oblique multiwavelets respectively as

$$g(x, y(x)) = H^T.P^2.\tilde{\Lambda}(x), \tag{3.10}$$

and

$$f(x) = F^T.\tilde{\Lambda}(x), \tag{3.11}$$

in which

$$H = [h_1, h_2, ..., h_{(m+1)2^J}]^T, \tag{3.12}$$

is an unknown vector and F is the known vector of order $(m+1)2^J$ which is calculated like equation (2.27) as

$$F = \int_0^1 f(x).\Lambda^T(x)dt, \tag{3.13}$$

In order to approximate the solution of equation (3.1), we use $(m+1)2^J$ collocation points. First, set $N = (m+1)2^J$, so the suitable collocation points are Newoton-Cotes nodes as

$$x_i = \frac{2i-1}{2N}, \quad i = 1, 2, ..., N. \tag{3.14}$$

Suppose the matrix L_N is a square matrix of dimension $N \times N$ whose columns are the value of $\tilde{\Lambda}(x)$ at the above points as

$$L_N = [\tilde{\Lambda}(x_1), \tilde{\Lambda}(x_2), ..., \tilde{\Lambda}(x_N)]. \tag{3.15}$$

Furthermore, using equation (3.15), we will have

$$\tilde{\Lambda}(x_i) = L_N.e_i, \tag{3.16}$$

where e_i is $N \times 1$ vector that entire i is 1 and the others are 0.

Now, by substituting equation (3.2) and equations (3.8) - (3.11) in (3.1) we can write

$$\mathbf{C}^T.\tilde{\Lambda}(x) + \frac{2}{x}.\mathbf{C}^T.P.T^{-1}.x.\tilde{\Lambda}(x) + H^T.P^2.\tilde{\Lambda}(x) = F^T.\tilde{\Lambda}(x), \tag{3.17}$$

and applying collocation points and equation (3.16), equation (3.17) can be rewritten as

$$\mathbf{C}^T.L_N.e_i + 2.\mathbf{C}^T.P.T^{-1}.\tilde{\Lambda}(x) + H^T.P^2.\tilde{\Lambda}(x) = F^T.\tilde{\Lambda}(x). \tag{3.18}$$

The equation (3.18) makes a set of algebraic equations with $(m+1)2^J$ equations and $2 \times (m+1)2^J$ unknowns $\{c_1, c_2, ..., c_N\}$ and $\{h_1, h_2, ..., h_N\}$. So we need an equation system with $(m+1)2^J$ other equations that is provided by

$$g(x, \mathbf{C}^T.P^2.\tilde{\Lambda}(x) + \alpha) = H^T.\tilde{\Lambda}(x), \tag{3.19}$$

and by solving (3.18) and (3.19) simultaneously, unknown values are obtained.

4 Error Analysis

Theorem 4.1. For flatlet oblique multiwavelet functions ψ_i of order m and $i = 0, 1, ..., m$ which have $m+i+1$ vanishing moments from Eq. (2.7), if $f(x) \in \mathbf{C}^{(m+i+1)}(\mathbb{R})$ then the coefficients in the expansion (2.25) will vanish as

$$\left|\tilde{d}_{i,j,k}\right| \le C.2^{-(m+i+1)} \max_{\zeta\in[0,1]} |f^p(\zeta)|, \tag{4.1}$$

in which C is a constant independent of j and f.

Proof . For each $x \in [0,1]$, the Taylor expansion of function $f(x)$ at the point $x = \frac{k}{2^j}$ is as follows

$$f(x) = \left(\sum_{l=0}^{(m+1)} f^{(l)}(\frac{k}{2^j})\frac{(x-\frac{k}{2^j})^l}{l!}\right) + f^{(m+i+1)}(\zeta)\frac{(x-\frac{k}{2^j})^{m+i+1}}{(m+i+1)!}, \tag{4.2}$$

in which $\zeta \in [\frac{k}{2^j}, x]$. By substituting (4.2) in (2.25) we can write

