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FUZZY APPROXIMATELY ADDITIVE MAPPINGS

H. KHODAEI* AND M. KAMYAR

Dedicated to the 70th Anniversary of S.M.Ulam's Problem for Approximate Homomorphisms

Abstract. Moslehian and Mirmostafaee, investigated the fuzzy stability problems for the Cauchy additive functional equation, the Jensen additive functional equation and the cubic functional equation in fuzzy Banach spaces. In this paper, we investigate the generalized Hyers-Ulam–Rassias stability of the generalized additive functional equation with n–variables, in fuzzy Banach spaces. Also, we will show that there exists a close relationship between the fuzzy continuity behavior of a fuzzy almost additive function, control function and the unique additive function which approximate the almost additive function.

1. Introduction and preliminaries

Studies on fuzzy normed linear spaces are relatively recent in the field of fuzzy functional analysis. In 1984, Katsaras [14] first introduced the notion of fuzzy norm on a linear space and at the same year Wu and Fang [32] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for a fuzzy topological linear space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [5, 16, 17, 33]. In 2003, Bag and Samanta [2] modified the definition of Cheng and Mordeson [4] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed linear spaces [3]. Next we define the notion of a fuzzy normed linear space.

Let X be a real linear space. A function $N : X \times \mathbb{R} \longrightarrow [0, 1]$ is said to be a fuzzy norm on X ([2]) iff the following conditions are satisfied:

(N_1) $N(x, t) = 0$ for all $x \in X$ and $t \leq 0$;

(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;

(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N_4) $N(x + y, s + t) \geq min\{N(x, s), N(y, t)\}$ for all $x, y \in X$ and all $s, t \in \mathbb{R}$;

(N_5) $N(x, \cdot)$ is non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$ for all $x \in X$.

In the following we will suppose that $N(x, \cdot)$ is left continuous for every x.

A fuzzy normed linear space is a pair (X, N), where X is a real linear space and N is a fuzzy norm on X.

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*: Corresponding author.

Let $(X, \| \, . \, \|)$ be a normed linear space. Then

$$N(x,t) = \begin{cases} 1, & t > 0, \; x \in X, \\ 0, & t \leq 0, \; x \in X \end{cases}$$

is a fuzzy norm on X.

Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we write $N-lim_{n\to\infty} x_n = x$.

A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $N(x_m - x_n, \delta) > 1 - \epsilon$ $(m, n \geq n_0)$. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

In 1940, Ulam gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. The first stability problem concerning group homomorphisms was raised by Ulam [31] in 1940 and affirmatively solved by Hyers [11].

Aoki [1] and Rassias [28] provided a generalization of the Hyers theorem for additive and linear functions, respectively, by allowing the Cauchy difference to be unbounded.

Theorem 1.1. (*Th.M. Rassias*). *Let $f : X \to Y$ be a function from a normed vector space X into a Banach space Y subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in X$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then there exists a unique additive function $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \leq \varepsilon \|x\|^p / (1 - 2^{p-1}) \tag{1.2}$$

for all $x \in X$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. Also, if for each fixed $x \in X$ the function $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then A is linear.

In 1991, Z. Gajda [9] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers–Ulam–Rassias stability of functional equations. During the last decades several stability problems of functional equations have been investigated in the spirt of Hyers–Ulam–Rassias. See [6]–[10], [12, 13] and [29, 30] for more detailed information on stability of functional equations.

The stability of different functional equations in fuzzy normed spaces and random normed spaces has been studied in, Mirmostafaee, Mirzavaziri and Moslehian [23, 24, 25, 26], Miheţ, Radu and Park [19, 20, 27], Miheţ, Saadati and Vaezpour [21, 22].

