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Error bounds of lower semi-continuous convex-along-rays functions

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Abstract

In this paper, we study Lipschitz global error bounds for lower semi-continuous convex-along-rays (l.s.c. CAR) functions. We find a condition that ensures the existence of a global error bound for a CAR function. Moreover, we find a condition under which an l.s.c. CAR function does not have a Lipschitz global error bound. Finally, we survey Lipschitz's global error bounds of an l.s.c. (in particular, an l.s.c. CAR) function from the perspective of abstract convexity.

Keywords: Convex-along-rays functions, Global error bound,Abstract convexity
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1 Introduction

An *error bound* is an inequality that restricts the distance from a vector x in a set X to a set S by a residual function. If X is the whole space, the error bound is called a *global error bound*, and if X is a neighborhood of the vector x, the error bound is said to be a *local error bound*. The study of error bounds has received growing attention from the community of mathematical programming. Error bounds for convex functions have been studied by many researchers. We refer the reader to [6, 7, 10, 11, 12, 16, 18, 19, 21, 22, 27]. For l.s.c. functions, error bounds have been studied in [2, 22, 26]. When the residual function has a linear form, the error bound is called a *Lipschitz error bound*. Throughout this paper, by an error bound we mean a Lipschitz error bound. When the residual function has a power form, the error bound is called a *Hölderian error bound* (or, an *error bound with exponent*). See [4, 8, 15, 17, 24]. To the best of our knowledge, error bounds for CAR functions have not been studied so far. Therefore, we investigate Lipschitz error bounds of l.s.c. CAR functions on $\mathbb{R}^n$. The rest of this paper is organized as follows. In Section 2, the required preliminaries are presented. In Section 3, the properties of CAR functions are used to find a condition that ensures the existence of a Lipschitz global error bound. Also, a condition is found under which a lower semi-continuous CAR function does not have a global error bound. Finally, some examples are given in this section. In Section 4, error bounds are investigated from the perspective of abstract convexity.

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2 Preliminaries

This section defines the basic notions that will be used in the rest of this paper.

Definition 2.1. A function $f : Q \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ is called convex-along-rays if the functions $f_x : \mathbb{R}_+ \longrightarrow \mathbb{R}_{+\infty}$ defined by $f_x(t) = f(tx)$ are convex for all $x \in Q$, where Q is a cone and $\mathbb{R}_{+\infty} = (-\infty, +\infty]$.

Definition 2.2. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ has a Lipschitz global error bound with constant $\mu > 0$ if

$$d_S(x) \leq \mu f_+(x) \tag{2.1}$$

for every $x \in \mathbb{R}^n$, where $f_+(x) = \max\{0, f(x)\}$, $S = \{x \in Q : f(x) \leq 0\}$ and $d_S(x) = \inf\limits_{s \in S} \|x - s\|$. By (2.1), it suffices to have $d_S(x) \leq \mu f_+(x)$ for all $x \in \mathbb{R}^n \setminus S$, and in our argument, for all $x \in Q \setminus S$.

Definition 2.3. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ has a Lipschitz local error bound with constant $\mu > 0$ if there exists $\varepsilon > 0$ such that $d_S(x) \leq \mu f_+(x)$ for all $x \in f^{-1}(0, \varepsilon)$, where $f^{-1}(0, \varepsilon) = \{x \in \mathbb{R}^n : 0 < f(x) < \varepsilon\}$.

Definition 2.4. A set $U \subseteq \mathbb{R}^n$ is called downward if

$$x \in U,\ \bar{x} \in \mathbb{R}^n \ \textit{and}\ \bar{x} \leq x \ \textit{imply}\ \bar{x} \in U,$$

where $\bar{x} \leq x$, that is, $\bar{x}_i \leq x_i$ for all $i = 1, \ldots, n$. We let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0 \ \ \forall \ \ i = 1, \ldots, n\}$ and $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x_i > 0 \ \ \forall \ \ i = 1, \ldots, n\}$. Also, we say that the set $A \subset \mathbb{R}^n$ is *proper* if $A \neq \emptyset$ and $A \neq \mathbb{R}^n$.

Definition 2.5. A collection G of subsets V_i of $\mathbb{R}^n$, where i is in an index set I, is called linearly regular if there exists $\sigma > 0$ such that $d_{\bigcap\limits_{i \in I} V_i}(x) \leq \sigma \sup\limits_{i \in I} d_{V_i}(x)$ for all $x \in \mathbb{R}^n$. See[25]. The function $\delta_C(x) = \begin{cases} 0 & x \in C \\ +\infty, & o.w. \end{cases}$ is called the indicator function of the set C at the point x.

Definition 2.6. The Dini lower directional derivative of the function g at the point x in the direction v is defined by $\underline{d}^+ g(x, v) = \liminf\limits_{t \longrightarrow 0^+} \dfrac{g(x + tv) - g(x)}{t}$.

Definition 2.7. Let C be a subset of X, let $x \in X$ and $p \in C$. Then p is best approximation to x from C (or a projection of x onto C) if $\|x - p\| = d_C(x)$. If every point in X has at least one projection onto C, then C is called proximal. See [3].

Definition 2.8. Let X be a metric space with metric d. The diameter of a subset C of X is $diamC = Sup_{x,y \in C}\ d(x, y)$. See [3].

3 Error bounds for l.s.c. CAR functions

In this section, we find a condition that ensures the existence of an error bound for an l.s.c. CAR function. Also, we find some conditions under which an l.s.c. CAR function does not have a global error bound. For an l.s.c. CAR function f, consider $S = \{x \in Q : f(x) \leq 0\}$ and define $S_x = \{t \in \mathbb{R}_+ : f_x(t) \leq 0\}$. The following lemma reveals the relationship between S and S_x.

Lemma 3.1. Let $f : Q \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ be a proper l.s.c. CAR function. Then, $S \neq \emptyset$ if and only if $S_x \neq \emptyset$ for some $x \in Q$.

Proof . If $S \neq \emptyset$, then there exists $\bar{x} \in Q$ such that $f(\bar{x}) \leq 0$. Then $f_{\bar{x}}(1) \leq 0$, that is, $1 \in S_{\bar{x}}$. This shows that $S_{\bar{x}} \neq \emptyset$. Conversely, let $S_{\bar{x}} \neq \emptyset$ for some $\bar{x} \in Q$. Then, there exists $\bar{t} \in \mathbb{R}_+$ such that $f_{\bar{x}}(\bar{t}) \leq 0$, that is, $f(\bar{t}\bar{x}) \leq 0$. Since Q is a cone, $\bar{t}\bar{x} \in Q$. Therefore, $\bar{t}\bar{x} \in S$. □

Throughout this paper, we assume that S is non-empty. It is well-known that on a normed space, a proper convex function f has a local error bound if and only if it has a global error bound. See [26, Proposition 2]. But, this is not true for a CAR function. The following example reveals this fact.

Example 3.1. Consider $Q = \mathbb{R}$ and the CAR function

$$f(x) = \begin{cases} x+1, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

If $\varepsilon = \frac{1}{2}$, then $d_S(x) \leq \mu f_+(x)$ for all $x \in f^{-1}(0,\varepsilon)$ and $\mu = 1$. But, by taking $(x_k)_{k\geq 1} = (k)_{k\geq 1}$ we obtain $d_S(x_k) \longrightarrow +\infty$ as $k \longrightarrow +\infty$, while $f_+(x_k) = 1$ for all $k \geq 1$. Then f has a local error bound, but it does not have a global error bound. Define $A_x = x.S_x$ for each $x \in Q$. Then, $A_x \subseteq S$ and A_x is on the ray $R_x = \{tx : t \geq 0\}$.

Lemma 3.2. (i) Given $0 \neq x'$ and $0 \neq x''$ on the ray $R_{\bar{x}} = \{t\bar{x} : t \geq 0\}$. Then $A_{x'} = A_{x''}$.

(ii) If $0 \neq x'$ and $0 \neq x''$ satisfy $A_{x'} = A_{x''} \neq \{0\}$, then x' and x'' are on the same ray, that is, $R_{x''} = R_{x'}$.

Proof . (i) Let $0 \neq x'$ and $0 \neq x''$ be on the ray $R_{\bar{x}}$. Then, $x'' = \hat{t}\bar{x}$ and $x' = \bar{t}\bar{x}$ for some $\hat{t} > 0$ and $\bar{t} > 0$, respectively. If $y' \in A_{x'}$, then there exists $t' \in S_{x'}$ such that $y' = t'x'$. Then $y' = t'\bar{t}\bar{x}$. Since $t' \in S_{x'}$, thus, $f_{x'}(t') \leq 0$ which implies $f(t'\bar{t}\bar{x}) = f(t'x') = f_{x'}(t') \leq 0$. Since $x'' = \hat{t}\bar{x}$, it follows that $\bar{x} = \frac{1}{\hat{t}}x''$. This implies $y' = \frac{t'\bar{t}}{\hat{t}}x''$. Since $f_{\frac{t'\bar{t}}{\hat{t}}}(x'') = f(\frac{t'\bar{t}}{\hat{t}}x'') = f(t'\bar{t}\bar{x}) \leq 0$, we obtain $\frac{t'\bar{t}}{\hat{t}} \in S_{x''}$, which implies $y' = \frac{t'\bar{t}}{\hat{t}}x'' \in A_{x''}$. Then, $A_{x'} \subseteq A_{x''}$. The inclusion $A_{x''} \subseteq A_{x'}$ can be proved similarly.

(ii) Assume, on the contrary, that x' and x'' are not on the same ray. Then, no $t > 0$ exists such that $x' = tx''$. Consider $0 \neq y \in A_{x'} = A_{x''}$. Then $y = t'x'$ for some $t' > 0$ and $y = t''x''$ for some $t'' > 0$. So, $x' = \frac{t''}{t'}x'' = \hat{t}x''$, where $\hat{t} = \frac{t''}{t'} > 0$. This contradiction shows that x' and x'' are on the same ray. □ The following lemma gives us a relationship between the distance from a point on the ray R_x to the set A_x and the distance from a point on $\mathbb{R}_+$ to the set S_x.

Lemma 3.3. For each $x \in Q$, let A_x and S_x be defined as above. If $S_x \neq \emptyset$ for some $x \in Q$, then

$$d_{A_x}(tx) = \|x\| d_{S_x}(t) \quad \forall\, t \in \mathbb{R}_+. \tag{3.1}$$

Here, $||.||$ refers to the Euclidean norm.

Proof . Let $\bar{x} \in Q$ and $S_{\bar{x}} \neq \emptyset$. If $\bar{x} = 0$, then $A_{\bar{x}} = \{0\}$ and $t\bar{x} = 0$ for all $t \geq 0$. Thus, (3.1) holds. Let $\bar{x} \neq 0$. Since $S_{\bar{x}} \neq \emptyset$, $S_{\bar{x}} \subseteq \mathbb{R}_+$ and $f_{\bar{x}}$ is l.s.c., $S_{\bar{x}}$ is closed and has one of the following forms.

