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New results on coefficient estimates for subclasses of bi-univalent functions related by a new integral operator

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Abstract

In the present paper, we introduce two new subclasses of the function class $\sum$ of bi-univalent functions defined in the open unit disc U. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses.

Keywords: Bi-univalent function, Analytic function, Coefficient bounds, starlike and convex function
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1 Introduction

Let $\mathbb{G}(U)$ be a class of all analytic functions f in the open unit disk $U = \{z : |z| < 1\}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad (z \in U) \tag{1.1}$$

Let $\mathbb{G}_U$ be the class of all functions in $\mathbb{G}(U)$, which are univalent in U. A function $f \in \mathbb{G}(U)$ is said to be starlike if $f(U)$ is a starlike domain with respect to the origin i.e.,the line segment joining any point of $f(U)$ to the origin lies entirely in $f(U)$ and a function $f \in \mathbb{G}(U)$ is said to be convex if $f(U)$ is convex domain i.e., the line segment joining any two points in $f(U)$ lies entirely in $f(U)$. Analytically $f \in \mathbb{G}(U)$ is starlike, denoted by $\mathcal{S}^*$ if and only if $Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0$, whereas $f \in \mathbb{G}(U)$, is convex,denoted by $\mathcal{C}$ if and only if $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$. The classes $\mathcal{S}^*(\tau)$ and $\mathcal{C}(\tau)$ of starlike and convex functions of order $\tau (0 \leq \tau \leq 1)$ are respectively characterized by

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \tau, \quad z \in U, \tag{1.2}$$

and

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$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \tau, \quad z \in U. \tag{1.3}$$

Determination of the bounds for coefficients a_n is an important problem of geometric function theory as it give information about the geometric properties of these functions. For example, the bound for the second coefficient a_2 of functions in $\mathbb{G}_U$ gives the growth and distortion bounds as well as covering theorems. It is well known that the n-th coefficient a_n is bounded by n for each $f \in \mathbb{G}(U)$.

In this paper, we estimate the initial coefficients $|a_2|$ and $|a_3|$ coefficients problem for certain subclasses of bi-univalent functions.

The Koebe One-Quarter Theorem [17] proves that the image of U under every univalent function $f \in \mathbb{G}_U$, contains the disk of radius $\frac{1}{4}$. Therefore every function $f \in \mathbb{G}_U$ has an inverse f^{-1} defined by:

$$f^{-1}(f(z)) = z, \qquad (z \in U)$$

and

$$f(f^{-1}(\omega)) = \omega, \qquad (|\omega| < \mathfrak{r}_0(f), \mathfrak{r}_0(f) \geq \frac{1}{4}),$$

where

$$g(\omega) = f^{-1}(\omega) = \omega + \sum_{j=2}^{\infty} b_j \omega^j = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + ... \tag{1.4}$$

A function $f \in \mathbb{G}(U)$ is said to be bi-univalent in the open unit disk U if both the functions f and f^{-1} are univalent there. Let $\sum$ denote the class of all bi-univalent functions defined in the unit disk U. Examples of functions in the class $\sum$ are:

$$\frac{z}{1-z}, \ \log\frac{z}{1-z}, \ \log\sqrt{\frac{1+z}{1-z}}.$$

However, the familiar Koebe function is not a member of $\sum$. Other common examples of functions in U such as:

$$\frac{2z-z^2}{2} \text{ and } \frac{z}{1-z^2},$$

are not members of $\sum$ either. Finding bounds for the coefficients of classes of bi-univalent functions dates back to 1967 (see Lewin [21]). Brannan and Taha [14] (see also [30]) introduced certain subclasses of the bi-univalent functions class $\sum$ similar to the familiar subclasses $\mathcal{S}^*(\tau)$ and $\mathcal{C}(\tau)$ (see [14]). Thus, following Brannan and Taha [14] (see also [30]) a function $f \in \mathbb{G}(U)$ is in the class $\mathcal{S}^*_{\sum}(\tau)$ of strongly bi-starlike functions of order $\tau(0 < \tau \leq 1)$, if each of the following conditions are satisfied:

$$f \in \sum \text{ and } \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\tau\pi}{2}, \quad (0 < \tau \leq 1, \ z \in U)$$