The functional equation

$$\begin{aligned} &\sum_{k=2}^{n} \Big(\sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^{n} \Big) f\Big(\sum_{i=1, i \neq i_1, \ldots, i_{n-k+1}}^{n} a_i x_i - \sum_{r=1}^{n-k+1} a_{i_r} x_{i_r} \Big) \\ &\quad + f\Big(\sum_{i=1}^{n} a_i x_i\Big) = 2^{n-1} a_1 f(x_1) \end{aligned} \tag{1.3}$$

where $a_1, ..., a_n \in \mathbb{Z} - \{0\}$ with $a_1 \neq \pm 1$, is called the generalized additive functional equation with n–variables, since the function $f(x) = ax$ is its solution. The stability problem for the generalized additive functional equation with n–variables was proved by Khodaei and Rassias [15]. As a special case, if $n = 2$ in (1.3), then the functional equation (1.3) reduces to

$$f(a_1x_1 - a_2x_2) + f(a_1x_1 + a_2x_2) = 2a_1f(x_1)$$

also by putting $n = 3$ in (1.3), we obtain

$$\begin{aligned} &f(a_1x_1 - a_2x_2 - a_3x_3) + f(a_1x_1 - a_2x_2 + a_3x_3) + f(a_1x_1 + a_2x_2 - a_3x_3) \\ &\quad + f(a_1x_1 + a_2x_2 + a_3x_3) = 2^2a_1f(x_1) \end{aligned}$$

In this paper, we prove the generalized Hyers–Ulam–Rassias stability and the fuzzy continuity of the functional equation (1.3) in in fuzzy Banach spaces. Throughout this paper, assume that $a_1, ..., a_n$ are nonzero fixed integers with $a_1 \neq \pm 1$.

2. Fuzzy stability of the functional equation (1.3)

In the rest of this paper, unless otherwise explicitly stated, we will assume that X is a linear space, (Z, N') is a fuzzy normed space and (Y, N) is a fuzzy Banach space. For convenience, we use the following abbreviation for a given function $f : X \to Y$,

$$\begin{aligned} Df(x_1, .., x_n) := &\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^{n}\Big)f\Big(\sum_{i=1, i\neq i_1,...,i_{n-k+1}}^{n} a_ix_i - \sum_{r=1}^{n-k+1} a_{i_r}x_{i_r}\Big) \\ &+ f\Big(\sum_{i=1}^{n} a_ix_i\Big) - 2^{n-1}a_1f(x_1). \end{aligned}$$

In this section, we prove the generalized Hyers–Ulam–Rassias stability of the functional equation (1.3) in fuzzy Banach spaces for additive functions. Later, we will show that there exists a close relationship between the fuzzy continuity behavior of a fuzzy almost additive function, control function and the unique additive function which approximates the almost additive function.

Lemma 2.1. [15]. *Let V_1 and V_2 be real vector spaces. A function $f : V_1 \to V_2$ satisfies the functional equation (1.3) if and only if $f : V_1 \to V_2$ is additive.*

Theorem 2.2. *Let $\ell \in \{-1, 1\}$ be fixed and let $\varphi : \underbrace{X \times X \times ... \times X}_{n-times} \longrightarrow Z$ be a function such that*

$$\varphi(a_1x_1, ..., a_1x_n) = \alpha\varphi(x_1, ..., x_n) \tag{2.1}$$

for all $x_1, ..., x_n \in X$ and for some positive real number α with $\alpha\ell < a_1\ell$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N\left(D_f(x_1, ..., x_n), t\right) \geq N'\left(\varphi(x_1, ..., x_n), t\right) \tag{2.2}$$

for all $x_1, ..., x_n \in X$ and all $t > 0$. Then the limit

$$A(x) = N - \lim_{k\to\infty} \frac{f(a_1^{\ell k}x)}{a_1^{\ell k}})$$

exists for all $x \in X$ and $A : X \to Y$ is a unique additive function satisfying

$$N(A(x) - f(x), t) \geq N^{'}(\varphi(x, \underbrace{0, ..., 0}_{n-1}), 2^{n-2}\ell(a_1 - \alpha)t) \tag{2.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $x_1 = x$ and $x_i = 0$ $(i = 2, ..., n)$ in (2.2), we obtain

$$N\left(\left(\binom{n-1}{n-1} + \binom{n-1}{n-2} + ... + \binom{n-1}{1} + 1\right)f(a_1x) - 2^{n-1}a_1f(x), t\right) \\ \geq N^{'}(\varphi(x, 0, 0, ..., 0), t)$$

for all $x \in X$ and all $t > 0$. It follows from last inequality that

$$N((1 + \sum_{\ell=1}^{n-1}\binom{n-1}{\ell})f(a_1x) - 2^{n-1}a_1f(x), t) \geq N^{'}(\varphi(x, 0, 0, ..., 0), t)$$