(i) $S_{\bar{x}} = [a, b]$ with $0 \leq a \leq b < \infty$. Then $A_{\bar{x}} = \bar{x}[a, b]$. Now, let $t \in \mathbb{R}_+ \setminus S_{\bar{x}}$. If $t > b$, then $d_{S_{\bar{x}}}(t) = t - b = |t - b|$ and $d_{A_{\bar{x}}}(t\bar{x}) = \|t\bar{x} - b\bar{x}\| = \|\bar{x}\||t-b| = \|\bar{x}\| d_{S_{\bar{x}}}(t)$. If $t < a$, a similar reasoning shows $d_{A_{\bar{x}}}(t\bar{x}) = \|\bar{x}\||a-t| = \|\bar{x}\| d_{S_{\bar{x}}}(t)$.

(ii) $S_{\bar{x}} = [a, +\infty)$ with $a \geq 0$. If $a = 0$, then $S_{\bar{x}} = \mathbb{R}_+$ and $A_{\bar{x}} = R_{\bar{x}}$, which imply (3.1). If $a > 0$, then similar to (i), for each $t < a$ we obtain $d_{A_{\bar{x}}}(t\bar{x}) = \|\bar{x}\||a - t| = \|\bar{x}\| d_{S_{\bar{x}}}(t)$. □ To find a condition under which f has a global error bound, we need to find some conditions under which f_x has a global error bound. First, similar to the notations used in [22], for each $t \in \mathbb{R}_+$ and $\bar{x} \in Q$ define

$$N^1_{S_{\bar{x}}}(t) = \{v \in \{-1, 1\} : d_{S_{\bar{x}}}(t + \alpha v) = \alpha \text{ for some } \alpha > 0\},$$

$$\partial^-_N S_{\bar{x}} = \{t \in \partial S_{\bar{x}} : N^1_{S_{\bar{x}}}(t) = \{-1\}\},$$

$$\partial^+_N S_{\bar{x}} = \{t \in \partial S_{\bar{x}} : N^1_{S_{\bar{x}}}(t) = \{+1\}\}$$

and

$$\partial_N S_{\bar{x}} = \{t \in \partial S_{\bar{x}} : N^1_{S_{\bar{x}}}(t) \neq \emptyset\},$$

where $\partial S_{\bar{x}}$ denotes the boundary of the set $S_{\bar{x}}$.

Theorem 3.1. Let $f : Q \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ be a proper l.s.c. CAR function, $\bar{x} \in Q$ and $S_{\bar{x}}$ be proper. Then, for a constant $\gamma_{\bar{x}} > 0$, the following statements are equivalent.

(i) The global error bound

$$d_{S_{\bar{x}}}(t) \leq \gamma_{\bar{x}} (f_{\bar{x}})_+(t) \quad \forall \;\; t \in \mathbb{R}_+ \tag{3.2}$$

holds.

(ii) For each $\bar{t} \in \partial_N^+ S_{\bar{x}}$, $f_{\bar{x}}^+(\bar{t}) \geq \gamma^{-1}$ for all $\gamma \geq \gamma_{\bar{x}}$, and for each $\bar{t} \in \partial_N^- S_{\bar{x}}$, $f_{\bar{x}}^-(\bar{t}) \leq -\gamma^{-1}$ for all $\gamma \geq \gamma_{\bar{x}}$. Here, $f_{\bar{x}}^+(\bar{t}) = \lim_{\alpha \longrightarrow 0^+} \frac{f_{\bar{x}}(\bar{t}+\alpha)-f_{\bar{x}}(\bar{t})}{\alpha}$ and $f_{\bar{x}}^-(\bar{t}) = \lim_{\alpha \longrightarrow 0^+} \frac{f_{\bar{x}}(\bar{t})-f_{\bar{x}}(\bar{t}-\alpha)}{\alpha}$ are the right and left derivatives of $f_{\bar{x}}$ at the point $\bar{t}$, respectively.

Proof . Let $\bar{x} \in Q$ and $S_{\bar{x}}$ be proper. Consider all the cases in which $S_{\bar{x}}$ is proper: $S_{\bar{x}} = [t_1, t_2]$ with $0 \leq t_1 < t_2 < +\infty$, $S_{\bar{x}} = [t_1, +\infty)$ with $0 < t_1$, $S_{\bar{x}} = \{0\}$, and $S_{\bar{x}} = \{\bar{t}\}$ with $\bar{t} > 0$.

(i)⇒(ii): Let (i) hold for $\gamma_{\bar{x}} > 0$ and $S_{\bar{x}} = [t_1, t_2]$ with $t_2 > 0$. If $t_1 = 0$, then $t_1 \notin \partial_N S_{\bar{x}}$. Thus, let $t_1 > 0$ and $\bar{t} = t_1$. Then, $N^1_{S_{\bar{x}}}(\bar{t}) = \{-1\}$ and for each $\alpha \in (0, \bar{t}]$, $(\bar{t}-\alpha) \in \mathbb{R}_+ \setminus S_{\bar{x}}$. Hence, by (3.2), $f_{\bar{x}}(\bar{t}-\alpha) \geq \gamma_{\bar{x}}^{-1} d_{S_{\bar{x}}}(\bar{t}-\alpha) = \gamma_{\bar{x}}^{-1}\alpha$, for each $\alpha \in (0, \bar{t}]$. Then, $-f_{\bar{x}}(\bar{t}-\alpha) \leq -\gamma_{\bar{x}}^{-1}\alpha$. Since $f_{\bar{x}}(\bar{t}) = 0$, we obtain $f_{\bar{x}}^-(\bar{t}) = \lim_{\alpha \longrightarrow 0^+} \frac{f_{\bar{x}}(\bar{t})-f_{\bar{x}}(\bar{t}-\alpha)}{\alpha} \leq -\gamma_{\bar{x}}^{-1} \leq -\gamma^{-1}$ for all $\gamma \geq \gamma_{\bar{x}} > 0$.

If $\bar{t} = t_2$, then $N^1_{S_{\bar{x}}}(\bar{t}) = \{+1\}$ and for each $\alpha > 0$, $\bar{t}+\alpha \in \mathbb{R}_+ \setminus S_{\bar{x}}$. Thus, by (3.2), $f_{\bar{x}}(\bar{t}+\alpha) \geq \gamma_{\bar{x}}^{-1} d_{S_{\bar{x}}}(\bar{t}+\alpha) = \gamma_{\bar{x}}^{-1}\alpha$, for each $\alpha > 0$. Since $f_{\bar{x}}(\bar{t}) = 0$, similar to the above argument we can write $f_{\bar{x}}^+(\bar{t}) \geq \gamma^{-1}$ for all $\gamma \geq \gamma_{\bar{x}} > 0$.

For the other cases considered for $S_{\bar{x}}$, similar arguments allow us to conclude (ii). Note that for the case $S_{\bar{x}} = \{\bar{t}\}$, where $\bar{t} > 0$, one has $N^1_{S_{\bar{x}}}(\bar{t}) = \{-1, +1\}$.

(ii)⇒(i): Let (ii) hold and, $S_{\bar{x}} = [t_1, t_2]$ with $t_1 > 0$ and $t_2 < +\infty$. If $\bar{t} = t_2$, then $N^1_{S_{\bar{x}}}(\bar{t}) = \{+1\}$. Since by the hypothesis $\lim_{\alpha \longrightarrow 0^+} \frac{f_{\bar{x}}(\bar{t}+\alpha)-f_{\bar{x}}(\bar{t})}{\alpha} \geq \gamma^{-1}$, $f_{\bar{x}}$ is convex, and the fraction $\frac{f_{\bar{x}}(\bar{t}+\alpha)-f_{\bar{x}}(\bar{t})}{\alpha}$ is increasing relative to α, $f_{\bar{x}}(\bar{t}+\alpha) - f_{\bar{x}}(\bar{t}) \geq \gamma^{-1}\alpha$ for all $\alpha > 0$ and all $\gamma \geq \gamma_{\bar{x}}$. Consequently, $f_{\bar{x}}(\bar{t}+\alpha) \geq \gamma^{-1}\alpha = \gamma^{-1} d_{S_{\bar{x}}}(\bar{t}+\alpha)$ for all $\alpha > 0$ and all $\gamma \geq \gamma_{\bar{x}}$. Set $\bar{t}+\alpha = t > t_2$. Then, $f_{\bar{x}}(t) \geq \gamma^{-1} d_{S_{\bar{x}}}(t)$ for all $\gamma \geq \gamma_{\bar{x}}$. As $\gamma \longrightarrow \gamma_{\bar{x}}$ one has

$$d_{S_{\bar{x}}}(t) \leq \gamma_{\bar{x}} f_{\bar{x}}(t) = \gamma_{\bar{x}} (f_{\bar{x}})_+(t) \quad \forall \quad t > t_2. \tag{3.3}$$

Now, let $\bar{t} = t_1 > 0$. Then $N^1_{S_{\bar{x}}}(\bar{t}) = \{-1\}$, which implies $(\bar{t}-\alpha) \in \mathbb{R}_+ \setminus S_{\bar{x}}$ for each $\alpha \in (0, \bar{t}]$. Since by the hypothesis $\lim_{\alpha \longrightarrow 0^+} \frac{f_{\bar{x}}(\bar{t})-f_{\bar{x}}(\bar{t}-\alpha)}{\alpha} \leq -\gamma^{-1}$ for all $\gamma \geq \gamma_{\bar{x}}$, and the fraction $\frac{f_{\bar{x}}(\bar{t})-f_{\bar{x}}(\bar{t}-\alpha)}{\alpha}$ is decreasing relative to α, $f_{\bar{x}}(\bar{t}-\alpha) \geq \gamma^{-1}\alpha = \gamma^{-1} d_{S_{\bar{x}}}(\bar{t}-\alpha)$ for all $\alpha \in (0, \bar{t}]$ and all $\gamma \geq \gamma_{\bar{x}}$. Set $\bar{t}-\alpha = t \in [0, t_1)$. As $\gamma \longrightarrow \gamma_{\bar{x}}$, one has $d_{S_{\bar{x}}}(t) \leq \gamma_{\bar{x}} f_{\bar{x}}(t) = \gamma_{\bar{x}} (f_{\bar{x}})_+(t)$, for all $t \in [0, t_1)$. Then, by (3.3), we obtain $d_{S_{\bar{x}}}(t) \leq \gamma_{\bar{x}} f_{\bar{x}}(t) = \gamma_{\bar{x}} (f_{\bar{x}})_+(t)$, for all $t \in \mathbb{R}_+ \setminus S_{\bar{x}}$. Since this inequality holds for each $t \in S_{\bar{x}}$, (3.2) holds.

For the other cases considered for $S_{\bar{x}}$, arguments similar to the one above imply (3.2). Hence, (i) holds. □

The following theorem, which can be applied to find a global error bound for each function f_x, is a special case of [22, Theorem 3.3].