$$\begin{aligned}\tilde{d}_{i,j,k} &= \int_0^1 f(x).\psi_{i,j,k}(x)dx \\ &= \sum_{l=0}^{m+i} f^{(l)}(\frac{k}{2^j})\frac{1}{l!}\int_0^1 (x-\frac{k}{2^j})^l.\psi_{i,j,k}(x)dx \\ &+ \frac{1}{(m+i+1)!}\int_0^1 f^{(m+i+1)}(\zeta)(x-\frac{k}{2^j})^{m+i+1}.\psi_{i,j,k}(x)dx.\end{aligned} \tag{4.3}$$

Suppose that $t = 2^j x - k$ so

$$\begin{aligned} \int_0^1 (x - \frac{k}{2^j})^l \psi_{i,j,k}(x)dx &= \int_0^1 (x - \frac{k}{2^j})^l \psi_i(2^j x - k)dx \\ &= \int_0^1 \left(\frac{t}{2^j}\right) \psi_i(t).2^{-j}dt \\ &= 2^{-j(l+1)} \int_0^1 t^l \psi_i(t)dt \quad , \quad l = 0, 1, ..., m + i. \end{aligned} \tag{4.4}$$

Now, using vanishing moments property we will have

$$\int_0^1 (x - \frac{k}{2^j})^l \psi_{i,j,k}(x)dx = 0 \quad , \quad l = 0, 1, ..., m + i, \tag{4.5}$$

and equation (4.3) can be written as

$$\begin{aligned} \left|\tilde{d}_{i,j,k}\right| &= \frac{1}{(m+i+1)!} \left| \int_0^1 f^{(m+i+1)}(\zeta)(x - \frac{k}{2^j})^{m+i+1} \psi_i(2^j x - k)dx \right| \\ &\le \frac{1}{(m+i+1)!} \max_{\zeta\in[0,1]} \left|f^{(m+i+1)}(\zeta)\right| \int_0^1 \left|(x - \frac{k}{2^j})^{m+i+1} \psi_i(2^j x - k)\right| dx \\ &= 2^{-(m+i+2)j} \frac{1}{(m+i+1)!} \max_{\zeta\in[0,1]} \left|f^{(m+i+1)}(\zeta)\right| \int_0^1 \left|t^{m+i+1}\psi_i(t)\right| dt, \end{aligned} \tag{4.6}$$

and assuming

$$C = \frac{1}{(m+i+1)!} \int_0^1 \left|t^{m+i+1}\psi_i(t)\right| dt, \tag{4.7}$$

the proof ends. □

Theorem 4.2. For flatlet oblique multiwavelet functions ψ_i of order m and $i = 0, 1, ..., m$ with compact support which have $(m + i + 1)$ vanishing moments we can say

$$|\epsilon_J(x)| = \mathbf{O}\left(2^{-(m+i+1)J}\right). \tag{4.8}$$

Proof . Since

$$f(x) = \sum_{i=0}^{m} \tilde{c}_i \tilde{\phi}_i(x) + \sum_{i=0}^{m} \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \tilde{d}_{i,j,k} \tilde{\psi}_i(2^j x - k), \tag{4.9}$$

therefore, the error of this approximation will be as follows

$$\epsilon_J(x) = \sum_{J}^{\infty} \sum_{k=0}^{2^j-1} \tilde{d}_{i,j,k} \tilde{\psi}_{i,j,k}(x). \tag{4.10}$$

Suppose

$$C_{\tilde{\psi}} = \max_{x\in[0,1]} \left|\tilde{\psi}_i(2^j x - k)\right| = \max_{t\in[0,1]} \left|\tilde{\psi}_i(t)\right|, \tag{4.11}$$

and using (4.11) we can write

$$\left|\tilde{d}_{i,j,k}\tilde{\psi}_{i,j,k}(x)\right| \le C2^{-(m+i+2)j} \max_{\zeta\in[0,1]} \left|f^{(m+i+1)}(\zeta)\right| C_{\tilde{\psi}}. \tag{4.12}$$

