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On cubic convex functions and applications in information theory

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Abstract

In this paper, we introduce the cubic convex function and investigate Jensen type inequality, Fejér-Hermite-Hadamard type inequality and Mercer type inequality for cubic convex functions. Also, we give some applications in means and information theory by applying those inequalities.

Keywords: cubic convex function, entropy, Fejér-Hermite-Hadamard inequality, Jensens inequality, Mercer inequality
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1 Introduction

Let I be an interval in $\mathbb{R}$. A function $\varphi : I \to \mathbb{R}$ is named convex if

$$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y) \tag{1.1}$$

for all $x, y \in I$ and $0 \leq t \leq 1$.

Jensen's inequality [4], Hermite-Hadamard inequality [3] and mercer inequality [5] for convex functions are some of the best known used inequalities in various fields of optimization and mathematics.

In [2], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality.

Be reason of the extensive applications of the Jensen's, Hermite-Hadamard and mercer inequalities, some authors generalized their works via mappings of different kinds. For example, we can refer to [1, 6, 13] and the references therein.

The concept of convexity is important in information theory [9] and Shannon entropy [7]-[12].

The aim of this article is to introduce cubic convex function and establish some new Jensen, Fejér-Hermite-Hadamard and Mercer type inequality for cubic convex functions. Next, applying those inequalities in analysis and information theory to obtain a new bound for Shannon entropy.

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2 Jensen inequality for cubic convex functions

Definition 2.1. Let I be an interval in $\mathbb{R}$. A function $\varphi : I \to \mathbb{R}$ is named cubic convex if the following inequality

$$\varphi\left(\sqrt[3]{tx^3 + (1-t)y^3}\right) \leq t\varphi(x) + (1-t)\varphi(y) \tag{2.1}$$

for all $x, y \in I$ and $t \in [0,1]$.

Example 2.2. Let a be a positive real number and $\varphi : \mathbb{R} \to \mathbb{R}$ define by $\varphi(x) = ax^3$. Then φ is cubic convex but is not convex.

Example 2.3. Let a be a positive real number and $\varphi : \mathbb{R} \to \mathbb{R}$ define by $\varphi(x) = ax$. Then φ is convex but is not cubic convex because for $x = -1, y = 2$ and $t = \frac{1}{2}$ we have $\sqrt[3]{\frac{7}{2}} > \frac{1}{2}$.

Example 2.4. Let $\varphi : (0,\infty) \to \mathbb{R}$ define by $\varphi(x) = -\log(x)$. Then φ is convex and cubic convex function.

Remark 2.5. Let φ be an increasing function on $[a,b]$. Then

1. if $a < b \leq 0$ and φ is convex on $[a,b]$, then φ is cubic convex on $[a,b]$,
2. if $0 \leq a < b$ and φ is cubic convex on $[a,b]$, then φ is convex on $[a,b]$.

Lemma 2.6. Let $\varphi : [a,b] \to \mathbb{R}$ be a cubic convex function. If $a \leq u < v < w \leq b$. Then

1. $\frac{\varphi(v)-\varphi(u)}{2} \leq \varphi\left(\sqrt[3]{\frac{v^3+w^3}{2}}\right) - \varphi\left(\sqrt[3]{\frac{u^3+w^3}{2}}\right)$.
2. $\frac{\varphi(w)-\varphi(v)}{2} \geq \varphi\left(\sqrt[3]{\frac{u^3+w^3}{2}}\right) - \varphi\left(\sqrt[3]{\frac{u^3+v^3}{2}}\right)$.

Proof . Let $a \leq u < v < w \leq b$.