Theorem 3.2. Let $f : Q \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ be a proper l.s.c. CAR function, and $\bar{x} \in Q$ with $S_{\bar{x}} \neq \emptyset$. Then, $f_{\bar{x}}$ has a global error bound if and only if there exists a constant $\beta > 0$ such that

$$|\xi| \geq \beta \quad \forall\, \xi \in \partial f_{\bar{x}}(t) \quad \forall t \in \mathbb{R}_+ \setminus S_{\bar{x}}, \tag{3.4}$$

where $\partial f_{\bar{x}}(t)$ is the classical subdifferential of the function $f_{\bar{x}}$ at the point t.

Proof . Let (3.4) hold. Consider $\bar{t} \in \mathbb{R}_+ \setminus S_{\bar{x}}$ fixed and arbitrary. By (3.4), $(-\beta, \beta) \cap \partial f_{\bar{x}}(\bar{t}) = \emptyset$. Then, for some $h_{\bar{t}} \in \{1, -1\}$ one has

$$\sup_{\xi \in \partial f_{\bar{x}}(\bar{t})} \xi h_{\bar{t}} \leq \inf_{u \in (-\beta, +\beta)} u h_{\bar{t}} = -\beta. \tag{3.5}$$

Since $f_{\bar{x}}$ is convex, for each $\bar{t} \in int(dom(f_{\bar{x}}))$,

$$f_{\bar{x}}^+(\bar{t}, h_{\bar{t}}) = \sup_{\xi \in \partial f_{\bar{x}}(\bar{t})} \xi h_{\bar{t}}. \tag{3.6}$$

Here, $f_{\bar{x}}^+(\bar{t}, h_{\bar{t}})$ is the right directional derivative of the function $f_{\bar{x}}$ at the point $\bar{t}$ in the direction $h_{\bar{t}}$. Thus, by (3.5), $f_{\bar{x}}^+(\bar{t}, h_{\bar{t}}) \leq -\beta$. For each $\bar{t} \notin dom(f_{\bar{x}})$ one has $f_{\bar{x}}(\bar{t}) = +\infty$. Then, $f_{\bar{x}}^+(\bar{t}, h_{\bar{t}}) = -\infty$ for each $h_{\bar{t}} \in \{-1, +1\}$ and we can write $\partial f_{\bar{x}}(\bar{t}) = \emptyset$. Consequently, by (3.5) and (3.6) one has $f_{\bar{x}}^+(\bar{t}, h_{\bar{t}}) = -\infty \leq -\beta$. Hence, for each point $\bar{t}$ mentioned above there exists $h_{\bar{t}} \in \{-1, +1\}$ such that $f_{\bar{x}}^+(\bar{t}, h_{\bar{t}}) \leq -\beta$. Then, by [22, Theorem 3.1], $d_{S_{\bar{x}}}(\bar{t}) \leq \dfrac{1}{\beta}(f_{\bar{x}})_+(\bar{t})$.

Now, consider the case $\bar{t} \in bd(dom(f_{\bar{x}}))$, where bd refers to boundary. Since $f_{\bar{x}}$ is proper, l.s.c. and convex, $dom(f_{\bar{x}})$ is a non-empty, closed and convex subset of $\mathbb{R}_+$, and it has the form $[c, d]$ or $[c, +\infty)$ with $d < +\infty$ and $0 \leq c < +\infty$.

Then, $\mathrm{bd}(\mathrm{dom}(f_{\bar{x}}))$ has at most two points. Consider $\bar{t} = c$ and $\bar{t} = d$. Since by the hypothesis $\bar{t} \in \mathbb{R}_+ \setminus S_{\bar{x}}$, we find that $(f_{\bar{x}})_+(c) = f_{\bar{x}}(c) > 0$ and $(f_{\bar{x}})_+(d) = f_{\bar{x}}(d) > 0$. Consider $\mu' = \max\{\frac{d_{S_{\bar{x}}}(c)}{(f_{\bar{x}})_+(c)}, \frac{d_{S_{\bar{x}}}(d)}{(f_{\bar{x}})_+(d)}\} < +\infty$ and $\mu_{\bar{x}} = \max\{\frac{1}{\beta}, \mu'\}$. On the other hand, since $d_{S_{\bar{x}}}(t) = 0$ for all $t \in S_{\bar{x}}$, we obtain $d_{S_{\bar{x}}}(t) \le \mu_{\bar{x}}(f_{\bar{x}})_+(t)$ for all $t \in \mathbb{R}_+$, that is, $f_{\bar{x}}$ has a global error bound.

Conversely, let $f_{\bar{x}}$ have a global error bound with constant $\gamma > 0$. Then, by [22, Theorem 3.1 (i)$\Rightarrow$ (v)], for each $t \in \mathbb{R}_+ \setminus S_{\bar{x}}$ there exists $h_t \in \{-1, +1\}$ such that $f^+_{\bar{x}}(t, h_t) \le \frac{-1}{\gamma}$. Then for each $h_t \in \{-1, +1\}$,

$$\sup\{-|\xi||h_t| : \xi \in \partial f_{\bar{x}}(t)\} \le \sup\{\xi h_t : \xi \in \partial f_{\bar{x}}(t)\} \le f^+_{\bar{x}}(t, h_t) \le \frac{-1}{\gamma}.$$

Thus,

$$\inf\{|\xi| : \xi \in \partial f_{\bar{x}}(t)\} = -\sup\{-|\xi| : \xi \in \partial f_{\bar{x}}(t)\} = -\sup\{-|\xi||h_t| : \xi \in \partial f_{\bar{x}}(t)\} \ge \frac{1}{\gamma}.$$

Then, by choosing $\beta = \frac{1}{\gamma}$, (3.4) holds. $\square$ In the following example, we use the above theorem to find μ_x for each $x \in Q$. Also, we show that if f_x has a global error bound for each $x \in Q$, then the condition that f has a global error bound is not necessary.

Example 3.2. Consider $Q = \mathbb{R}^2$ and $f(x) = x_1 + \sqrt{x_1^2 + x_2^2}$. Then, $S = \{x \in \mathbb{R}^2 : x_1 \le 0, x_2 = 0\}$. Consider the following cases for any $x \in Q$.

(i) $\{x = 0\}$. (ii) $\{x \in \mathbb{R}^2 : x_1 > 0\}$. (iii) $\{x \in \mathbb{R}^2 : x_1 < 0, x_2 = 0\}$. (iv) $\{x \in \mathbb{R}^2 : x_1 < 0, x_2 \neq 0\}$. (v) $\{x \in \mathbb{R}^2 : x_1 = 0, x_2 \neq 0\}$.

In (i), (ii), (iii), (iv) and (v), $S_x = \mathbb{R}_+$, $S_x = \{0\}$, $S_x = \mathbb{R}_+$, $S_x = \{0\}$, and $S_x = \{0\}$ for each x, respectively. Then, $S_x \neq \emptyset$ for each $x \in Q$. In (i) and (iii), for each x, f_x has a global error bound for all constants $\mu_x > 0$. For the other cases, we use the subdifferential. For each x in the other cases, $\partial f_x(t) = \{x_1 + \|x\|\}$. By Theorem 3.2, for each such x, f_x has a global error bound with constant $\mu_x = \frac{1}{x_1 + \|x\|}$. Thus, for each x in each case, f_x has a global error bound.

On the other hand, by taking $x_k = (-k, 2)_{k \ge 1}$, $d_S(x_k) \longrightarrow 2$ and $f(x_k) = -k + \sqrt{k^2 + 4} \longrightarrow 0$ as $k \longrightarrow +\infty$. Thus, f does not have a global error bound, while for each $x \in Q$, f_x has a global error bound.

Assume that for each $x \in Q$, f_x has a global error bound. Here, we consider the collection $\{B_i : i \in I\}$, where I is an index set, as a partition of $Q \setminus S$, that is, $Q \setminus S = \bigcup_{i \in I} B_i$, $B_i \cap B_j = \emptyset$, for all $i \neq j$, where each B_i is a set of elements of $Q \setminus S$ such that for each $x \in B_i$, μ_x is attained in the same form. For instance, in Example 3.2, $Q \setminus S = \bigcup_{1 \le i \le 3} B_i$, where B_1, B_2, B_3 contain all x in the cases (ii), (iv) and (v), respectively.

In the following theorem, a condition is given that ensures the existence of a global error bound for an l.s.c. CAR function f.

Theorem 3.3. Let $f : Q \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ be a proper l.s.c. CAR function. Also, assume that for each $x \in Q$, f_x has a global error bound with constant $\mu_x > 0$. Finally, let $\sup_{i \in I} \sup_{x \in B_i} (\|x\| \mu_x) < +\infty$. Then, f has a global error bound. Here, the collection $\{B_i : i \in I\}$ is a partition of $Q \setminus S$, mentioned above.

Proof . One has $S = \bigcup_{x \in Q} A_x$. Indeed, let $x' \in S$. Then $f_{x'}(1) = f(x') \le 0$ which implies $1 \in S_{x'}$ and accordingly, $x' = x'.1 \in A_{x'} \subseteq \bigcup_{x \in Q} A_x$.

Conversely, let $x' \in \bigcup_{x \in Q} A_x$. Then, there exists $\bar{x} \in Q$ such that $x' \in A_{\bar{x}} = \bar{x}.S_{\bar{x}}$. Therefore, there exists $\bar{t} \in S_{\bar{x}}$ such that $x' = \bar{t}\bar{x}$. Since $\bar{t} \in S_{\bar{x}}$, it follows that $f_{\bar{x}}(\bar{t}) \le 0$. Thus $f(x') = f(\bar{t}\bar{x}) = f_{\bar{x}}(\bar{t}) \le 0$ and so, $x' \in S$. Hence, $S = \bigcup_{x \in Q} A_x$.

Now, suppose that $S \neq \mathbb{R}^n$ and $x' \in Q \setminus S$. Otherwise, the inequality of error bound holds. Consider the collection $\{B_i : i \in I\}$ as a partition of $Q \setminus S$. Then, there exists an index $i' \in I$ such that $x' \in B_{i'}$. By the equality $S = \bigcup_{x \in Q} A_x$, Lemma 3.3 and the hypothesis, one has

$$\begin{aligned} d_S(x') \leq & d_{A_{x'}}(x') = \|x'\| d_{S_{x'}}(1) \\ \leq & \|x'\| \mu_{x'} (f_{x'})_+(1) = \|x'\| \mu_{x'} f_+(x') \\ \leq & \sup_{x \in B_{i'}} (\|x\| \mu_x) f_+(x') \\ \leq & \sup_{i \in I} \sup_{x \in B_i} (\|x\| \mu_x) f_+(x'). \end{aligned}$$

Since by the hypothesis $\sup_{i \in I} \sup_{x \in B_i} (\|x\| \mu_x) < +\infty$, and since $d_S(x') = 0$ for all $x' \in S$, we conclude that $d_S(x') \leq \mu f_+(x')$, for all $x' \in Q$ with $\mu = \sup_{i \in I} \sup_{x \in B_i} (\|x\| \mu_x) < +\infty$. $\square$

Note that using the non-emptiness of S and the hypothesis of the theorem, $\mu > 0$. Also, when the set S is not clearly given (in which case $Q \setminus S$ cannot be clearly determined), we may use a collection $\{B_i : i \in I\}$ for the whole Q.