for all $x \in X$ and all $t > 0$. Hence by using the relation $1 + \sum_{\ell=1}^{n-1}\binom{n-1}{\ell} = 2^{n-1}$, gives

$$N\left(2^{n-1}f(a_1x) - 2^{n-1}a_1f(x), t\right) \geq N^{'}(\varphi(x, 0, 0, ..., 0), t) \tag{2.4}$$

for all $x \in X$ and all $t > 0$. So

$$N\left(\frac{f(a_1x)}{a_1} - f(x), \frac{t}{2^{n-1}a_1}\right) \geq N^{'}(\varphi(x, 0, 0, ..., 0), t) \tag{2.5}$$

for all $x \in X$ and all $t > 0$. Then by our assumption,

$$N^{'}(\varphi(a_1x, 0, 0, ..., 0), t) = N^{'}(\varphi(x, 0, 0, ..., 0), \frac{t}{\alpha}) \tag{2.6}$$

for all $x \in X$ and all $t > 0$. Replacing x by a_1^kx in (2.5) and using (2.6), we obtain

$$\begin{aligned} N(\frac{f(a_1^{k+1}x)}{a_1^{k+1}} - \frac{f(a_1^kx)}{a_1^k}, \frac{t}{2^{n-1}a_1^{k+1}}) &\geq N^{'}(\varphi(a_1^kx, 0, 0, ..., 0), t) \\ &= N^{'}(\varphi(x, 0, 0, ..., 0), \frac{t}{\alpha^k}) \end{aligned} \tag{2.7}$$

for all $x \in X$, $t > 0$ and $k \geq 0$. Replacing t by α^kt in (2.7), we see that

$$N(\frac{f(a_1^{k+1}x)}{a_1^{k+1}} - \frac{f(a_1^kx)}{a_1^k}, \frac{\alpha^kt}{2^{n-1}a_1^{k+1}}) \geq N^{'}(\varphi(x, 0, 0, ..., 0), t) \tag{2.8}$$

for all $x \in X$, $t > 0$ and $k > 0$. It follows from $\frac{f(a_1^kx)}{a_1^k} - f(x) = \sum_{j=0}^{k-1}(\frac{f(a_1^{j+1}x)}{a_1^{j+1}} - \frac{f(a_1^jx)}{a_1^j})$ and (2.8) that

$$\begin{aligned} N(\frac{f(a_1^kx)}{a_1^k} - f(x), \sum_{j=0}^{k-1}\frac{\alpha^jt}{2^{n-1}a_1^{j+1}}) &\geq min\bigcup_{j=0}^{k-1}N(\frac{f(a_1^{j+1}x)}{a_1^{j+1}} - \frac{f(a_1^jx)}{a_1^j}, \frac{\alpha^jt}{2^{n-1}a_1^{j+1}}) \\ &\geq N^{'}(\varphi(x, 0, 0, ..., 0), t) \end{aligned} \tag{2.9}$$

for all $x \in X$, $t > 0$ and $k > 0$. Replacing x by $a_1^m x$ in (2.9), we observe that

$$N(\frac{f(a_1^{k+m}x)}{a_1^{k+m}} - \frac{f(a_1^m x)}{a_1^m}, \sum_{j=0}^{k-1} \frac{\alpha^j t}{2^{n-1}a_1^{j+m+1}} \geq N^{'}(\varphi(a_1^m x, 0, 0, ..., 0), t)$$
$$= N^{'}(\varphi(x, 0, 0, ..., 0), \frac{t}{\alpha^m})$$

for all $x \in X$, all $t > 0$ and all $m \geq 0,\ k > 0$. Hence

$$N(\frac{f(a_1^{k+m}x)}{a_1^{k+m}} - \frac{f(a_1^m x)}{a_1^m}, \sum_{j=m}^{k+m-1} \frac{\alpha^j t}{2^{n-1}a_1^{j+1}}) \geq N^{'}(\varphi(x, 0, 0, ..., 0), t)$$

for all $x \in X$, all $t > 0$ and all $m \geq 0,\ k > 0$. By last inequality, we obtain