1. Since $u^3 < v^3 < \frac{v^3+w^3}{2} < w^3$, there exist $t, s \in [0,1]$, $t + s = 1$ such that

$$v = \sqrt[3]{tu^3 + s\frac{v^3+w^3}{2}}.$$

So

$$\begin{aligned}
\frac{\varphi(u)-\varphi(v)}{2} + \varphi\left(\sqrt[3]{\frac{v^3+w^3}{2}}\right) &= \frac{1}{2}\left[\varphi(u) - \varphi\left(\sqrt[3]{tu^3 + s\frac{v^3+w^3}{2}}\right)\right] + \varphi\left(\sqrt[3]{\frac{v^3+w^3}{2}}\right) \\
&\geq \frac{1}{2}\left[\varphi(u) - \left(t\varphi(u) + s\varphi\left(\sqrt[3]{\frac{v^3+w^3}{2}}\right)\right)\right] + \varphi\left(\sqrt[3]{\frac{v^3+w^3}{2}}\right) \\
&= \frac{s}{2}\varphi(u) + \left(1 - \frac{s}{2}\right)\varphi\left(\sqrt[3]{\frac{v^3+w^3}{2}}\right) \\
&\geq \varphi\left(\sqrt[3]{\frac{s}{2}u^3 + \left(1-\frac{s}{2}\right)\frac{v^3+w^3}{2}}\right) \\
&= \varphi\left(\sqrt[3]{\frac{s}{2}u^3 + \frac{v^3+w^3}{2} - \frac{1}{2}\left(v^3 - tu^3\right)}\right) = \varphi\left(\sqrt[3]{\frac{u^3+w^3}{2}}\right).
\end{aligned}$$

2. Since $u^3 < \frac{u^3+v^3}{2} < v^3 < w^3$, there exist $t, s \in [0,1]$, $t + s = 1$ such that $v^3 = t\frac{u^3+v^3}{2} + sw^3$. Thus $v = \sqrt[3]{t\frac{u^3+v^3}{2} + sw^3}$ and the proof is similar to the proof of part (1).

This completes the proof. □

Theorem 2.7. Let $\varphi : [a, b] \to \mathbb{R}$ be a cubic convex function and let

$$\Delta(p, q) := \varphi(p) + \varphi(q) - 2\varphi\left(\sqrt[3]{\frac{p^3 + q^3}{2}}\right)$$

for all $p, q \in [a, b]$. Then

$$\max_{p,q} \Delta(p, q) = \Delta(a, b).$$

Proof . Setting $u = a, v = p$ and $w = b$ in the part (i) of Lemma 2.6 and $u = p, v = q$ and $w = b$ in the part (ii) of Lemma 2.6, we get

$$\frac{\varphi(p) - \varphi(a)}{2} \leq \varphi\left(\sqrt[3]{\frac{p^3 + b^3}{2}}\right) - \varphi\left(\sqrt[3]{\frac{a^3 + b^3}{2}}\right), \tag{2.2}$$

$$\frac{\varphi(b) - \varphi(q)}{2} \geq \varphi\left(\sqrt[3]{\frac{p^3 + b^3}{2}}\right) - \varphi\left(\sqrt[3]{\frac{p^3 + q^3}{2}}\right), \tag{2.3}$$

respectively.

From (2.3) we arrive at

$$\frac{\varphi(q) - \varphi(b)}{2} \leq \varphi\left(\sqrt[3]{\frac{p^3 + q^3}{2}}\right) - \varphi\left(\sqrt[3]{\frac{p^3 + b^3}{2}}\right). \tag{2.4}$$

Adding (2.2) from (2.4), we obtain the desired result. □

In the following theorem we get Jensen type inequality for cubic convex functions.

Theorem 2.8. Let $\varphi : I \to \mathbb{R}$ be a cubic convex function, let $x_i \in I$, and let $t_i \geq 0$ for all $i = 1, \ldots, n$, with $\sum_{i=1}^{n} t_i = 1$. Then

$$\varphi\left(\sqrt[3]{\sum_{i=1}^{n} t_i x_i^3}\right) \leq \sum_{i=1}^{n} t_i \varphi(x_i). \tag{2.5}$$

Proof . The proof can be done by using method induction. For $n = 2$ (2.5) is holds. To prove the step of induction for $n \geq 2$, suppose that (2.5) satisfies for all convex combinations having less than or equal to n elements. Without loss of generality, we assume that $t_n + t_{n+1} \neq 0$. Therefore,