The above theorem is applicable and gives us the constant of the global error bound. In the following examples, we obtain the constant of the global error bound using the above theorems.

Example 3.3. Consider $Q = \mathbb{R}^2$ and $f(x) = \begin{cases} \dfrac{x_1^2 + 3\|x\|^2}{|x_2|}, & x_2 \neq 0, \\ 0 & x_2 = 0. \end{cases}$ Then, $S = \{x \in \mathbb{R}^2 : x_1 \in \mathbb{R}, x_2 = 0\}$. Here, the partition of $Q \setminus S$ has one element $\hat{B} = \{x \in \mathbb{R}^2 : x_1 \in \mathbb{R}, x_2 \neq 0\} = Q \setminus S$. For each $x \in \hat{B}$, $\partial f_x(t) = \{\dfrac{x_1^2 + 3\|x\|^2}{|x_2|}\}$. Then for each $x \in \hat{B}$, f_x has a global error bound with constant $\mu_x = \dfrac{|x_2|}{x_1^2 + 3\|x\|^2}$. Now, consider $\sup_{x \in \hat{B}} (\|x\| \mu_x) = \sup_{x_2 \neq 0} (\dfrac{\|x\| |x_2|}{x_1^2 + 3\|x\|^2}) = \dfrac{1}{3}$.

Then, f has a global error bound with constant $\mu = \dfrac{1}{3}$.

Example 3.4. Consider $Q = \mathbb{R}^3$ and $f(x) = \begin{cases} \dfrac{x_1^2 + x_2^2 + x_3^2}{x_3 - x_2}, & x_3 > x_2, \\ 0, & x_3 = x_2, \\ +\infty. & x_3 < x_2 \end{cases}$

Then, one has $S = \{x \in \mathbb{R}^3 : x_2 = x_3\}$. Consider $B_1 = \{x \in \mathbb{R}^3 : x_3 > x_2\}$ and $B_2 = \{x \in \mathbb{R}^3 : x_3 < x_2\}$. For each $x \in B_1$ we can write $S_x = \{0\}$. Also, for each $x \in B_2$ one has $S_x = \{0\}$. Since $f(x) = +\infty$ for each $x \in B_2$, the inequality of error bound holds for such points. Thus, it suffices to check theorems 3.2 and 3.3 for each $x \in B_1$. For such x we can write $\partial f_x(t) = \{\dfrac{x_1^2 + x_2^2 + x_3^2}{x_3 - x_2}\}$. Then, by Theorem 3.2, f_x has a global error bound with constant $\mu_x = \dfrac{x_3 - x_2}{x_1^2 + x_2^2 + x_3^2}$, for each $x \in B_1$. Now, consider $\sup_{x \in B_1} (\|x\| \mu_x) = \sup_{x \in B_1} \dfrac{\|x\|(x_3 - x_2)}{x_1^2 + x_2^2 + x_3^2} = \sup_{x \in B_1} \dfrac{x_3 - x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}} = \sqrt{2}$. Then, f has a global error bound with constant $\sqrt{2}$.

Note: It is clear that $x \in S$ if and only if $1 \in S_x$. In particular, when $S_x = \mathbb{R}_+$, we conclude that $x \in S$.

Example 3.5. Consider $Q = \{x \in \mathbb{R}^2 : 0 \leq x_2 \leq x_1\} \cup \{x \in \mathbb{R}^2 : x_1 \leq x_2 \leq 0\}$ and $f(x) = x_1^2 - x_2^2 + x_1$. Then, $S = \{x \in Q : -1 \leq x_1 \leq 0, x_1 \leq x_2 \leq 0\} \cup \{x \in Q : x_1 \leq -1, x_1 \leq x_2 \leq -\sqrt{x_1^2 + x_1}\}$. Consider $B_1 = \{x \in Q : 0 \leq x_2 \leq x_1\}$ and $B_2 = \{x \in Q : x_1 \leq -1, x_1 \leq -\sqrt{x_1^2 + x_1} \leq x_2 \leq 0\}$. Here, $Q \setminus S \subseteq B_1 \cup B_2$. Then, it is sufficient to apply Theorem 3.2 and Theorem 3.3 for B_1 and B_2. For each $x \in B_1$ one has $S_x = \{0\}$, and for each $x \in B_2$ one has $S_x = [0, \bar{t}]$, where $\bar{t} = \dfrac{-x_1}{x_1^2 - x_2^2} > 0$. For each $x \in Q$, $\partial f_x(t) = \{2t(x_1^2 - x_2^2) + x_1\}$. By Theorem 3.2, for each $x \in B_1$, $|\xi| \geq x_1$ for all $\xi \in \partial f_x(t)$ and each $t \in \mathbb{R}_+ \setminus S_x$.

For each $x \in B_2$, $|\xi| \geq -x_1 > 0$, for all $\xi \in \partial f_x(t)$ and each $t \in \mathbb{R}_+ \setminus [0, \bar{t}]$, with $\bar{t} = \dfrac{-x_1}{x_1^2 - x_2^2} > 0$. Then for each $x \in B_1$, f_x has a global error bound with constant $\mu_x = \dfrac{1}{x_1} > 0$, and for each $x \in B_2$, f_x has a global error bound with constant $\mu_x = \dfrac{-1}{x_1} > 0$. Now, consider $\sup\limits_{x\in B_1}(\|x\|\mu_x) = \sup\limits_{x\in B_1}(\dfrac{\|x\|}{x_1}) = \sqrt{2}$ and $\sup\limits_{x\in B_2}(\|x\|\mu_x) = \sup\limits_{\substack{x_1\le -1 \\ x_1\le -\sqrt{x_1^2+x_1}\le x_2\le 0}}(\dfrac{-\|x\|}{x_1}) = \sqrt{2}$. Then,

$$\sup_{i\in I}\sup_{x\in B_i}(\|x\|\mu_x) = \max\{\sup_{x\in B_1}(\|x\|\mu_x), \sup_{x\in B_2}(\|x\|\mu_x)\} = \max\{\sqrt{2},\sqrt{2}\} = \sqrt{2}.$$

Thus, f has a global error bound with constant $\sqrt{2}$.

Example 3.6. Consider $Q = \mathbb{R}_- \times \mathbb{R} \times \mathbb{R}_+$ and $f(x) = -x_1^3 - x_1x_2^2 + \|x\| + x_3$. By checking that $S_x = \{0\}$ for each $x \in Q$, we obtain $A_x = \{0\}$ for each $x \in Q$. Then, $S = \bigcup\limits_{x\in Q} A_x = \{0\}$. We obtain $Q \setminus S = B = \{x \in Q : x \neq 0\}$. Consider $x \in Q \setminus S$. Then, $\partial f_x(t) = \{3t^2(-x_1^3 - x_1x_2^2) + \|x\| + x_3\}$ for each $x \in Q \setminus S$. Therefore, $|\xi| \geq \|x\| + x_3$ for each $\xi \in \partial f_x(t)$ and each $t \in \mathbb{R}_+ \setminus S_x$. This implies that for each $x \in Q \setminus S = B$, f_x has a global error bound with constant $\mu_x = \dfrac{1}{\|x\| + x_3}$ and $\sup\limits_{x\in B}(\|x\|\mu_x) = \sup\limits_{x\in B}(\dfrac{\|x\|}{\|x\| + x_3}) = 1$. Consequently, f has a global error bound with constant $\mu = 1$.

It is important to determine the conditions under which an l.s.c. CAR function does not have a global error bound. The following theorem presents a necessary condition for an l.s.c. CAR function f to have a global error bound (or equivalently, a sufficient condition for f to have no global error bounds).

Similar to the definitions of $N^1_{S_{\bar{x}}}(t)$ and $\partial_N S_{\bar{x}}$, we define

$$N^1_S(x) = \{v \in \mathbb{R}^n : \|v\| = 1 \text{ and } d_S(x + \alpha v) = \alpha \text{ for some } \alpha > 0\}$$

and $\partial_N S = \{x \in \partial S : N^1_S(x) \neq \emptyset\}$. Here, ∂S refers to the boundary of the set S.

Theorem 3.4. Let $f : Q \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ be a proper l.s.c. CAR function that has a global error bound with constant $\mu > 0$. Let $0 \neq \bar{x} \in Q, \partial_N S_{\bar{x}} \neq \emptyset$ and for each $t \in \partial_N S_{\bar{x}}$, $\dfrac{\bar{x}}{\|\bar{x}\|}$ or $\dfrac{-\bar{x}}{\|\bar{x}\|} \in N^1_S(t\bar{x})$. Then, $f_{\bar{x}}$ has a global error bound.

Proof . Let $t \in \partial_N S_{\bar{x}}$. Then $t \in \partial S_{\bar{x}}$ and so, $t\bar{x} \in \partial S$. Since $\dfrac{\bar{x}}{\|\bar{x}\|}$ or $\dfrac{-\bar{x}}{\|\bar{x}\|} \in N^1_S(t\bar{x})$, we find that $t\bar{x} \in \partial_N S$. Since f has a global error bound with constant $\mu > 0$, by [22, Corollary 2.8] one has

$$\inf\{\underline{d}^+ f(t\bar{x}, u) : u \in N^1_S(t\bar{x})\} \geq \frac{1}{\mu}. \tag{3.7}$$

Here $\underline{d}^+ f(t\bar{x}, u)$ is lower Dini directional derivative of the function f at the point $t\bar{x}$ in the direction u. Now, if $\dfrac{\bar{x}}{\|\bar{x}\|} \in N^1_S(t\bar{x})$, then by (3.7),

$$\underline{d}^+ f(t\bar{x}, \frac{\bar{x}}{\|\bar{x}\|}) \geq \frac{1}{\mu}.$$

Thus,

$$\liminf_{\beta\longrightarrow 0^+} \frac{f(t\bar{x} + \beta\frac{\bar{x}}{\|\bar{x}\|}) - f(t\bar{x})}{\beta} \geq \frac{1}{\mu}$$

which implies

$$\liminf_{\beta\longrightarrow 0^+} \frac{f_{\bar{x}}(t + \frac{\beta}{\|\bar{x}\|}(1)) - f_{\bar{x}}(t)}{\beta} \geq \frac{1}{\mu}.$$

Set $\alpha = \dfrac{\beta}{\|\bar{x}\|}$. Then,

$$\liminf_{\alpha\longrightarrow 0^+} \frac{f_{\bar{x}}(t + \alpha(1)) - f_{\bar{x}}(t)}{\alpha} \geq \frac{\|\bar{x}\|}{\mu}.$$