Now, we obtain

$$\begin{aligned}\sum_{k=0}^{2^j-1}\left|\tilde{d}_{i,j,k}\tilde{\psi}_{i,j,k}(x)\right| &\le C_{\tilde{\psi}}C2^{-(m+i+2)j}2^j \max_{\zeta\in[0,1]}\left|f^{(m+i+1)}(\zeta)\right| \\ &= C_{\tilde{\psi}}C2^{-(m+i+1)j}\max_{\zeta\in[0,1]}\left|f^{(m+i+1)}(\zeta)\right|,\end{aligned} \tag{4.13}$$

by substituting (4.13) in (4.10) we can write

$$\begin{aligned}|\epsilon_J(x)| &\le C_{\tilde{\psi}}C\max_{\zeta\in[0,1]}\left|f^{(m+i+1)}(\zeta)\right|\sum_{j=J}^{\infty}2^{-(m+i+1)j} \\ &= C_{\tilde{\psi}}C\max_{\zeta\in[0,1]}\left|f^{(m+i+1)}(\zeta)\right|\frac{2^{-(m+i+1)J}}{1-2^{-(m+i+1)}}.\end{aligned} \tag{4.14}$$

Therefore, we conclude that for any desired x, the approximation error will be as follows

$$|\epsilon_J(x)| = \mathbf{O}\left(2^{-(m+i+1)J}\right), \tag{4.15}$$

and as m and J increase, the error decreases. □

5 Numerical examples

In order to illustrate the performance of the proposed method and justify the accuracy and efficiently of the presented method, we consider the following examples.

Example 5.1. First we consider the Lane-Emden equation of index α given by, [21]

$$\begin{aligned}&y''(x) + \frac{2}{x}y'(x) + y^{\alpha} = 0, \\ &\quad y(0) = 1, \quad y'(0) = 0.\end{aligned} \tag{5.1}$$

When $\alpha = 0$ and 1, this equation is linear and the exact solutions exist only for $\alpha = 0, 1$ and 5 which are given in Bender *et al.* [4] respectively by

$$y(x) = 1 - \frac{1}{6}x^2, \qquad y(x) = \frac{\sin(x)}{x}, \qquad y(x) = \left(1 + \frac{x^2}{3}\right)^{-\frac{1}{2}}. \tag{5.2}$$

In Table 1, the absolute error values for different m and J is presented. Also, in Table 5, we compare the maximum amount of error values of our method with B-spline expansion method [19], cubic Hermite spline functions [22] and Laguerre collocation method [30]. Also Figure.1, 2 and 3 briefly shows the approximation processes and the error rate in this example.

Example 5.2. Consider the following nonlinear Lane-Emden equation

$$\begin{aligned}y''(x) + \frac{2}{x}y'(x) + 4(2e^{y} + e^{\frac{y}{2}}) = 0, &\qquad 0 < x \le 1 \\ y(0) = 0, &\qquad y'(0) = 0.\end{aligned} \tag{5.3}$$

The exact solution of above equation is

$$y(x) = -2\ln(1 + x^2), \tag{5.4}$$

and you can see the absolute values of errors for different values of m and J in Table 2. Also, table 5 provides a comparison of the maximum error values between the proposed method in the current paper and other methods. The approximation and error diagrams are shown in figure 4.

Example 5.3. Consider Lane-Emden equation

$$y'' + \frac{2}{x}y' + e^{y} = 0, \quad 0 < x \leq 1, \qquad (5.5)$$
$$y(0) = 0, \quad y'(0) = 0.$$

This model can be used to view the isothermal gas spheres, where the temperature remaines constant. In [25] *Wazwaz AM* obtaned a series solution as follows:

$$y(x) = -\frac{1}{6}x^2 + \frac{1}{5.4!}x^4 - \frac{8}{21.6!}x^6 + \frac{122}{81.81}x^8 - \frac{61.67}{495.10!}x^{10}. \qquad (5.6)$$

The absolute values of errors for different values of m and J are presented in Table 3 and we provide a comparison of the maximum error values between the proposed method in the current paper and other methods in Table 5. Also the approximation and error diagrams are shown in figure 5.