and

$$\left|\arg\left(\frac{\omega g'(z)}{g(z)}\right)\right| < \frac{\tau\pi}{2}, \quad (0 < \tau \leq 1, \ \omega \in U),$$

where g is the extension of f^{-1} to U. The classes $\mathcal{S}^*_{\sum}(\tau)$ and $\mathcal{C}_{\sum}(\tau)$ of bi-starlike functions of order τ, and bi-convex functions of order τ, corresponding (respectively) to the function classes defined by equations (1.2) and (1.3) were also introduced analogously. For each of the function classes $\mathcal{S}^*_{\sum}(\tau)$ and $\mathcal{C}_{\sum}(\tau)$, it found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (see [14, 30]).

Motivated by the earlier works of Atshan et al. [6, 7, 8, 9, 10, 11, 12], Srivastava et al. [29] and Frasin and Aouf [18] (see also [2, 3, 15, 16, 20, 22, 24, 25, 26, 32] and [1, 4, 5, 19, 23, 28, 31, 33]). In this paper, we introduce two new subclasses $\mathcal{J}^{\propto}_{\sum}(\lambda, m, n, \tau)$ and $\mathcal{J}^{\propto}_{\sum}(\lambda, m, n, \delta)$ of the function class $\sum$, that generalize the previous defined classes. This subclasses are defined with the aid of new integral operator $\mathcal{T}^{\propto}_{m,n}$ of analytic functions involving binomial series in the open unit disk U. In addition, upper bounds for the second and third coefficients for functions in this new subclasses are derive.

We introduce the following integral operator in the class $\mathcal{J}^{\propto}_{\sum}$ of analytic functions defined as follow:

Lemma 1.1. Let $f \in \mathbb{G}_U, m, n > 0$ and $\propto \in \mathbb{N}$. The integral operator denoted $\mathcal{T}^{\propto}_{m,n}$ defined as:

$$\begin{aligned}\mathcal{T}^{\propto}_{m,n} &: \mathbb{G}_U \longrightarrow \mathbb{G}_U\\ \mathcal{T}^{\propto}_{m,n} f(z) &= \frac{1}{\beta(m+1,n+1)}\int_0^{\infty}\frac{\mathfrak{t}^{m-1}}{(1-\mathfrak{t})^{m+n}}f(\mathfrak{t}z)dt\\ &= z+\sum_{j=2}^{\infty}\left(\frac{\beta(m+j,n+j)}{\beta(m+1,n+1)}\right)^{\propto}a_j z^j,\end{aligned}$$

where $\beta(m,n) = \int_0^1 \frac{\mathfrak{t}^{m+1}}{(1-\mathfrak{t})^{1-n}}d\mathfrak{t}$.

Proof .

$$\begin{aligned}\mathcal{T}_{m,n} f(z) &= \frac{1}{\beta(m+1,n+1)}\int_0^{\infty}\frac{\mathfrak{t}^{m-1}}{(1-\mathfrak{t})^{m+n}}f(\mathfrak{t}z)dt\\ &= \frac{1}{\beta(m+1,n+1)}\int_0^{\infty}\frac{\mathfrak{t}^{m-1}}{(1+\mathfrak{t})^{m+n}}\left(\mathfrak{t}z+\sum_{j=2}^{\infty}\mathfrak{t}^j a_j z^j\right)dt\\ &= \frac{1}{\beta(m+1,n+1)}\left[z\int_0^{\infty}\frac{\mathfrak{t}^{m}}{(1+\mathfrak{t})^{m+n}}d\mathfrak{t}+\left(\sum_{j=2}^{\infty}a_j z^j\right)\int_0^{\infty}\frac{\mathfrak{t}^{m+j-1}}{(1+\mathfrak{t})^{m+n}}d\mathfrak{t}\right].\end{aligned}$$

Let $x = \frac{\mathfrak{t}}{(1+\mathfrak{t})}$. Then $\mathfrak{t} = \frac{x}{1-x}$ and $d\mathfrak{t} = \frac{dx}{(1-x)^2}$. If $\mathfrak{t} = 0$, we obtain $x = 0$, while if $\mathfrak{t} = \infty$, we obtain $x = 1$.