$$\begin{aligned}
\varphi\left(\sqrt[3]{\sum_{i=1}^{n+1} t_i x_i^3}\right) &= \varphi\left(\sqrt[3]{\sum_{i=1}^{n-1} t_i x_i^3 + (t_n + t_{n+1}) \times \frac{t_n x_n^3 + t_{n+1} x_{n+1}^3}{t_n + t_{n+1}}}\right) \\
&\leq \sum_{i=1}^{n-1} t_i \varphi(x_i) + (t_n + t_{n+1}) \varphi\left(\sqrt[3]{\frac{t_n x_n^3 + t_{n+1} x_{n+1}^3}{t_n + t_{n+1}}}\right) \\
&\leq \sum_{i=1}^{n-1} t_i \varphi(x_i) + (t_n + t_{n+1}) \left(\frac{t_n}{t_n + t_{n+1}} \varphi(x_n) + \frac{t_{n+1}}{t_n + t_{n+1}} \varphi(x_{n+1})\right) \\
&= \sum_{i=1}^{n+1} t_i \varphi(x_i).
\end{aligned}$$

This completes the proof. □

Theorem 2.9. Let $\varphi : I \to \mathbb{R}$ be a cubic convex function, let $x_i \in I$, and let $t_i \geq 0$ for all $i = 1, \ldots, n$, with

$\sum_{i=1}^{n} t_i = 1$. Then

$$\begin{aligned}\mathcal{JC}_\varphi(\mathbf{t},\mathbf{x}) :&= \sum_{i=1}^{n} t_i\varphi(x_i) - \varphi\left(\sqrt[3]{\sum_{i=1}^{n} t_i x_i^3}\right)\\ &\geq \max_{1\leq r<s\leq n}\left\{t_r\varphi(x_r)+t_s\varphi(x_s)-(t_r+t_s)\varphi\left(\sqrt[3]{\frac{t_r x_r^3+t_s x_s^3}{t_r+t_s}}\right)\right\}\end{aligned}$$

where $\mathbf{x}=\{x_i\}, \mathbf{t}=\{t_i\}$.

Proof . Suppose that $r, s \in \{1,\dots,n\}$ are arbitrary. So

$$\begin{aligned}\varphi\left(\sqrt[3]{\sum_{i=1}^{n} t_i x_i^3}\right) &= \varphi\left(\sqrt[3]{\sum_{i\neq r,s} t_i x_i^3 + (t_r+t_s)\frac{t_r x_r^3+t_s x_s^3}{t_r+t_s}}\right)\\ &\leq \sum_{i\neq r,s} t_i\varphi(x_i) + (t_r+t_s)\varphi\left(\sqrt[3]{\frac{t_r x_r^3+t_s x_s^3}{t_r+t_s}}\right).\end{aligned}$$

Therefore,

$$\mathcal{JC}_\varphi(\mathbf{t},\mathbf{x}) \geq t_r\varphi(x_r)+t_s\varphi(x_s)-(t_r+t_s)\varphi\left(\sqrt[3]{\frac{t_r x_r^3+t_s x_s^3}{t_r+t_s}}\right).$$

This completes the proof. □

Theorem 2.10. Let $\varphi : [a,b] \to \mathbb{R}$ be a cubic convex function, let $\mathbf{x} := \{x_i\} \in I$, and let $\mathbf{t} := \{t_i\}, t_i \geq 0$ for all $i = 1,\dots,n$, with $\sum_{i=1}^{n} t_i = 1$ and $\lambda, \mu \geq 0, \lambda+\mu = 1$. Then

$$\mathcal{JC}_\varphi(\mathbf{t},\mathbf{x}) \leq \max_{\lambda}\left[\lambda\varphi(a)+\mu\varphi(b)-\varphi\left(\sqrt[3]{\lambda a^3+\mu b^3}\right)\right] := \Lambda_\varphi(a,b). \tag{2.6}$$

Proof . Let $x_i \in [a,b]$. Then there exists a $\gamma_i \in [0,1]$ such that $x_i^3 = \gamma_i a^3 + (1-\gamma_i)b^3$ for all $i = 1,2,\dots n$. So