Example 5.4. Consider Lane-Emden equation

$$y'' + \frac{2}{x}y' + y(x) = 6 + 12x + x^2 + x^3, \quad 0 < x \leq 1, \qquad (5.7)$$
$$y(0) = 0, \quad y'(0) = 0.$$

The analytical solution of this equation is

$$y(x) = x^2 + x^3. \qquad (5.8)$$

The absolute values of errors for different values of m and J are presented in Table 4. Also, we provide a comparison of the maximum error values between the proposed method in the current paper and other methods in Table 5. The plot of error function is shown in figure 6.

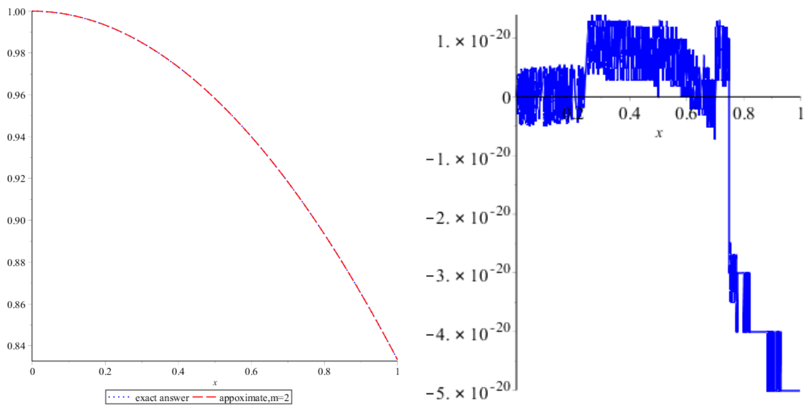

Figure 1: The numerical solution of Example 5.1 for $m = 2$ and $\alpha = 0$ and the exact answer and error values

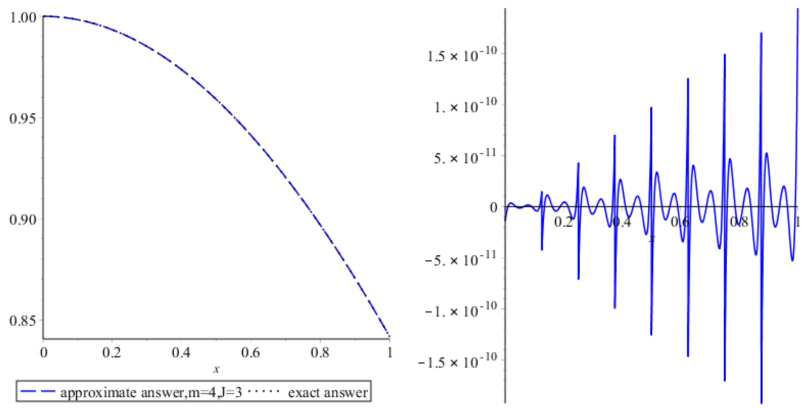

Figure 2: The numerical solution of Example 5.1 for $m = 2$ and $\alpha = 1$ and the exact answer and error values

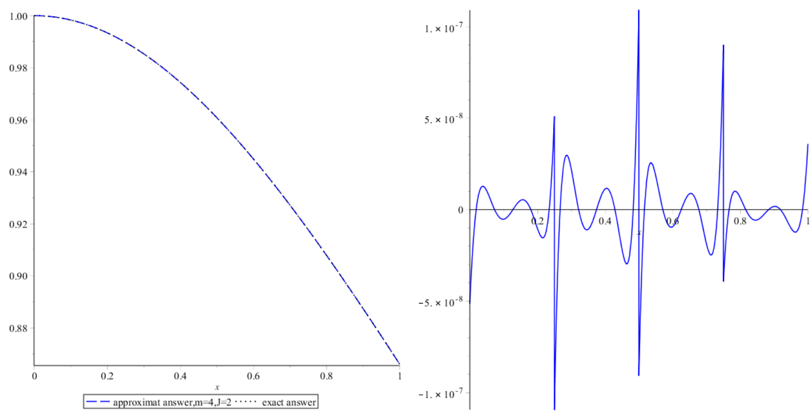

Figure 3: The numerical solution of Example 5.1 for $m = 2$ and $\alpha = 5$ and the exact answer and error values