$$\begin{aligned}&= \frac{1}{\beta(m+1,n+1)}\left[z\int_0^1\frac{\left(\frac{x}{1-x}\right)^m}{\left(1+\frac{x}{1-x}\right)^{m+n}}\frac{dx}{(1-x)^2}+\left(\sum_{j=2}^{\infty}a_j z^j\right)\int_0^1\frac{\left(\frac{x}{1-x}\right)^{m+j-1}}{\left(1+\frac{x}{1-x}\right)^{m+n}}\frac{dx}{(1-x)^2}\right]\\ &= \frac{1}{\beta(m+1,n+1)}\left[z\int_0^1\frac{x^m}{(1-x)^{2-n}}dx+\left(\sum_{j=2}^{\infty}a_j z^j\right)\int_0^1\frac{x^{m+j-1}}{(1-x)^{1+j-n}}dx\right]\\ &= \frac{1}{\beta(m+1,n+1)}\left[z\beta(m+1,n+1)+\left(\sum_{j=2}^{\infty}a_j z^j\right)\beta(m+j,n+j)\right]\\ &= z+\sum_{j=2}^{\infty}\frac{\beta(m+j,n+j)}{\beta(m+1,n+1)}a_j z^j.\end{aligned}$$

In general,

$$\mathcal{T}^{\propto}_{m,n} f(z) = z+\sum_{j=2}^{\infty}\left(\frac{\beta(m+j,n+j)}{\beta(m+1,n+1)}\right)^{\propto}a_j z^j = z+\sum_{j=2}^{\infty}\left(\mathcal{K}^j_{m,n}\right)^{\propto}a_j z^j.$$

A function $f \in \mathbb{G}_U$ is called bi-univalent in the open unit disk U if both f and f^{-1} are univalent in U. □

In order to derive our main results, we have to recall here the following Lemma [13, 27].

Lemma 1.2. If $p \in P$, then $|p_i| \leq 2$ for each i, where P is the family of all analytic functions p, for which $Re\{p(z) > 0\}$ where: $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$.

2 Coefficient Bounds for the Function Class $\mathcal{J}^{\propto}_{\sum}(\lambda, m, n, \tau)$

Definition 2.1. A function f given by 2.1 is said to be in the class $\mathcal{J}^{\propto}_{\sum}(\lambda, m, n, \tau)$, if the following are holds such that $0 \leq \tau \leq 1,\ m, n > 0$ and $\propto \in \mathbb{N}$:

$$f \in \sum \quad \text{and} \quad \left|\arg\left((1-\lambda)\frac{\mathcal{T}^{\propto}_{m,n}f(z)}{z} + \lambda(\mathcal{T}^{\propto}_{m,n}f(z))'\right)\right| < \frac{\tau\pi}{2}, \tag{2.1}$$

and

$$g \in \sum \quad \text{and} \quad \left|\arg\left((1-\lambda)\frac{\mathcal{T}^{\propto}_{m,n}g(\omega)}{\omega} + \lambda(\mathcal{T}^{\propto}_{m,n}g(\omega))'\right)\right| < \frac{\tau\pi}{2}, \tag{2.2}$$

where $\lambda \geq 1,\ z, \omega \in U$, and $g = f^{-1}$.

Theorem 2.2. Let a function (z) given by 2.1, be in the class $\mathcal{J}^{\propto}_{\sum}(\lambda, m, n, \tau)$ $0 \leq \tau \leq 1, \lambda \geq 1$ and $m, n > 0$. Then:

$$|a_2| \leq \frac{2\tau}{\sqrt{2\tau(1+2\lambda)(\mathcal{K}^3_{m,n})^{2\propto} + (1-\tau)(1+\lambda)^2(\mathcal{K}^2_{m,n})^{2\propto}}}$$

and

$$|a_3| \leq \frac{4\tau^2}{(1+\lambda)^2(\mathcal{K}^2_{m,n})^{2\propto}} + \frac{2\tau}{(1+\lambda)(\mathcal{K}^3_{m,n})^{\propto}}.$$