$$N(\frac{f(a_1^{k+m}x)}{a_1^{k+m}} - \frac{f(a_1^m x)}{a_1^m}, t) \geq N^{'}(\varphi(x, 0, 0, ..., 0), \frac{t}{\sum_{j=m}^{k+m-1} \frac{\alpha^j}{2^{n-1}a_1^{j+1}}}) \tag{2.10}$$

for all $x \in X$, all $t > 0$ and all $m \geq 0,\ k > 0$. Since $0 < \alpha < a_1$ and $\sum_{j=0}^{\infty}(\frac{\alpha}{a_1})^j < \infty$, the Cauchy criterion for convergence and (N_5) imply that $\{\frac{f(a_1^k x)}{a_1^k}\}$ is a Cauchy sequence in Y. Since Y is a fuzzy Banach space, this sequence converges to some point $A(x) \in Y$. So one can define the function $A : X \to Y$ by

$$A(x) = N - \lim_{k\to\infty} \frac{f(a_1^k x)}{a_1^k}) \tag{2.11}$$

for all $x \in X$. Fix $x \in X$ and put m=0 in (2.10) to obtain

$$N(\frac{f(a_1^k x)}{a_1^k} - f(x), t) \geq N^{'}(\varphi(x, 0, 0, ..., 0), \frac{t}{\sum_{j=0}^{k-1} \frac{\alpha^j}{2^{n-1}a_1^{j+1}}})$$

for all $x \in X$, all $t > 0$ and all $k > 0$. From which we obtain

$$\begin{aligned} N(A(x) - f(x), t) &\geq min\{N(A(x) - \frac{f(a_1^k x)}{a_1^k}, \frac{t}{2}), N(\frac{f(a_1^k x)}{a_1^k} - f(x), \frac{t}{2})\} \\ &\geq N^{'}(\varphi(x, 0, 0, ..., 0), \frac{t}{\sum_{j=0}^{k-1} \frac{\alpha^j}{2^{n-2}a_1^{j+1}}}) \end{aligned} \tag{2.12}$$

for k large enough. Taking the limit as $k \to \infty$ in (2.12), we obtain

$$N(A(x) - f(x), t) \geq N^{'}(\varphi(x, 0, 0, ..., 0), 2^{n-2}(a_1 - \alpha)t) \tag{2.13}$$

for all $x \in X$ and all $t > 0$. It follows from (2.8) and (2.11) that

$$\begin{aligned} N(\frac{A(a_1 x)}{a_1} - A(x), t) &\geq min\{N(\frac{A(a_1 x)}{a_1} - \frac{f(a_1^{k+1}x)}{a_1^{k+1}}, \frac{t}{3}), N(\frac{f(a_1^k x)}{a_1^k} - A(x), \frac{t}{3}) \\ &, N(\frac{f(a_1^{k+1}x)}{a_1^{k+1}} - \frac{f(a_1^k x)}{a_1^k}, \frac{t}{3})\} \geq N^{'}(\varphi(x, 0, 0, ..., 0), \frac{2^{n-1}a_1^{k+1}t}{3\alpha^k}) \end{aligned}$$

for all $x \in X$ and all $t > 0$. Therefore

$$A(a_1 x) = a_1 A(x) \tag{2.14}$$

for all $x \in X$. Replacing $x_1, ..., x_n$ by $a_1^k x_1, ..., a_1^k x_n$ in (2.2), respectively, we obtain

$$N(\frac{1}{a_1^k} D_f(a_1^k x_1, ..., a_1^k x_n), t) \geq N^{'}(\varphi(a_1^k x_1, ..., a_1^k x_n), a_1^k t) = N^{'}(\varphi(x_1, ..., x_n), \frac{a_1^k t}{\alpha^k})$$

which tends to 1 as $k \to \infty$ for all $x_1, ..., x_n \in X$ and all $t > 0$. So, we see that A satisfies (1.3). Thus by Lemma 2.1, the function $A : X \to Y$ is additive. Now, to prove the uniqueness property of A, let $A^{'} : X \to Y$ be another additive function satisfying (2.3). It follows from (2.3), (2.6) and (2.14) that