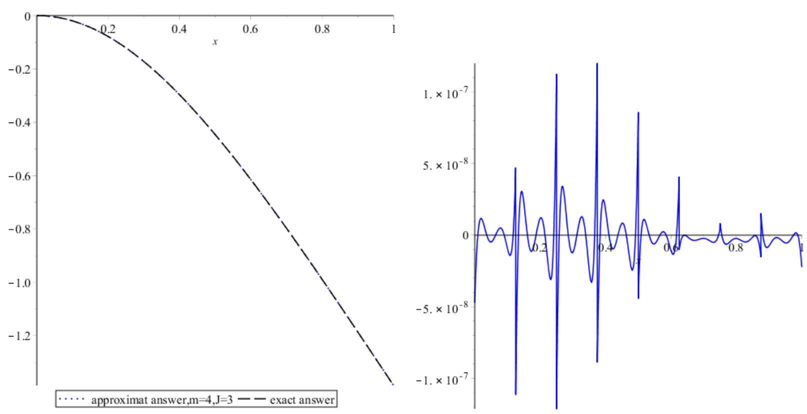

Figure 4: The numerical solution of Example 5.2 for $m = 4, J = 3$ and the exact answer and error values

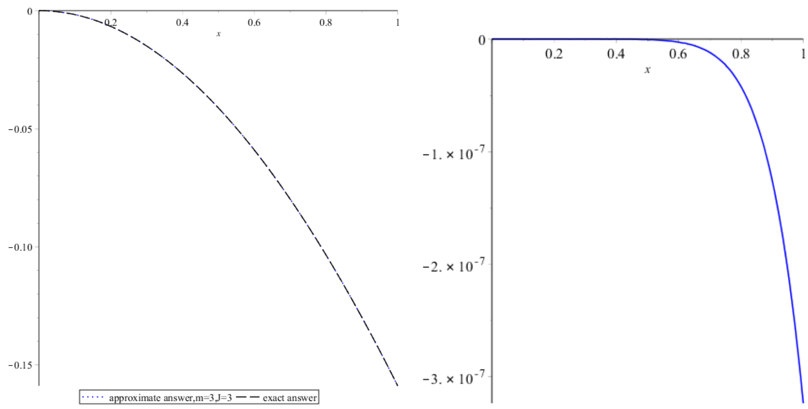

Figure 5: The numerical solution of Example 5.3 for $m = 3, J = 3$ and the exact answer and error values

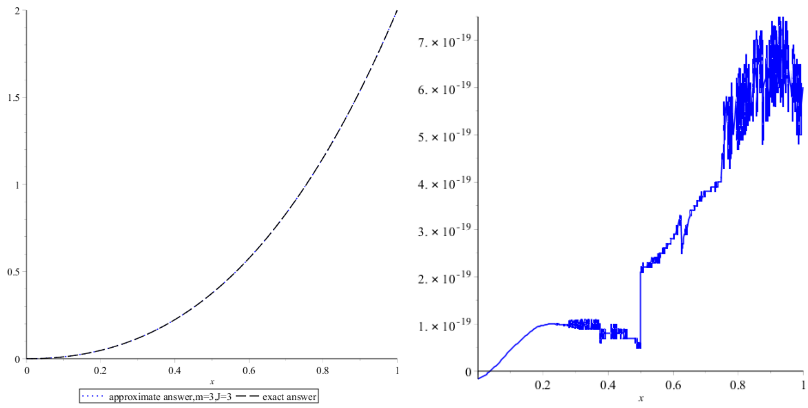

Figure 6: The numerical solution of Example 5.4 for $m = 3, J = 3$ and the exact answer and error values