Proof . It follows from (2.1) and (2.2):

$$(1-\lambda)\frac{\mathcal{T}^{\propto}_{m,n}f(z)}{z} + \lambda(\mathcal{T}^{\propto}_{m,n}f(z))' = [u(z)]^{\tau} \tag{2.3}$$

$$(1-\lambda)\frac{\mathcal{T}^{\propto}_{m,n}g(\omega)}{\omega} + \lambda(\mathcal{T}^{\propto}_{m,n}g(\omega))' = [v(\omega)]^{\tau}, \tag{2.4}$$

where $u(z)$ and $v(\omega)$ in P and have the form:

$$u(z) = 1 + u_1 z + u_2 z^2 + ... \tag{2.5}$$

$$v(\omega) = 1 + v_1 \omega + v_2 \omega^2 + ... \tag{2.6}$$

Now, equating the coefficients in (2.3) and (2.4), we get:

$$(1+\lambda)(\mathcal{K}^2_{m,n})^{\propto} a_2 = \tau u_1 \tag{2.7}$$

$$(1+2\lambda)(\mathcal{K}^3_{m,n})^{\propto} a_3 = \tau u_2 + \frac{\tau(\tau-1)}{2} u_1^2 \tag{2.8}$$

$$-(1+\lambda)(\mathcal{K}^2_{m,n})^{\propto} a_2 = \tau v_1 \tag{2.9}$$

$$(1+2\lambda)(\mathcal{K}^3_{m,n})^{\propto}(2a_2^2 - a_3) = \tau v_2 + \frac{\tau(\tau-1)}{2} v_1^2 \tag{2.10}$$

From (2.7) and (2.9) we get:

$$u_1 = -v_1, \tag{2.11}$$

and

$$2(1+\lambda)^2(\mathcal{K}^2_{m,n})^{2\propto} a_2^2 = \tau^2(u_1^2 + v_1^2), \tag{2.12}$$

now by adding (2.8), (2.10):

$$2(1+2\lambda)(\mathcal{K}^3_{m,n})^{\propto}(a_2^2) = \tau(u_2 + v_2) + \frac{\tau(\tau-1)}{2}(u_1^2 + v_1^2).$$

By using (2.12):

$$2(1+2\lambda)(\mathcal{K}^3_{m,n})^{\propto}(a_2^2) = \tau(u_2 + v_2) + \frac{\tau(\tau-1)}{2}\frac{2(1+2\lambda)^2(\mathcal{K}^2_{m,n})^{2\propto} a_2^2}{\tau^2}.$$

Therefore, we have:

$$a_2^2 = \frac{\tau^2(u_2+v_2)}{2\tau(1+2\lambda)(\mathcal{K}_{m,n}^3)^{\propto} + (1-\tau)(1+\lambda)^2(\mathcal{K}_{m,n}^2)^{2\propto}}.$$

Applying Lemma 1.2 for the coefficients u_2 and v_2, we have:

$$|a_2| \leq \frac{2\tau}{\sqrt{2\tau(1+2\lambda)(\mathcal{K}_{m,n}^3)^{\propto} + (1-\tau)(1+\lambda)^2(\mathcal{K}_{m,n}^2)^{2\propto}}}$$

Next, in order to find the bound on $|a_3|$ by subtracting (2.10) from (2.8), we get:

$$2(1+2\lambda)(\mathcal{K}_{m,n}^3)^{\propto} a_3 - 2(1+2\lambda)(\mathcal{K}_{m,n}^3)^{\propto}(a_2^2) = \tau(u_2+v_2) + \frac{\tau(\tau-1)}{2}(u_1^2 - v_1^2).$$

Or equivalent:

$$a_3 = \frac{\tau^2(u_1^2 - v_1^2)}{2(1+\lambda)^2(\mathcal{K}_{m,n}^2)^{2\propto}} + \frac{\tau(u_2 - v_2)}{2(1+\lambda)(\mathcal{K}_{m,n}^3)^{\propto}}$$

Applying Lemma 1.2 for the coefficients u_1, u_2, v_1 and v_2, we have:

$$|a_3| \leq \frac{4\tau^2}{(1+\lambda)^2(\mathcal{K}_{m,n}^2)^{2\propto}} + \frac{2\tau}{(1+2\lambda)(\mathcal{K}_{m,n}^3)^{\propto}}.$$