This implies

$$\underline{d}^+ f_{\bar{x}}(t, 1) \geq \frac{\|\bar{x}\|}{\mu}. \tag{3.8}$$

If $\frac{-\bar{x}}{\|\bar{x}\|} \in N_S^1(t\bar{x})$, then similar to the above argument we obtain

$$\liminf_{\alpha \longrightarrow 0^+} \frac{f_{\bar{x}}(t+\alpha(-1)) - f_{\bar{x}}(t)}{\alpha} \geq \frac{\|\bar{x}\|}{\mu}.$$

Then,

$$\underline{d}^+ f_{\bar{x}}(t,-1) \geq \frac{\|\bar{x}\|}{\mu}. \tag{3.9}$$

Since $f_{\bar{x}}^+(t,1) \geq \underline{d}^+ f_{\bar{x}}(t,1)$, and $f_{\bar{x}}^+(t,-1) \geq \underline{d}^+ f_{\bar{x}}(t,-1)$ (where $f_{\bar{x}}^+(t,.)$ is the right directional derivative of $f_{\bar{x}}$), and since for each $t \in \partial_N S_{\bar{x}}$, $\emptyset \neq N_{S_{\bar{x}}}^1(t) \subseteq \{1,-1\}$,

$$\inf\{f_{\bar{x}}^+(t,u) : u \in N_{S_{\bar{x}}}^1(t)\} = \min\{f_{\bar{x}}^+(t,1), f_{\bar{x}}^+(t,-1)\} \geq \frac{\|\bar{x}\|}{\mu}.$$

Since $t \in \partial_N S_{\bar{x}}$ is arbitrary and the function $f_{\bar{x}}$ is proper, l.s.c. and convex, by [22, Theorem 3.1 (iii)⇒(i)], $f_{\bar{x}}$ has a global error bound with constant $\mu_{\bar{x}} = \frac{\mu}{\|\bar{x}\|}$. $\square$

The following corollary has an applicable role for an l.s.c. CAR function.

Corollary 3.1. Let $f : Q \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ be a proper l.s.c. CAR function. Also, let $0 \neq \bar{x} \in Q, \partial_N S_{\bar{x}} \neq \emptyset$ and for each $t \in \partial_N S_{\bar{x}}$, $\frac{\bar{x}}{\|\bar{x}\|}$ or $\frac{-\bar{x}}{\|\bar{x}\|} \in N_S^1(t\bar{x})$. If $f_{\bar{x}}$ does not have a global error bound, then f does not have a global error bound.

The following two examples show that under the conditions presented in Corollary 3.1, when f_x does not have a global error bound for some $x \in Q$, f also does not have a global error bound.

Example 3.7. Consider $Q = \mathbb{R}^2$ and $f(x) = \begin{cases} e^{x_1x_2} + x_2^2 - 1, & x_1 \geq 0, x_2 \leq 0, \\ +\infty. & otherwise. \end{cases}$ Then $S = \{x \in \mathbb{R}^2 : x_1 \geq \frac{\ln(1-x_2^2)}{x_2}, -1 < x_2 < 0\} \cup \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\}$. Consider $B = \{x \in Q : x_1 = 0, x_2 < 0\}$. For each $x \in B$ one has $S_x = \{0\}$ and $\partial f_x(t) = \{2tx_2^2\}$. If $\bar{x} \in B$, then by Theorem 3.2, no $\mu_{\bar{x}} > 0$ exists such that $|\xi| \geq \mu_{\bar{x}}$ for each $\xi \in \partial f_{\bar{x}}(t)$ and each $t \in \mathbb{R}_+ \setminus S_{\bar{x}} = \mathbb{R}_{++}$. Then, for such $\bar{x}$ (and for each $x \in B$), $f_{\bar{x}}$ (also, each f_x) does not have a global error bound. Now, consider $\bar{x} = (0,-1) \in B$. Then, $\frac{\bar{x}}{\|\bar{x}\|} = (0,-1)$ and $S_{\bar{x}} = \{0\}$. Since $N_{S_{\bar{x}}}^1(0) = \{1\}$, $\bar{t} = 0 \in \partial_N S_{\bar{x}} = \{0\}$. By checking, there exist $\alpha > 0$ with $d_S(\bar{t}\bar{x} + \alpha\frac{\bar{x}}{\|\bar{x}\|}) = d_S(\alpha\frac{\bar{x}}{\|\bar{x}\|}) = d_S((0,-\alpha)) = \alpha$. Then $\frac{\bar{x}}{\|\bar{x}\|} \in N_S^1(\bar{t}\bar{x})$ for $\bar{t} = 0 \in \partial_N S_{\bar{x}} = \{0\}$. Since $f_{\bar{x}}$ does not have a global error bound, by Corollary 3.1, f does not have a global error bound.

Example 3.8. Consider $Q = \mathbb{R}_- \times \mathbb{R}_-$ and $f(x) = x_1x_2 + x_1^2$. Then, $S = \{x \in Q : x_1 = 0\}$. Consider $B = \{x \in Q : x_1 \neq 0\} = Q \backslash S$. Then for each $x \in B$, $S_x = \{0\}$ and $\partial f_x(t) = \{2t(x_1x_2 + x_1^2)\}$. By Theorem 3.2, for each $x \in B$, f_x does not have a global error bound. Consider $\bar{x} = (-2,0) \in B$. Then, $\frac{\bar{x}}{\|\bar{x}\|} = (-1,0)$ and $S_{\bar{x}} = \{0\}$. Since $N_{S_{\bar{x}}}^1(0) = \{1\}$, $\bar{t} = 0 \in \partial_N S_{\bar{x}} = \{0\}$. For $\alpha = 1$ one has $d_S(\bar{t}\bar{x} + \alpha\frac{\bar{x}}{\|\bar{x}\|}) = d_S((-\alpha,0)) = \alpha$. Then, $\frac{\bar{x}}{\|\bar{x}\|} = (-1,0) \in N_S^1(\bar{t}\bar{x})$ and $f_{\bar{x}}$ does not have a global error bound. Thus, f does not have a global error bound.

Note that if $S_{\bar{x}} = \emptyset$ for some $\bar{x} \in Q$, then $f_{\bar{x}}$ does not have a global error bound ($d_\emptyset(t) = +\infty$ by convention). But, f may or may not have a global error bound. To see this, consider the following two examples.

Example 3.9. Consider $Q = \mathbb{R}^2$ and $f(x) = \begin{cases} -|x_1| + 1, & x_1 \leq 1, \\ 0, & x_1 > 1. \end{cases}$ Then, $S = \{x \in \mathbb{R}^2 : x_1 \leq -1, x_2 \in \mathbb{R}\} \cup \{x \in \mathbb{R}^2 : x_1 \geq +1, x_2 \in \mathbb{R}\}$. Consider $B = \{x \in \mathbb{R}^2 : x_1 = 0, x_2 \in \mathbb{R}\}$. For each $x \in B$, $S_x = \emptyset$. Then, f_x does not have a global error bound for such x. But, it is easy to check that f has a global error bound with constant $\mu = 1$.

Example 3.9 shows that if f has a global error bound,it may happen that for some $x \in Q \setminus S$, f_x does not have a global error bound.

Example 3.10. Consider $Q = \mathbb{R}^2_+ \cup \mathbb{R}^2_-$ and $f(x) = x_1x_2 - x_1 + 2$. Then, $S = \{x \in Q : 0 \leq x_2 < 1, x_1 \geq \frac{-2}{x_2 - 1}\}$. Consider $B = \{x \in Q : x_1 \leq 0, x_2 = 0\}$. For each $x \in B$, $S_x = \emptyset$. Then for such x, the function f_x does not have a global error bound. On the other hand, by taking $x_k = (0, k)_{k \geq 1}$, as $k \longrightarrow +\infty$ one has $d_S(x_k) \longrightarrow +\infty$, while $f(x_k) = 2$ for all $k \geq 1$. Thus, f does not have a global error bound.

The following theorem shows that under some conditions different from those presented in Theorem 3.4, when the function f has a global error bound, f_x also has a global error bound.

Theorem 3.5. Let $f : Q \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ be a proper l.s.c. CAR function which has a global error bound with constant $\mu > 0$. Assume that the set S is bounded, $0 \neq \bar{x} \in Q$, and there exists $M > 0$ such that $f_{\bar{x}}(t) \geq M$ for all $t \in \mathbb{R}_+ \setminus S_{\bar{x}}$. Then, $f_{\bar{x}}$ has a global error bound.

Proof . Let $0 \neq \bar{x} \in Q$ and $t \in \mathbb{R}_+ \setminus S_{\bar{x}}$ be arbitrary, and assume that the above condition holds. Then $t\bar{x} \in Q \setminus S$ and since S is proximal (is closed in $\mathbb{R}^n$), there exists $x' \in \partial S$ such that $d_S(t\bar{x}) = \|t\bar{x} - x'\|$. Since $S_{\bar{x}} \neq \emptyset$ and it is closed, there exists $\bar{t} \in \partial S_{\bar{x}}$ such that $\bar{t}\bar{x} \in \partial A_{\bar{x}}$ and $d_{A_{\bar{x}}}(t\bar{x}) = \|t\bar{x} - \bar{t}\bar{x}\|$. Then,
$d_{A_{\bar{x}}}(t\bar{x}) = \|t\bar{x} - \bar{t}\bar{x}\| \leq \|t\bar{x} - x'\| + \|x' - \bar{t}\bar{x}\| = d_S(t\bar{x}) + \|x' - \bar{t}\bar{x}\| \leq \mu f_+(t\bar{x}) + \|x' - \bar{t}\bar{x}\| \leq \mu f_+(t\bar{x}) + diam(S) = (\mu + \frac{diam(S)}{f_+(t\bar{x})}) f_+(t\bar{x}) = (\mu + \frac{diam(S)}{(f_{\bar{x}})_+(t)})(f_{\bar{x}})_+(t) \leq (\mu + \frac{diam(S)}{M})(f_{\bar{x}})_+(t)$. Thus, $d_{A_{\bar{x}}}(t\bar{x}) \leq (\mu + \frac{diam(S)}{M})(f_{\bar{x}})_+(t)$, where $diam(S)$ is the diameter of the set S. Since $d_{A_{\bar{x}}}(t\bar{x}) = \|\bar{x}\| d_{S_{\bar{x}}}(t)$, it follows that $d_{S_{\bar{x}}}(t) \leq (\frac{\mu + \frac{diam(S)}{M}}{\|\bar{x}\|})(f_{\bar{x}})_+(t)$ for all $t \in \mathbb{R}+ \setminus S_{\bar{x}}$. Setting $\mu_{\bar{x}} = \frac{\mu + \frac{diam(S)}{M}}{\|\bar{x}\|} > 0$ we obtain $d_{S_{\bar{x}}}(t) \leq \mu_{\bar{x}} (f_{\bar{x}})_+(t)$ for all $t \in \mathbb{R}_+ \setminus S_{\bar{x}}$. Since this inequality holds for all $t \in S_{\bar{x}}$, we conclude that $f_{\bar{x}}$ has a global error bound with constant $\mu_{\bar{x}}$. $\square$

Using Theorem 3.5 we obtain the following corollary. It shows that if $\bar{x} \neq 0$ and $f_{\bar{x}}$ does not have a global error bound, then the l.s.c. CAR function f does not have a global error bound.