Table 1: Absolute values of errors for Example 5.1

x	$\alpha = 0$ $m = 1$, $J = 4$	$m = 2, J = 1$	$m = 2, J = 2$
0.1	4.7×10^{-5}	3.3×10^{-21}	3.3×10^{-21}
0.2	4.3×10^{-6}	3.3×10^{-21}	3.3×10^{-21}
0.3	4.3×10^{-6}	2.1×10^{-20}	1.1×10^{-20}
0.4	4.7×10^{-5}	7.1×10^{-21}	3.1×10^{-21}
0.5	1.0×10^{-4}	7.0×10^{-21}	3.0×10^{-21}
0.6	4.7×10^{-5}	1.2×10^{-20}	1.0×10^{-20}
0.7	4.3×10^{-6}	3.8×10^{-21}	3.1×10^{-21}
0.8	4.3×10^{-6}	1.2×10^{-20}	1.0×10^{-20}
0.9	4.7×10^{-5}	1.1×10^{-20}	1.0×10^{-20}
x	$\alpha = 1$ $m = 3$, $J = 3$	$m = 4, J = 2$	$m = 4, J = 3$
0.1	1.1×10^{-8}	9.2×10^{-11}	2.9×10^{-12}
0.2	2.7×10^{-9}	1.8×10^{-10}	4.3×10^{-12}
0.3	2.6×10^{-9}	5.4×10^{-10}	7.1×10^{-12}
0.4	1.0×10^{-8}	2.7×10^{-10}	1.9×10^{-11}
0.5	3.5×10^{-8}	2.7×10^{-9}	9.7×10^{-11}
0.6	1.0×10^{-8}	4.4×10^{-10}	2.5×10^{-11}
0.7	2.2×10^{-9}	8.8×10^{-10}	1.5×10^{-11}
0.8	2.1×10^{-9}	1.1×10^{-9}	1.7×10^{-11}
0.9	8.1×10^{-9}	5.8×10^{-10}	3.8×10^{-11}
x	$\alpha = 5$ $m = 3$, $J = 3$	$m = 2, J = 3$	$m = 4, J = 2$
0.1	4.7×10^{-6}	3.9×10^{-7}	5.2×10^{-9}
0.2	6.1×10^{-6}	7.0×10^{-7}	1.1×10^{-8}
0.3	1.0×10^{-5}	1.1×10^{-6}	2.2×10^{-8}
0.4	9.8×10^{-6}	1.5×10^{-6}	1.1×10^{-8}
0.5	4.4×10^{-5}	6.0×10^{-6}	1.0×10^{-7}
0.6	1.2×10^{-5}	1.7×10^{-6}	9.2×10^{-9}
0.7	1.4×10^{-5}	1.5×10^{-6}	1.8×10^{-8}
0.8	1.2×10^{-5}	1.4×10^{-6}	6.4×10^{-9}
0.9	1.1×10^{-5}	1.4×10^{-6}	1.7×10^{-9}

Table 2: Absolute values of errors for Example 5.2

x	$m = 3$, $J = 3$	$m = 3, J = 4$	$m = 4, J = 3$
0.1	1.3×10^{-6}	1.7×10^{-8}	9.4×10^{-9}
0.2	1.8×10^{-8}	5.3×10^{-8}	1.1×10^{-8}
0.3	8.5×10^{-8}	3.6×10^{-8}	1.1×10^{-8}
0.4	2.5×10^{-8}	1.3×10^{-9}	1.7×10^{-8}
0.5	2.4×10^{-7}	3.3×10^{-8}	8.6×10^{-8}
0.6	4.3×10^{-7}	2.1×10^{-9}	1.0×10^{-8}
0.7	2.4×10^{-8}	3.8×10^{-8}	2.2×10^{-9}
0.8	9.6×10^{-8}	4.2×10^{-8}	2.3×10^{-9}
0.9	6.6×10^{-7}	1.1×10^{-8}	7.3×10^{-9}