This completes the proof. □

Corollary 2.3. Let a function $f(z)$ given by 2.1, be in the class $\mathcal{J}_{\sum}^{\propto}(1, m, n, \tau)$ $0 \leq \tau \leq 1$ and $m, n > 0$. Then:

$$|a_2| \leq \frac{\sqrt{2}\tau}{\sqrt{3\tau(\mathcal{K}_{m,n}^3)^{\propto} + 2(1-\tau)(\mathcal{K}_{m,n}^2)^{2\propto}}}$$

and

$$|a_3| \leq \frac{\tau^2}{(\mathcal{K}_{m,n}^2)^{2\propto}} + \frac{2\tau}{3(\mathcal{K}_{m,n}^3)^{\propto}}.$$

Definition 2.4. A function f given by 2.1 is said to be in the class $\mathcal{J}_{\sum}^{\propto}(\lambda, m, n, \delta)$, if the following are holds such that $\lambda \geq 1, 0 \leq \delta \leq 1, m, n > 0$ and $\propto \in \mathbb{N}$:

$$f \in \sum \quad \text{and} \quad Re\left((1-\lambda)\frac{\mathcal{T}_{m,n}^{\propto} f(z)}{z} + \lambda(\mathcal{T}_{m,n}^{\propto} f(z))'\right) > \delta, \tag{2.13}$$

and

$$g \in \sum \quad \text{and} \quad Re\left((1-\lambda)\frac{\mathcal{T}_{m,n}^{\propto} g(\omega)}{\omega} + \lambda(\mathcal{T}_{m,n}^{\propto} g(\omega))'\right) > \delta, \tag{2.14}$$

where $z, \omega \in U$, and $g = f^{-1}$.

Theorem 2.5. Let a function $f(z)$ given by 2.1, be in the class $\mathcal{J}_{\sum}^{\propto}(\lambda, m, n, \delta)$ $0 \leq \delta \leq 1, \lambda \geq 1$ and $m, n > 0$. Then:

$$|a_2| \leq \sqrt{\frac{2(1-\delta)}{(1+2\lambda)(\mathcal{K}_{m,n}^3)^{\propto}}}$$

and

$$|a_3| \leq \frac{4(1-\delta)^2}{(1+\lambda)^2(\mathcal{K}_{m,n}^2)^{2\propto}} + \frac{2(1-\delta)}{(1+2\lambda)(\mathcal{K}_{m,n}^3)^{\propto}}$$

Proof . It follows from (2.13) and (2.14):

$$(1-\lambda)\frac{\mathcal{T}^{\propto}_{m,n}f(z)}{z}+\lambda(\mathcal{T}^{\propto}_{m,n}f(z))'=\delta+(1-\delta)u(z), \tag{2.15}$$

$$(1-\lambda)\frac{\mathcal{T}^{\propto}_{m,n}g(\omega)}{\omega}+\lambda(\mathcal{T}^{\propto}_{m,n}g(\omega))'=\delta+(1-\delta)v(\omega), \tag{2.16}$$

where $u(z)$ and $v(\omega)$ have the form (2.5) and (2.6), respectively. Now, equating the coefficients in (2.3) and (2.4), equating coefficients in (2.15) and (2.16), we get:

$$(1+\lambda)(\mathcal{K}^2_{m,n})^{\propto}a_2=(1-\delta)u_1 \tag{2.17}$$

$$(1+2\lambda)(\mathcal{K}^3_{m,n})^{\propto}a_3=(1-\delta)u_2 \tag{2.18}$$

$$-(1+\lambda)(\mathcal{K}^2_{m,n})^{\propto}a_2=(1-\delta)v_1 \tag{2.19}$$

$$(1+2\lambda)(\mathcal{K}^3_{m,n})^{\propto}(2a_2^2-a_3)=(1-\delta)v_2. \tag{2.20}$$

From (2.17) and (2.19) we get:

$$u_1=-v_1, \tag{2.21}$$

and

$$2(1+\lambda)^2(\mathcal{K}^2_{m,n})^{2\propto}a_2^2=(1-\delta)^2(u_1^2+v_1^2). \tag{2.22}$$