$$\begin{aligned} N(A(x) - A^{'}(x), t) &= N(\frac{A(a_1^k x)}{a_1^k} - \frac{A^{'}(a_1^k x)}{a_1^k}, t) \\ &\geq min\{N(\frac{A(a_1^k x)}{a_1^k} - \frac{f(a_1^k x)}{a_1^k}, \frac{t}{2}), N(\frac{f(a_1^k x)}{a_1^k} - \frac{A^{'}(a_1^k x)}{a_1^k}, \frac{t}{2})\} \\ &\geq N^{'}(\varphi(a_1^k x, 0, 0, ..., 0), \frac{a_1^k 2^{n-2}(a_1 - \alpha)t}{2}) \\ &= N^{'}(\varphi(x, 0, 0, ..., 0), \frac{a_1^k 2^{n-2}(a_1 - \alpha)t}{2\alpha^k}) \end{aligned}$$

for all $x \in X$ and all $t > 0$. Since $\alpha < a_1$, we obtain

$$\lim_{k \to \infty} N^{'}(\varphi(x, 0, 0, ..., 0), \frac{a_1^k 2^{n-2}(a_1 - \alpha)t}{2\alpha^k}) = 1$$

Thus, $A(x) = A^{'}(x)$.

Case (2): $\ell = -1$. We can state the proof in the same pattern as we did in the first case. Replacing x by $\frac{x}{a_1}$ in (2.4), we obtain

$$N(f(x) - a_1 f(\frac{x}{a_1}), \frac{t}{2^{n-1}}) \geq N^{'}(\varphi(\frac{x}{a_1}, 0, 0, ..., 0), t) \tag{2.15}$$

for all $x \in X$ and all $t > 0$. Replacing x and t by $\frac{x}{a_1^k}$ and $\frac{t}{a_1^k}$ in (2.15), respectively, we obtain

$$\begin{aligned} &N(a_1^k f(\frac{x}{a_1^k}) - a_1^{k+1} f(\frac{x}{a_1^{k+1}}), \frac{t}{2^{n-1}}) \geq N^{'}(\varphi(\frac{x}{a_1^{k+1}}, 0, 0, ..., 0), \frac{t}{a_1^k}) \\ &= N^{'}(\varphi(x, 0, 0, ..., 0), (\frac{\alpha}{a_1})^k t\alpha) \end{aligned}$$

for all $x \in X$, all $t > 0$ and all $k > 0$. One can deduce

$$N(a_1^{k+m} f(\frac{x}{a_1^{k+m}}) - a_1^m f(\frac{x}{a_1^m}), t) \geq N^{'}(\varphi(x, 0, 0, ..., 0), \frac{t}{\sum_{j=m+1}^{k+m} \frac{a_1^j}{2^{n-1} a_1 \alpha^j}}) \tag{2.16}$$

for all $x \in X$, all $t > 0$ and all $m \geq 0$, $k \geq 0$. From which we conclude that $\{a_1^k f(\frac{x}{a_1^k})\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N). Therefore, there is a function $A : X \longrightarrow Y$ defined by $A(x) := N - \lim_{k\to\infty} a_1^k f(\frac{x}{a_1^k})$. Employing (2.16) with $m = 0$, we obtain

$$N(A(x) - f(x), t) \geq N^{'}(\varphi(x, 0, 0, ..., 0), 2^{n-2}(\alpha - a_1)t)$$

for all $x \in X$ and all $t > 0$. The proof for uniqueness of A for this case proceeds similarly to that in the previous case, hence it is omitted. □

Remark 2.3. Let $0 < \alpha < a_1$. Suppose that the function $t \longmapsto N(f(x) - A(x), .)$ from $(0, \infty)$ into $[0, 1]$ is right continuous. Then we obtain a better fuzzy (2.13) as follows.

We obtain

$$N(A(x) - f(x), t+s) \geq min\{N(A(x) - \frac{f(a_1^k x)}{a_1^k}, s), N(\frac{f(a_1^k x)}{a_1^k} - f(x), t)\}$$
$$\geq N'(\varphi(x, 0, 0, ..., 0), \frac{t}{\sum_{j=0}^{k-1} \frac{\alpha^j}{2^{n-1} a_1^{j+1}}})$$

Tending s to zero we infer that

$$N(A(x) - f(x), t) \geq N'(\varphi(x, 0, 0, ..., 0), 2^{n-1}(a_1 - \alpha)t)$$

for all $x \in X$ and all $t > 0$.