Corollary 3.2. Let $0 \neq \bar{x} \in Q$ with $S_{\bar{x}} \neq \emptyset$ and $f_{\bar{x}}(t) \geq M$, for all $t \in \mathbb{R}_+ \setminus S_{\bar{x}}$ and some $M > 0$. Let S be bounded, and assume that $f_{\bar{x}}$ does not have a global error bound. Then, f does not have a global error bound.

4 Error bounds from the perspective of abstract convexity

In this section, we focus on the error bounds of an l.s.c. (and in particular, an l.s.c. CAR) function from the perspective of abstract convexity. The following notions and theorems can be found in the literature. We refer the reader to [10] and [25]. Consider $H_k = \{h : h(x) = \min_{i=1...,j}(\langle l(i), x\rangle - c_i),\ j \leq k,\quad c_i \in \mathbb{R}\ \forall\ i = 1, 2, ..., j\}$, where $\langle ., . \rangle$ refers to the inner product. Consider the vector $\ell = (l(1), ..., l(j)), j \leq k$, where each $l(i)$ belongs to $\mathbb{R}^n$ and $k = 1, 2, ...$ Define L_k as the collection of all vectors $\ell = (l(1), ..., l(j)), j \leq k$, such that each ℓ makes (at least) one elementary function $h \in H_k$.

A function f is called abstract convex with respect to H_k (H_k-convex), if there exists a subset $U \subseteq H_k$ such that $f(x) = \sup_{h \in U} h(x)$ for all $x \in \mathbb{R}^n$. A particular case of H_k is when all the values c_i are the same, that is, $c_i = c$ for some $c \in \mathbb{R}$ and each $i = 1, 2, ..., j$. Consider $\ell : x \longrightarrow \langle \ell, x \rangle_* = \min_{i=1...,j} \langle l(i), x\rangle,\ j \leq k,\ \ell = (l(1), ..., l(j))$ and the vectors $l(i) \in \mathbb{R}^n, i = 1, 2, ..., j$. Each function $\ell : x \longrightarrow \langle \ell, x \rangle_* = \min_{i=1...,j} \langle l(i), x\rangle$ is called a min-type function. We focus on the cases $k = n + 1$ and $k = n + 2$ (on $\mathbb{R}^n$) in this section.

We consider $H_{n+1} = \{h : h(x) = \min_{i=1...,j}(\langle l(i), x\rangle - c_i),\ j \leq n+1,\ c_i \in \mathbb{R}\ \forall\ i = 1, 2, ..., j\}$ } and $H_{n+2} = \{h : h(x) = \min_{i=1...,j}(\langle l(i), x\rangle - c_i),\ j \leq n + 2,\quad c_i \in \mathbb{R}\quad \forall\ i = 1, 2, ..., j\}$. The following two theorems hold for these cases.

Theorem 4.1. ([25, Theorem 5.16]). If $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ is an l.s.c. CAR function with $f(0) < +\infty$, then f is H_{n+1}-convex.

Theorem 4.2. ([25, Theorem 5.21]). A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ is H_{n+2}-convex if and only if f is proper and l.s.c. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ be a proper l.s.c. (in particular, a proper l.s.c. CAR) function. We consider U_f as the collection of all $h \in H_k$, $k = n + 1$ or $k = n + 2$, such that $f(x) = \sup_{h \in U_f} h(x)$ for all $x \in \mathbb{R}^n$, and L_f is the collection of all $\ell \in L_k$ such that each ℓ makes (at least) one elementary function $h \in U_f$.

Theorem 4.3. ([10]). Let A, B be matrices in $\mathbb{R}^{m\times n}$, and a, b be vectors in $\mathbb{R}^m$. Assume that $S = \{x \in \mathbb{R}^n : Ax \leq a, Bx = b\}$ is non-empty. Then, there exists $c > 0$ such that $d_S(x) \leq c(\|[Ax - a]_+\| + \|Bx - b\|)$ for all $x \in \mathbb{R}^n$. If $S = \{x \in \mathbb{R}^n : Ax \leq a\}$, the following characterization of c in Theorem 4.3 is often used in the optimization literature. We refer the reader to [9], [13], [20] and [23].

$$c = \max_{\substack{J\subseteq\{1,2,\ldots m\} \\ A_J:\ full\ row\ rank}} \frac{1}{\min\limits_{\substack{v\in\mathbb{R}^J_+ \\ \|v\|^*=1}} \|A_J^T v\|^*}. \tag{4.1}$$

Here, A^T refers to the transpose of A and $\|.\|^*$ refers to the dual norm.

Note that the Euclidean norm and its dual are the same. Thus, we use $\|.\|$ instead of $\|.\|^*$. Assumption Λ: There exists $M > 0$ such that $\|l(i)\| \geq M$ for all $l(i) \in \ell$ and all $\ell \in L_f$.

Under some conditions, the following theorem presents an error bound for a proper l.s.c. (in particular, a proper l.s.c. CAR) function using abstract convexity.

Theorem 4.4. Let $f : Q \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ be a proper l.s.c. (in particular, a proper l.s.c. CAR) function. Suppose that Assumption Λ holds, and $l(i) \in \mathbb{R}^n_+$ for all $l(i)$ in Assumption Λ. Then, f has a global error bound.

Proof . Since f is a proper l.s.c. function, by Theorem 4.2, f is H_{n+2}-convex, that is, $f(x) = \sup\limits_{h\in U_f} h(x)$ for all $x \in Q$, where U_f is as above. Consider $h \in U_f$ fixed (and arbitrary). Then $h(x) = \min\limits_{i=1,\ldots,j}(\langle l(i), x\rangle - c_i)$, where $l(i) \in \mathbb{R}^n_+$ for $i = 1, \ldots, j$, $j \leq n + 2$. Set $h_i(x) = \langle l(i), x\rangle - c_i$ for all $i = 1, \ldots, j$, $S_{h_i} = \{x \in Q : h_i(x) \leq 0\}$ and $S_h = \{x \in Q : h(x) \leq 0\}$. Then $S_h = \bigcup\limits_{i=1,\ldots,j} S_{h_i}$, and S_{h_i} is downward for each $i = 1, ..., j$. Indeed, for a fixed $i \in \{1, ..., j\}$, let $x \in S_{h_i}$ and $\bar{x} \in Q$ with $\bar{x} \leq x$. Since $l(i) \in \mathbb{R}^n_+$, $\sum\limits_{r=1,\ldots,n} l_r(i)\bar{x}_r \leq \sum\limits_{r=1,\ldots,n} l_r(i)x_r$. Then, $h_i(\bar{x}) = \langle l(i), \bar{x}\rangle - c_i \leq \langle l(i), x\rangle - c_i = h_i(x) \leq 0$. Thus, $\bar{x} \in S_{h_i}$. Then for each $i \in \{1, \ldots, j\}$, S_{h_i} is downward and since $S_h = \bigcup\limits_{i=1,\ldots,j} S_{h_i}$, it is clear that S_h is downward for each $h \in U_f$. Now, suppose that $x' \in Q \setminus S$ is fixed (and arbitrary), where $S = \{x \in Q : f(x) \leq 0\}$ is non-empty by the hypothesis. Since $h(x') = \min\limits_{i=1,\ldots,j} h_i(x')$, there exists an index $i' \in \{1, ..., j\}$ such that $h(x') = h_{i'}(x')$. Then, by the characterization of Theorem 4.3 with $A = l(i')$ in (4.1), since $S_h = \bigcup\limits_{i=1,\ldots,j} S_{h_i}$, and using Assumption Λ one has,

$$d_{S_h}(x') \leq d_{S_{h_{i'}}}(x') \leq \frac{1}{\|l(i')\|}(h_{i'}(x'))_+ = \frac{1}{\|l(i')\|}h_+(x') \leq \frac{1}{M}h_+(x') \leq \frac{1}{M}f_+(x'). \tag{4.2}$$

Since $h \in U_f$ is arbitrary, $\sup\limits_{h\in U_f} d_{S_h}(x') \leq \dfrac{1}{M}f_+(x')$ for all $x' \in Q \setminus S$. On the other hand, since $S = \bigcap\limits_{h\in U_f} S_h$ and each S_h is downward, the collection $\{S_h : h \in U_f\}$ is linearly regular (see [25]). Thus, there exists $\sigma > 0$ such that $d_S(x') \leq \sigma \sup\limits_{h\in U_f} d_{S_h}(x') \leq \dfrac{\sigma}{M}f_+(x')$. Since $x' \in Q \setminus S$ is arbitrary and $d_S(x') = 0$ for each $x' \in S$, $d_S(x') \leq \dfrac{\sigma}{M}f_+(x')$ for each $x' \in Q$. $\square$

Example 4.1. Consider $Q = \mathbb{R}^3$ and $f(x) = \begin{cases} 0.5x_1 + 0.125x_2 + 0.1x_3 & x_3 \leq 0, \\ 0.5x_1 + 0.125x_2 & x_3 > 0. \end{cases}$ Then, f is an l.s.c. CAR function. Consider $\ell = ((0.5, 0.125, 0.1), (0.5, 0.125, 0))$ and $h(x) = \min\{h_1(x), h_2(x)\}$, where $h_1(x) = 0.5x_1 + 0.125x_2 + 0.1x_3, h_2(x) = 0.5x_1 + 0.125x_2$. Then, $\min\{\|l(1)\|, \|l(2)\|\} = \dfrac{\sqrt{17}}{8} = M$ and by checking that $f(x) = h(x)$ for all $x \in Q$,we obtain $S = S_h$. Since $0_3 \neq l(i) \in \mathbb{R}^3_+$ for $i = 1, 2$, by Theorem 4.4, $d_S(x) = d_{S_h}(x) \leq \dfrac{1}{M}f_+(x)$ and f has a global error bound with constant $\mu = \dfrac{1}{M} = \dfrac{8}{\sqrt{17}}$.

Note that if $\{S_h : h \in U_f\}$ is a linearly regular collection, then using Assumption Λ, and in the absence of the condition "$l(i) \in \mathbb{R}^n_+$", Theorem 4.4 holds.

Corollary 4.1. Let $f : Q \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ be a proper l.s.c. (in particular, a proper l.s.c. CAR) function. Suppose that Assumption Λ holds and the collection $\{S_h : h \in U_f\}$ is linearly regular. Then, f has a global error bound.