Table 3: Absolute values of errors for Example 5.3

x	$m = 3$, $J = 3$	$m = 3$,$J = 4$	$m = 4$,$J = 3$
0.1	1.1×10^{-8}	1.6×10^{-10}	7.7×10^{-12}
0.2	2.6×10^{-9}	6.8×10^{-10}	1.1×10^{-11}
0.3	2.4×10^{-9}	6.6×10^{-10}	2.1×10^{-11}
0.4	9.6×10^{-9}	1.9×10^{-10}	3.5×10^{-12}
0.5	3.1×10^{-8}	2.3×10^{-9}	2.2×10^{-10}
0.6	5.8×10^{-9}	2.8×10^{-9}	2.7×10^{-9}
0.7	1.3×10^{-8}	1.1×10^{-8}	1.1×10^{-8}
0.8	4.3×10^{-8}	4.1×10^{-8}	4.2×10^{-8}
0.9	3.3×10^{-7}	1.2×10^{-7}	1.2×10^{-7}

Table 4: Absolute values of errors for Example 5.4

x	$m = 2$,$J = 4$	$m = 3$,$J = 2$	$m = 3$, $J = 3$
0.1	3.1×10^{-6}	2.6×10^{-19}	4.4×10^{-20}
0.2	3.5×10^{-6}	4.5×10^{-19}	9.7×10^{-20}
0.3	3.5×10^{-6}	6.5×10^{-19}	1.0×10^{-19}
0.4	3.1×10^{-6}	9.0×10^{-19}	8.0×10^{-20}
0.5	1.3×10^{-5}	1.1×10^{-18}	5.0×10^{-20}
0.6	3.1×10^{-6}	1.3×10^{-17}	2.9×10^{-19}
0.7	3.5×10^{-6}	1.2×10^{-17}	3.8×10^{-19}
0.8	3.5×10^{-6}	7.0×10^{-18}	6.0×10^{-19}
0.9	3.1×10^{-6}	1.0×10^{-18}	7.0×10^{-19}

Table 5: Comparison of the maximum absolute Error values

	Flatlet Multiwavelets (m=4,J=3)	B-spline wavelet ($M = 7$)	Cubic Hermit spline function ($m = 4$,$J = 4$)	Laguerre collocation ($N = 4$)
Ex 5.1, $\alpha = 0$	5.2×10^{-20}	1.1×10^{-5}	2.4×10^{-9}	———
Ex 5.1, $\alpha = 1$	9.7×10^{-11}	1.1×10^{-5}	2.2×10^{-9}	3.8×10^{-7}
Ex 5.1, $\alpha = 5$	1.0×10^{-7}	8.8×10^{-6}	7.8×10^{-7}	1.1×10^{-2}
Ex 5.2	8.6×10^{-8}	9.0×10^{-5}	4.3×10^{-5}	———
Ex 5.3	1.2×10^{-7}	1.1×10^{-5}	3.1×10^{-7}	3.2×10^{-7}

6 Conclusions

In this paper, the flatlet oblique multiwavelets (FOM) is used to solve the Lane-Emden equation. In this way, $y(x)$ was found from the expansion of $y''(x)$ by twice integrating and after any integrating, we expand the results by the dual functions of flatlet oblique multiwavelets. Aalthough these expantions have errors in our approximation, but becouse of good-conditioned behavior of integration, the results of approximation are acceptable. Considering the properties of flatlet oblique multiwavelets and using dual functions, the solution of Lane-Emden equation is converted to the solution of a sparse linear system of algebraic equations. The obtained results are presented in different values of m and J and also compared with B-spline expansion method, cubic Hermit spline functions method and Laguerre collocation method. This method is computationally attractive and its practicality is demonstrated through illustrative examples. In this research we have seen that by increasing the order of Flatlet multiwavelet (m) even using low-dimensional matrices, an acceptable approximation of answer with less error values can be achieved.

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