$$\begin{aligned}&\sum_{i=1}^{n} t_i\varphi(x_i) - \varphi\left(\sqrt[3]{\sum_{i=1}^{n} t_i x_i^3}\right)\\ &= \sum_{i=1}^{n} t_i\varphi\left(\sqrt[3]{\gamma_i a^3+(1-\gamma_i)b^3}\right) - \varphi\left(\sqrt[3]{\sum_{i=1}^{n} t_i\left(\gamma_i a^3+(1-\gamma_i)b^3\right)}\right)\\ &\leq \sum_{i=1}^{n} t_i\left(\gamma_i\varphi(a)+(1-\gamma_i)\varphi(b)\right) - \varphi\left(\sqrt[3]{a^3\sum_{i=1}^{n} t_i\gamma_i + b^3\sum_{i=1}^{n} t_i(1-\gamma_i)}\right)\\ &= \lambda\varphi(a)+\mu\varphi(b)-\varphi\left(\sqrt[3]{\lambda a^3+\mu b^3}\right),\end{aligned}$$

where $\lambda = \sum_{i=1}^{n} t_i\gamma_i$ and $\mu = 1-\lambda$. Hence

$$\mathcal{JC}_\varphi(\mathbf{t},\mathbf{x}) \leq \max_{\lambda}\left[\lambda\varphi(a)+\mu\varphi(b)-\varphi\left(\sqrt[3]{\lambda a^3+\mu b^3}\right)\right], \tag{2.7}$$

which ends the proof. □

In the following theorem we obtain Fejér-Hermite-Hadamard type inequality for cubic convex functions.

Theorem 2.11. Let φ be a real-valued and cubic convex function on $[a,b]$ and $\omega : [a^3,b^3] \to \mathbb{R}$ be a nonnegative, integrable and symmetric with respect to $\frac{a^3+b^3}{2}$. Then

$$\varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right)\int_{a^3}^{b^3}\omega(x)dx \le \int_{a^3}^{b^3}\varphi\left(\sqrt[3]{x}\right)\omega(x)dx \le \frac{\varphi(a)+\varphi(b)}{2}\int_{a^3}^{b^3}\omega(x)dx.$$

Proof . Assume that $0 \le t \le 1$, we have

$$\varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right) \le \frac{\varphi\left(\sqrt[3]{ta^3+(1-t)b^3}\right)+\varphi\left(\sqrt[3]{(1-t)a^3+tb^3}\right)}{2} \le \frac{\varphi(a)+\varphi(b)}{2}.$$

Multiply the above inequalities by $\omega(ta^3+(1-t)b^3)$ and integrating with respect to t over $[0,1]$, we arrive at

$$\varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right)\int_{a^3}^{b^3}\omega(x)dx \le \int_{a^3}^{b^3}\varphi\left(\sqrt[3]{x}\right)\omega(x)dx \le \frac{\varphi(a)+\varphi(b)}{2}\int_{a^3}^{b^3}\omega(x)dx,$$

and the proof is complete. □

If we set $\omega(x)=1$ in Theorem 2.11, then it implies that the Hermite–Hadamard type inequality for cubic convex functions.

Corollary 2.12. Let φ be a real-valued and cubic convex function on $[a,b]$. Then

$$\varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right) \le \frac{1}{b^3-a^3}\int_{a^3}^{b^3}\varphi\left(\sqrt[3]{x}\right)dx \le \frac{\varphi(a)+\varphi(b)}{2}. \tag{2.8}$$

Now, we show that Hermite-Hadamard inequality has a self-improving property.

Corollary 2.13. Let φ be a real-valued and cubic convex function on $[a,b]$. Then

$$\varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right) \le \frac{1}{b^3-a^3}\int_{a^3}^{b^3}\varphi\left(\sqrt[3]{x}\right)dx \le \frac{1}{2}\left[\varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right)+\frac{\varphi(a)+\varphi(b)}{2}\right]. \tag{2.9}$$

Proof . Using twice the right part of (2.8) yield

$$\frac{2}{b^3-a^3}\int_{a^3}^{\frac{a^3+b^3}{2}}\varphi\left(\sqrt[3]{x}\right)dx \le \frac{1}{2}\left[\varphi(a)+\varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right)\right], \tag{2.10}$$

and

$$\frac{2}{b^3-a^3}\int_{\frac{a^3+b^3}{2}}^{b^3}\varphi\left(\sqrt[3]{x}\right)dx \le \frac{1}{2}\left[\varphi\left(\sqrt[3]{\frac{a^3+b^3}{2}}\right)+\varphi(b)\right]. \tag{2.11}$$

From (2.10) and (2.11) clearly leads to (2.9). □

Theorem 2.14. Let φ be a cubic convex function on $[a,b]$. Then

$$\varphi\left(\sqrt[3]{a^3+b^3-\sum_{i=1}^{n}t_ix_i^3}\right) \le \varphi(a)+\varphi(b)-\sum_{i=1}^{n}t_i\varphi(x_i), \tag{2.12}$$

where $x_i \in [a,b]$ and $t_i \ge 0$ for all $i=1,\ldots,n$ with $\sum_{i=1}^{n}t_i=1$.