In the following two examples, Assumption Λ holds, but the assumption "$l(i) \in \mathbb{R}^n_+$" is not satisfied; instead, "linear regularity" holds.

Example 4.2. Consider $Q = \mathbb{R}^3$ and $f(x) = -0.2|x_1| - 0.4|x_2| + 0.5x_3 - 5$,

$$\ell = ((-0.2, 0.4, 0.5), (-0.2, -0.4, 0.5), (0.2, -0.4, 0.5), (0.2, 0.4, 0.5))$$

and $h(x) = \min_{i\in\{1,\ldots,4\}} h_i(x)$, where $h_1(x) = -0.2x_1 + 0.4x_2 + 0.5x_3 - 5, h_2(x) = -0.2x_1 - 0.4x_2 + 0.5x_3 - 5, h_3(x) = 0.2x_1 - 0.4x_2 + 0.5x_3 - 5, h_4(x) = 0.2x_1 + 0.4x_2 + 0.5x_3 - 5$. Then, $M = \min\{\|l(i)\| : i = 1, 2, 3, 4\} = \frac{\sqrt{45}}{10}$, $f(x) = h(x)$ for all $x \in Q$, and by Theorem 4.4 and Corollary 4.1, $d_S(x) = d_{S_h}(x) \leq \frac{1}{M} f_+(x)$. The function f has a global error bound with constant $\mu = \frac{10}{\sqrt{45}}$.

Example 4.3. Consider $Q = \mathbb{R}^3$ and

$$f(x) = \begin{cases} x_1 - 0.3x_2 - x_3, & x_3 \geq 0 \\ x_1 - 0.3x_2 & x_3 < 0. \end{cases}$$

Then, f is an l.s.c. CAR function. Now, consider $\ell = (l(1), l(2))$, where $l(1) = (1, -0.3, -1), l(2) = (1, -0.3, 0)$, and $h(x) = \min\{h_1(x), h_2(x)\}$, where $h_1(x) = x_1 - 0.3x_2 - x_3, h_2(x) = x_1 - 0.3x_2$. Then, $M = \min\{\|l(1)\|, \|l(2)\|\} = \sqrt{1.09}$. Since $f(x) = h(x)$ for all $x \in Q$, by Theorem 4.4 and Corollary 4.1, f has a global error bound with constant $\mu = \frac{1}{M} = \frac{1}{\sqrt{1.09}}$.

The above two examples show that in some cases, U_f is a finite collection of elementary functions h. The argument of the rest of this section guarantees that when the collection $\{S_h : h \in U_f\}$ is finite, this collection is linearly regular. The following notions and definitions can be found in the literature. We refer the reader to [5] and [14].

The radial cone and the closed radial cone of a set C at a point $x \in C$ are defined as $R_C(x) = \underset{t>0}{\cup} \frac{C - x}{t}$ and $\bar{R}_C(x) = \overline{\underset{t>0}{\cup} \frac{C - x}{t}}$, respectively. The contingent cone of the set C at the point $x \in C$ is defined by $K_C(x) = \{u \in \mathbb{R}^n : \exists t_k \longrightarrow 0^+, u_k \longrightarrow u \ \textit{such that}\ x + t_k u_k \in C \ \ \forall\, k \geq 0\}$. It is easy to check that $K_C(x) \subseteq \bar{R}_C(x)$ for all $x \in C$. If the last inclusion is an equality then C is said to be pseudo-convex at x. One finds that C is pseudo-convex at $x \in C$ if and only if $C - x \subseteq K_C(x)$ (see [1]).

When g is locally Lipschitz around x, the Clarke directional derivative of the function g at the point x in the direction v is defined by $g^\circ(x, v) = \limsup_{\substack{y \longrightarrow x \\ t \longrightarrow 0^+}} \frac{g(y + tv) - g(y)}{t}$, and the Clarke tangent cone of the set C at the point $x \in C$ is defined by $T_C^c(x) = \{v \in \mathbb{R}^n : d_C^\circ(x, v) = 0\}$ or $T_C^c(x) = \{v \in \mathbb{R}^n : \forall t \longrightarrow 0^+, x_k \xrightarrow{C} x \ \exists\ v_k \longrightarrow v \ \textit{such that}\ x_k + t_k v_k \in C \ \forall\ k \geq 1\}$. Here, $x_k \xrightarrow{C} x$, that is, $x_k \longrightarrow x$ and $x_k \in C$ for all $k \geq 0$. Also, d refers to the distance function. (Note that the distance function d is Lipschitz continuous. See [5].)

The Hadamard directional derivative of the function g at the point x in the direction v is defined by $g_H^\downarrow(x, v) = \liminf_{\substack{v' \longrightarrow v \\ t \longrightarrow 0^+}} \frac{g(x + tv') - g(x)}{t}$. If g is locally Lipschitz around x, then one has $g_H^\downarrow(x, v) = \underline{d}^+ g(x, v)$.Where $\underline{d}^+ g(x, v)$ is Dini lower directional derivative. See [Definition2.6]. The Dini normal cone of the set C at the point $x \in C$ is defined by $N_C^-(x) = \partial^- \delta_C(x) = \{y \in X : \langle y, v\rangle \leq \underline{d}^+ \delta_C(x, v)\ \forall\ v \in X\}$, where X is the whole space (here, $\mathbb{R}^n$) and $\partial^- \delta_C(x)$ is the Dini subdifferential of the indicator function of the set C at x.

The Frechet normal cone of the set C at the point $x \in C$ is defined by $N_C^F(x) = \partial^F \delta_C(x)$. Here $\partial^F \delta_C(x)$ is the Frechet subdifferential of the indicator function of the set C at x. One has

$$N_C^F(x) = \{y \in \mathbb{R}^n : \limsup_{x' \xrightarrow{C} x} \frac{\langle y, x' - x\rangle}{\|x' - x\|} \leq 0\}. \tag{4.3}$$

Remark 4.1. The following argument can be found in [14]. When C is closed, by [14, Lemma 2.1], (in a finite-dimensional space, in particular, in $\mathbb{R}^n$) $N_C^F(x) = N_C^-(x)$, and whenever C is closed and convex, $N_C^F(x) = N_C^c(x) = N_C^-(x) = N_C(x)$, where $N_C^F(x)$, $N_C^c(x)$, $N_C^-(x)$ and $N_C(x)$ are the Frechet, Clarke, Dini and classical normal cones, respectively. Also, one has $T_C(x) = K_C(x) = T_C^c(x) = \bar{R}_C(x)$, where $T_C(x)$ is the classical tangent cone of the set C at x. By (4.3), $N_C^F(x)$ is convex and since (in our argument) $N_C^F(x) = N_C^-(x)$, $N_C^-(x)$ is convex. When U_f is a finite collection, since $S = \underset{h\in U_f}{\cap} S_h$ and $N_S^-(x)$ is convex, it is easy to check that

$$\sum_{h\in U_f} N_{S_h}^-(x) \subseteq N_S^-(x). \tag{4.4}$$

By the definitions of the sets S, S_h, S_{h_i}, for each $h \in U_f$, define $A_h = \{x \in \partial S : x \in \partial S_h, x \notin S_{h_i} \ for\ some\ i \in \{1, 2, ..., j\}\}$. Define $A = \underset{h\in U_f}{\cup} A_h$. Then, $A \subseteq \partial S \subseteq S$.

The following theorem presents a property of the set S_h for each $h \in U_f$.

Theorem 4.5. For each $h \in U_f$, S_h is pseudo-convex at each point of $S \setminus A$.

Proof . Let $h \in U_f$ be fixed. For each $x \in S \setminus A$ we must show that

$$S_h - x \subseteq K_{S_h}(x). \tag{4.5}$$

Since $S = \underset{h\in U_f}{\cap} S_h$, it follows that $T_S^c(x) \subseteq T_{S_h}^c(x)$ for each $x \in S$, and since $S_h = \underset{i=1,...j}{\cup} S_{h_i}$, we obtain $T_{S_{h_i}}^c(x) \subseteq T_{S_h}^c(x)$ for each $x \in S_h$. (Note that if $x \notin S_{h_i}$, then $T_{S_{h_i}}^c(x) = \emptyset$.) Also, $T_{S_h}^c(x) \subseteq K_{S_h}(x)$ (see [5]). On the other hand, since S_{h_i} is closed and convex for each $i \in \{1, .., j\}$, then $T_{S_{h_i}}(x) = T_{S_{h_i}}^c(x)$, where $T_{S_{h_i}}(x)$ is the classical tangent cone of the set S_{h_i} at the point x. Thus,

$$T_{S_{h_i}}(x) = T_{S_{h_i}}^c(x) \subseteq T_{S_h}^c(x) \subseteq K_{S_h}(x) \ \ \forall x \in S, \forall i \in \{1, ..., j\}. \tag{4.6}$$

$$T_S^c(x) \subseteq T_{S_h}^c(x) \subseteq K_{S_h}(x) \ \forall\ x \in S. \tag{4.7}$$

Let $x \in S \setminus A$ be fixed (S is closed). Then, if $x \in int(S)$, one has $T_S^c(x) = \mathbb{R}^n$. Thus, by (4.7), $K_{S_h}(x) = \mathbb{R}^n$ and inequality (4.5) holds. If $x \in \partial S$, since $x \in S = \underset{h'\in U_f}{\cap} S_{h'}$, $x \in S_h$. Now, if $x \in int(S_h)$, then $T_{S_h}^c(x) = \mathbb{R}^n$ and by (4.7), inequality (4.5) holds. If $x \in \partial(S_h)$, since $\partial(S_h) \subseteq \underset{i=1,...,j}{\cup} \partial S_{h_i}$, we consider two cases.

Case 1: There exists an index $i \in \{1, ..., j\}$ such that $x \notin \partial S_{h_i}$. Then, if $x \in int(S_{h_i})$ we obtain $T_{S_{h_i}}(x) = \mathbb{R}^n$. Thus, using (4.6), inequality (4.5) holds; otherwise $x \notin S_{h_i}$, that implies $x \in A$, a contradiction.