Now by adding (2.18), (2.20):

$$2(1+2\lambda)(\mathcal{K}^3_{m,n})^{\propto}(a_2^2)=(1-\delta)(u_2+v_2).$$

Therefore, we have:

$$a_2^2=\frac{(1-\delta)(u_2+v_2)}{2(1+2\lambda)(\mathcal{K}^3_{m,n})^{\propto}}$$

Applying Lemma 1.2 for the coefficients u_2 and v_2, we have:

$$|a_2|\le\sqrt{\frac{2(1-\delta)}{(1+2\lambda)(\mathcal{K}^3_{m,n})^{\propto}}}$$

Next, in order to find the bound on $|a_3|$ by subtracting (2.20) from (2.18), we get:

$$2(1+2\lambda)(\mathcal{K}^3_{m,n})^{\propto}a_3-2(1+2\lambda)(\mathcal{K}^3_{m,n})^{\propto}(a_2^2)=(1-\delta)^2(u_2-v_2).$$

Or equivalent:

$$a_3=\frac{(1-\delta)^2(u_1^2-v_1^2)}{2(1+\lambda)^2(\mathcal{K}^2_{m,n})^{2\propto}}+\frac{(1-\delta)^2(u_2-v_2)}{2(1+2\lambda)(\mathcal{K}^3_{m,n})^{\propto}}.$$

Applying Lemma 1.2 for the coefficients u_1,u_2,v_1 and v_2, we have:

$$|a_3|\le\frac{4(1-\delta)^2}{(1+\lambda)^2(\mathcal{K}^2_{m,n})^{2\propto}}+\frac{2(1-\delta)}{(1+2\lambda)(\mathcal{K}^3_{m,n})^{\propto}}$$

This completes the proof. □

Corollary 2.6. Let a function $f(z)$ given by 2.1, be in the class $\mathcal{J}^{\propto}_{\sum}(1,m,n,\delta)$ $0\le\delta\le1$ and $m,n>0$. Then:

$$|a_2|\le\sqrt{\frac{2(1-\delta)}{3(\mathcal{K}^3_{m,n})^{\propto}}}$$

and

$$|a_3|\le\frac{(1-\delta)^2}{(\mathcal{K}^2_{m,n})^{2\propto}}+\frac{2(1-\delta)}{3(\mathcal{K}^3_{m,n})^{\propto}}.$$

References

[1] R. Abd Al-Sajjad and W.G. Atshan, *Certain analytic function sandwich theorems involving operator defined by Mittag-Leffler function*, AIP Conf. Proc. **2398** (2022), 060065.

[2] S.A. Al-Ameedee, W.G. Atshan and F.A. Al-Maamori, *Second Hankel determinant for certain subclasses of bi-univalent functions*, J. Phys.: Conf. Ser. **1664** (2020), 012044.

[3] S.A. Al-Ameedee, W.G. Atshan and F.A. Al-Maamori, *Coefficients estimates of bi-univalent functions defined by new subclass function*, J. Phys.: Conf. Ser. **1530** (2020), 012105.

[4] S.A. Al-Ameedee, W.G. Atshan and F.A. Al-Maamori, *On sandwich results of univalent functions defined by a linear operator*, J. Interdiscip. Math. **23** (2020), no. 4, 803–809.

[5] S.A. Al-Ameedee, W.G. Atshan and F.A. Al-Maamori, *Some new results of differential subordinations for Higher-order derivatives of multivalent functions*, J. Phys.: Conf. Ser. **1804** (2021), 012111.

[6] W.G. Atshan and A.A.R. Ali, *On some sandwich theorems of analytic functions involving Noor-Sãlãgean operator*, Adv. Math.: Sci. J. **9** (2020), no. 10, 8455–8467.

[7] W.G. Atshan and A.A.R. Ali, *On sandwich theorems results for certain univalent functions defined by generalized operators*, Iraqi J. Sci. **62** (2021), no. 7, 2376–2383.

[8] W.G. Atshan and R.A. Al-Sajjad, *Some applications of quasi-subordination for bi-univalent functions using Jackson's convolution operator*, Iraqi J. Sci. **63** (2022), no. 10, 4417–4428.

[9] W.G. Atshan, A.H. Battor and A.F. Abaas, *Some sandwich theorems for meromorphic univalent functions defined by new integral operator*, J. Interdiscip. Math. 24 (2021), no. 3, 579–591.