Proof . Assume that $x \in [a,b]$. Then there exists $t \in [0,1]$ such that $x^3 = ta^3 + (1-t)b^3$. Thus

$$\varphi(x) + \varphi\left(\sqrt[3]{a^3+b^3-x^3}\right) = \varphi\left(\sqrt[3]{ta^3+(1-t)b^3}\right) + \varphi\left(\sqrt[3]{(1-t)a^3+tb^3}\right) \\ \leq \varphi(a) + \varphi(b).$$

Hence

$$\varphi(x) + \varphi\left(\sqrt[3]{a^3+b^3-x^3}\right) \leq \varphi(a) + \varphi(b),$$

for all $x \in [a,b]$. So

$$\varphi\left(\sqrt[3]{a^3+b^3-\sum_{i=1}^n t_i x_i^3}\right) = \varphi\left(\sqrt[3]{\sum_{i=1}^n t_i\left(a^3+b^3-x_i^3\right)}\right) \\ \leq \sum_{i=1}^n t_i \varphi\left(\sqrt[3]{a^3+b^3-x_i^3}\right) \leq \varphi(a)+\varphi(b) - \sum_{i=1}^n t_i\varphi(x_i).$$

This completes the proof of the inequality (2.12). □

3 Applications

Let $\mathbf{x} = \{x_i\} \subseteq [a,b], \mathbf{t} = \{t_i\}$ and $\sum_{i=1}^n t_i = 1$. Define

$$A(\mathbf{t},\mathbf{x}^3) = \sum_{i=1}^n t_i x_i^3, \qquad \tilde{A}(\mathbf{t},\mathbf{x}^3) := a+b-A(\mathbf{t},\mathbf{x}^3),$$

$$G(\mathbf{t},\mathbf{x}) = \prod_{i=1}^n x_i^{t_i} \quad \text{and} \quad \tilde{G}(\mathbf{t},\mathbf{x}) = \frac{ab}{G(\mathbf{t},\mathbf{x})}.$$

Definition 3.1. For probability distribution $\mathbf{t}$ the Shannon entropy of $\mathbf{t}$ is defined by

$$H(\mathbf{t}) := -\sum_{i=1}^n t_i \log(t_i).$$

Proposition 3.2. Let $0 < a \leq x_i \leq b$. Then

$$a\sqrt[3]{A(\mathbf{t},\mathbf{x}^3)} \leq bG(\mathbf{t},\mathbf{x}).$$

Proof . The inequality follows from Theorem 2.10 with $\varphi(x) := -\log(x)$ and

$$\log\left(\frac{\sqrt[3]{\lambda a^3+\mu b^3}}{a^\lambda b^\mu}\right) \leq \log\left(\frac{b}{a}\right)$$

where $0 \leq \lambda \leq 1$ and $\mu = 1-\lambda$. □

Proposition 3.3. Let $\eta := \max\{t_i : 1 \leq i \leq n\}$ and $\theta := \min\{t_i : 1 \leq i \leq n\}$. Then

$$0 \leq \log n - H(\mathbf{t}) \leq \log\left(\frac{\eta}{\theta}\right).$$

Proof . The inequality follows from Theorem 2.10 with $\varphi(x) := -\log(x)$ and $x_i := \frac{1}{\sqrt[3]{t_i}}$ for all $i = 1, \dots, n$. □

Proposition 3.4. Let $0 < a \leq x_i \leq b$. Then

$$\tilde{G}(\mathbf{t},\mathbf{x}) \leq \sqrt[3]{\tilde{A}(\mathbf{t},\mathbf{x}^3)}.$$

Proof . The inequality follows from Theorem 2.14 with $\varphi(x) := -\log(x)$. □

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