Case 2: $x \in \underset{i=1,...,j}{\cap} \partial S_{h_i}$. Then, it suffices to show that (4.5) holds for each $x \in \underset{i=1,...,j}{\cap} \partial S_{h_i}$. Let $x \in \underset{i=1,...,j}{\cap} \partial S_{h_i}$ and consider $y \in S_h - x$. Then there exists $u \in S_h$ such that $y = u - x$. Since $u \in S_h$, there exists $\bar{i} \in \{1, ..., j\}$ such that $\langle l(\bar{i}), u\rangle - c_{\bar{i}} = \underset{i\in\{1,...,j\}}{\min} (\langle l(i), u\rangle - c_i) = h(u) \leq 0$. Thus,

$$\langle l(\bar{i}), u\rangle \leq c_{\bar{i}}. \tag{4.8}$$

Since $x \in \underset{i=1,...,j}{\cap} \partial S_{h_i}$, $x \in \partial S_{h_{\bar{i}}}$ and $S_{h_{\bar{i}}}$ is closed. Therefore, $\langle l(\bar{i}), x\rangle = c_{\bar{i}}$. Consequently, by (4.8) one has

$$\langle l(\bar{i}), y\rangle = \langle l(\bar{i}), u - x\rangle \leq 0. \tag{4.9}$$

On the other hand, since $x \in \partial S_{h_{\bar{i}}}$ and $S_{h_{\bar{i}}}$ is closed and convex, $\emptyset \neq T_{S_{h_{\bar{i}}}}(x) = cl(F_{S_{h_{\bar{i}}}}(x))$. Here, "$cl$" refers to the closure, and $F_{S_{h_{\bar{i}}}}(x)$ refers to the collection of all feasible directions of $S_{h_{\bar{i}}}$ at the point x. Since $x \in \partial S_{h_{\bar{i}}}$, it is easy to check that $cl(F_{S_{h_{\bar{i}}}}(x)) = \{v \in \mathbb{R}^n : \langle l(\bar{i}), v\rangle \leq 0\}$. Hence, (4.9) implies $y \in T_{S_{h_{\bar{i}}}}(x)$. Consequently, by (4.6), we obtain (4.5). Since $x \in S \setminus A$ and $h \in U_f$ are arbitrary, the assertion is true. □

Let U_f be a finite collection of elementary functions $h \in H_{n+2}$ ($h \in H_{n+1}$). Then, the following theorem guarantees the existence of an error bound for a proper l.s.c. (proper l.s.c. CAR) function f.

Theorem 4.6. Let $f : Q \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ be a proper l.s.c. (in particular, a proper l.s.c. CAR) function, and U_f be the finite collection defined as above. Also, suppose that Assumption Λ and the following conditions are satisfied. Then, f has a global error bound.

(i) At each point $x \in S = \underset{h \in U_f}{\cap} S_h$, $N_S^-(x) \subseteq \sum_{h \in U_f} N_{S_h}^-(x)$.

(ii) At each point $x \in A = \underset{h \in U_f}{\cup} A_h$, each S_h is pseudo-convex.

(iii) There exists $r > 0$ such that for each $\bar{x} \in S$ and each $z \in \sum_{h \in U_f} N_{S_h}^-(\bar{x})$,

$$\min\{\max_{h \in U_f} \|z_h\| : z_h \in N_{S_h}^-(\bar{x}), \sum_{h \in U_f} z_h = z\} \leq r\|z\|.$$

Proof . Since by Theorem 4.5, each S_h is pseudo-convex at each point $x \in S \setminus A$, and by (4.4), $\sum_{h \in U_f} N_{S_h}^-(x) \subseteq N_S^-(x)$, and on $\mathbb{R}^n$ the Dini normal cone coincides with the Frechet normal cone, by [14, Theorem 3.3] the collection $\{S_h : h \in U_f\}$ is linearly regular. Since Assumption Λ holds, Corollary 4.1 completes the proof. $\square$

The following theorem provides a relationship between $N_{S_h}^F(x)$ and $N_{S_{h_i}}^F(x)$ for each $x \in S_h$.

Theorem 4.7. Let $f : Q \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ be a proper l.s.c. (in particular, a proper l.s.c. CAR) function. Then for each $h \in U_f$, $N_{S_h}^F(x) = \underset{i \in I(x)}{\cap} N_{S_{h_i}}^F(x)$ for each $x \in S_h$, where $I(x) = \{i \in \{1, ..., j\} : x \in S_{h_i}\}$.

Proof . First, we recall that S_h is the union of the collection of closed lower half spaces $S_{h_i}, i = 1, ..., j$. Then, each S_{h_i} is closed and convex, implying $N_{S_{h_i}}^F(x) = N_{S_{h_i}}(x)$, where $N_{S_{h_i}}(x)$ is the classical normal cone of S_{h_i} at the point x. Fix $h \in U_f$ and $\hat{x} \in S_h$. Since $S_{h_i} \subseteq S_h$, $N_{S_h}^F(\hat{x}) \subseteq N_{S_{h_i}}^F(\hat{x})$ for each $i \in I(\hat{x})$. Hence, $N_{S_h}^F(\hat{x}) \subseteq \underset{i \in I(\hat{x})}{\cap} N_{S_{h_i}}^F(\hat{x})$.

Conversely, let $u \in \underset{i \in I(\hat{x})}{\cap} N_{S_{h_i}}^F(\hat{x}) = \underset{i \in I(\hat{x})}{\cap} N_{S_{h_i}}(\hat{x})$. Then $u \in N_{S_{h_i}}(\hat{x})$ for each $i \in I(\hat{x})$. Since $\hat{x} \in S_{h_i}$ for each $i \in I(\hat{x})$, if there is $\bar{i} \in I(\hat{x})$ such that $\hat{x} \in int(S_{h_{\bar{i}}})$, then one has $N_{S_{h_{\bar{i}}}}(\hat{x}) = \{0\}$. Since $\hat{x} \in S_{h_i}$ for each $i \in I(\hat{x})$, $N_{S_{h_i}}(\hat{x}) \neq \emptyset$ and $0 \in N_{S_{h_i}}(\hat{x})$ that imply $\underset{i \in I(\hat{x})}{\cap} N_{S_{h_i}}(\hat{x}) = \{0\}$. Thus, $u = 0 \in N_{S_h}(\hat{x})$. (Note that since $\hat{x} \in S_h$, $N_{S_h}^F(\hat{x}) \neq \emptyset$ and it contains 0.) Otherwise, $\hat{x} \in \partial S_{h_i}$ for each $i \in I(\hat{x})$. Now, we consider the following cases.

Case 1: $|I(\hat{x})| = 1$, where $|I(\hat{x})|$ refers to the number of elements of the set $I(\hat{x})$. Then, for a single index $i' \in \{1, 2, ..., j\}, I(\hat{x}) = \{i'\}$ and $\hat{x} \in \partial S_{h_{i'}}$. Since $\hat{x} \in S_h$ and $|I(\hat{x})| = 1$, we obtain $\hat{x} \in \partial S_h$. Indeed, if $\hat{x} \in int(S_h)$, since $S_h = \underset{i=1,...,j}{\cup} S_{h_i}$ and $\hat{x} \in \partial S_{h_{i'}}$, there exists an index $\hat{i} \in \{1, ..., j\}$, $\hat{i} \neq i'$, such that $\hat{x} \in int(S_{h_{\hat{i}}})$. Then $|I(\hat{x})| > 1$, which is a contradiction. Hence $\hat{x} \in \partial S_h$, which implies $\hat{x} \in \partial S_h \cap \partial S_{h_{i'}}$. Then, there exists $\varepsilon > 0$ such that for the neighborhood $B(\hat{x}, \varepsilon)$ of $\hat{x}$ one has $B(\hat{x}, \varepsilon) \cap \partial S_h = B(\hat{x}, \varepsilon) \cap \partial S_{h_{i'}}$. On the other hand, since $S_{h_{i'}}$ is a lower half space and a subset of S_h, by the definition of the Frechet normal cone (4.3) we obtain $N_{S_h}^F(\hat{x}) = N_{S_{h_{i'}}}^F(\hat{x})$. (Note that by the above argument, in (4.3), $x' \xrightarrow{S_h} \hat{x}$ as $x' \xrightarrow{S_{hi'}} \hat{x}$.) Then, $u \in \underset{i \in I(\hat{x})}{\cap} N_{S_{h_i}}^F(\hat{x}) = N_{S_{h_{i'}}}^F(\hat{x}) = N_{S_h}^F(\hat{x})$.

Case 2: $|I(\hat{x})| \geq 2$. Since $\hat{x} \in \partial S_{h_i}$ for each $i \in I(\hat{x})$, $N_{S_{h_i}}^F(\hat{x}) = N_{S_{h_i}}(\hat{x}) = \mathbb{R}_+(l(i))$, where $l(i) \in \mathbb{R}^n$ is the vector for which $h_i(\hat{x}) = \langle l(i), \hat{x} \rangle - c_i$ holds for each $i \in I(\hat{x})$. Now, we consider two sub-cases for Case 2.

Case 2.1: There exists an index $\bar{i} \in I(\hat{x})$ such that $l(\bar{i}) \notin \mathbb{R}_+(l(i))$ for all $i \in I(\hat{x})$. Since $N_{S_{h_i}}(\hat{x}) = \mathbb{R}_+(l(i))$ and $N_{S_{h_{\bar{i}}}}(\hat{x}) = \mathbb{R}_+(l(\bar{i}))$, one has $N_{S_{h_{\bar{i}}}}(\hat{x}) \cap (\underset{i \in I(\hat{x}) \setminus \{\bar{i}\}}{\cap} N_{S_{h_i}}(\hat{x})) = \{0\}$. Then, $u = 0 \in N_{S_h}(\hat{x})$.

Case 2.2: $l(i) \in \mathbb{R}_+(l(\hat{i}))$ for all $i \in I(\hat{x})$ and for an index $\hat{i} \in I(\hat{x})$. Then, $l(i) = \alpha_i l(\hat{i})$ for each $i \in I(\hat{x})$, where $\alpha_i > 0$. Since $\hat{x} \in \partial S_{h_i}$ for each $i \in I(\hat{x})$, $\langle l(i), \hat{x} \rangle = c_i$ which implies $\langle \alpha_i l(\hat{i}), \hat{x} \rangle = c_i$, and therefore $\langle l(\hat{i}), \hat{x} \rangle = \dfrac{c_i}{\alpha_i}$. Consequently, all values $\dfrac{c_i}{\alpha_i}$ are the same, meaning that all the sets S_{h_i} are equal. Thus $|I(x)| = 1$, which is a contradiction. Since the inclusion holds for all cases, the proof is complete. $\square$

Since on $\mathbb{R}^n$ the Frechet normal cone coincides with the Dini normal cone and $S \subseteq S_h$ for each $h \in U_f$, by Theorem 4.7 we obtain $N_{S_h}^-(x) = \underset{i \in I(x)}{\cap} N_{S_{h_i}}^-(x)$ for each $x \in S$. We may replace (i) of Theorem 4.6 with the following statement.

At each point $x \in S = \underset{h \in U_f}{\cap} S_h$, $N_S^-(x) \subseteq \sum_{\substack{h \in U_f \\ h(x) = \min_{i=1...,j} \langle l(i), x \rangle}} \underset{i \in I(x) = \{i \in \{1,...,j\} : x \in S_{h_i}\}}{\cap} N_{S_{h_i}}^-(x)$.

The above argument guarantees the linear regularity of the finite collection U_f. If one guarantees the linear regularity whenever the collection U_f is infinite, then Theorem 4.4 guarantees the existence of a global error bound for a proper l.s.c. (in particular, proper l.s.c. CAR) function f in the absence of the condition "$l(i) \in \mathbb{R}^n_+$ and $l(i) \neq 0_n$".

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