[10] W.G. Atshan and S.R. Kulkarni, *On application of differential subordination for certain subclass of meromorphically p-valent functions with positive coefficients defined by linear operator*, J. Inequal. Pure Appl. Math. **10** (2009), no. 2, 11.

[11] W.G. Atshan, I.A.R. Rahman and A.A. Lupas, *Some results of new subclasses for bi-univalent functions using quasi-subordination*, Symmetry **13** (2021), no. 9, p. 1653.

[12] W.G. Atshan, S. Yalcin and R.A. Hadi, *Coefficient estimates for special subclasses of k-fold symmetric bi-univalent functions*, Math. Appl. **9** (2020), no. 2, 83–90.

[13] D.A. Brannan, J. Clunie and W.E. Kirwan, *Coefficient estimates for a class of starlike functions*, Canad. J. Math. **22** (1970), 476–485.

[14] D.A. Brannan and T.S. Taha, *On some classes of bi-univalent functions*, Stud. Univ. Babes-Bolyai Math. **31** (1986), no. 2, 70–77.

[15] S. Bulut, *Coefficient estimates for a class of analytic and bi-univalent functions*, Novi. Sad. J. Math. **43** (2013), 59–65.

[16] N.E. Cho, O.S. Kwon and S. Owa, *Certain subclasses of Sakaguchi functions*, SEA Bull. Math. **17** (1993), 121–126.

[17] P.L. Duren, *Univalent functions*, Springer Science & Business Media, 2001.

[18] B.A. Frasin and M.K. Aouf, *New subclasses of bi-univalent functions*, Appl. Math. Lett. **24** (2011), 1569–1573.

[19] I.A. Kadum, W.G. Atshan and A.T. Hameed, *Sandwich theorems for a new class of complete homogeneous symmetric functions by using cyclic operator*, Symmetry **14** (2022), no. 10, 2223.

[20] S. Kanas and H.E. Darwish, *Fekete-Szego problem for starlike and convex functions of complex order*, Appl. Math. Lett. **23** (2010), 777–782.

[21] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. 18 (1967), 63–68.

[22] W. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, Proc. Conf. Complex Anal., Tianjin, 1992, pp. 157–169.

[23] B.K. Mihsin, W.G. Atshan and S.S. Alhily, *On new sandwich results of univalent functions defined by a linear operator*, Iraqi J. Sci. **63** (2022), no. 12, 5467–5475.

[24] M.H. Mohd and M. Darus, *Fekete-Szegö problems for quasi-subordination classes*, Abstr. Appl. Anal. **2012** (2012).

[25] G. Murugusundaramoorthy, N. Magesh and V. Prameela, *Coefficient bounds for certain subclasses of bi-univalent functions*, Abstr. Appl. Anal. **2013** (2013), 573017.

[26] E. Netanyahu, *The minimal distance of the image boundary from the origin and the second coefficient of an univalent functions in:* $|z| < 1$, Arch. Rational Mech. Anal. **32** (1969), 100–112.

[27] C. Pommerenke, *Univalent functions, Vandenhoeck and Rupercht*, Gottingen, Germany, 1975.

[28] M.A. Sabri, W.G. Atshan and E. El-Seidy, *On sandwich-type results for a subclass of certain univalent functions using a new Hadamard product operator*, Symmetry **14** (2022), no. 5, 931.

[29] H.M. Srivastava, A.K. Mishra, P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. **23** (2010), 1188–1192.

[30] T.S. Taha, *Topics in univalent function theory*, Ph.D. Thesis, University of London, London, UK, 1981.

[31] S.D. Theyab, W.G. Atshan and H.K. Abdullah, *On some sandwich results of univalent functions related by differential operator*, Iraqi J. Sci. **63** (2022), no. 11, 4928–4936.

[32] S.D. Theyab, W.G. Atshan, A.A. Lupas and H.K. Abdullah, *New results on higher-order differential subordination and superordination for univalent analytic functions using a new operator*, Symmetry **14** (2022), no. 8, 1–12.

[33] S. Yalcin, W.G. Atshan and H.Z. Hassan, *Coefficients assessment for certain subclasses of bi-univalent functions related with quasi-subordination*, Pub. Inst. Math. **108** (2020), no. 122, 155–162.