From Theorem 2.2, we obtain the following corollary concerning the Hyers-Ulam-Rassias stability [28] of additive functions satisfies (1.3), in normed spaces.

Corollary 2.4. *Let X be a normrd space and Y be a Banach space. Let ε, λ be non-negative real numbers such that $\lambda \neq 1$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies*

$$\|D_f(x_1, ..., x_n)\| \leq \varepsilon \sum_{i=1}^{n} \|x_i\|^\lambda \tag{2.17}$$

for all $x_1, ..., x_n \in X$. Then the limit

$$A(x) = \lim_{k\to\infty} \frac{f(a_1^{\ell k} x)}{a_1^{\ell k}})$$

exists for all $x \in X$ and $A : X \to Y$ is a unique additive function satisfying

$$\|f(x) - A(x)\| \leq \frac{\varepsilon \|x\|^\lambda}{2^{n-2}\ell(a_1^\lambda - \alpha)} \tag{2.18}$$

for all $x \in X$, where $\lambda\ell < \ell$

Proof. Define the function N by

$$N(x, t) = \begin{cases} \frac{t}{t+\|x\|}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

It is easy to see that (X, N) is a fuzzy normed space and (Y, N) is a fuzzy Banach space. Denote $\varphi : X \times X \times ... \times X \longrightarrow R$, the function sending each $(x_1, ..., x_n)$ to $\varepsilon \sum_{i=1}^{n} \|x_i\|^\lambda$. By assumption

$$N(D_f(x_1, ..., x_n), t) \geq N'(\varphi(x_1, ..., x_n), t)$$

note that $N' : R \times R \longrightarrow [0, 1]$ given by

$$N'(x, t) = \begin{cases} \frac{t}{t+|x|}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

is a fuzzy norm on R. By Theorem 2.2, there exists a unique additive function $A : X \longrightarrow Y$ satisfies equation (1.3) and

$$\begin{aligned}\frac{t}{t+\|f(x)-A(x)\|} =& N(f(x)-A(x),t)\\ &\geq N^{'}(\varphi(x,0,0,...,0),2^{n-2}\ell(a_1^{\lambda}-\alpha)t)\\ &= N'(\varepsilon\|x\|^{\lambda},2^{n-2}\ell(a_1^{\lambda}-\alpha)t) = \frac{2^{n-2}\ell(a_1^{\lambda}-\alpha)t}{2^{n-2}\ell(a_1^{\lambda}-\alpha)t+\varepsilon\|x\|^{\lambda}}\end{aligned}$$

thus

$$\frac{t}{t+\|f(x)-A(x)\|} \geq \frac{2^{n-2}\ell(a_1^{\lambda}-\alpha)t}{2^{n-2}\ell(a_1^{\lambda}-\alpha)t+\varepsilon\|x\|^{\lambda}}$$

which implies that, $\|f(x)-A(x)\| \leq \frac{\varepsilon\|x\|^{\lambda}}{2^{n-2}\ell(a_1^{\lambda}-\alpha)}$ for all $x \in X$. □

We say that a function $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_k\}$ converging to x_0 in X, then the sequence $\{f(x_k)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [3]). Now we examine some conditions under which the additive function found in Theorem 2.2 to be continuous. In the following theorem, we investigate fuzzy continuity of additive functions in fuzzy normed spaces. In fact, we will show that under some extra conditions on Theorem 2.2, the additive function $r \longmapsto A(rx)$ is fuzzy continuous. It follows that in such a case, $A(rx) = rA(x)$ for all $r \in R$.

In the following result, we will use the terminologies of Theorem 2.2, we will also assume that all conditions of the theorem hold.

Theorem 2.5. *Denote N_1 the fuzzy norm obtained as Corollary 2.4 on R. Let for all $x \in X$, the functions $r \longmapsto f(rx)$ (from (R, N_1) into (Y, N)) and $r \longmapsto \varphi(rx, \underbrace{0, ..., 0}_{n-1})$ (from (R, N_1) into (Z, N')) be fuzzy continuous. Then for all $x \in X$, the function $r \longmapsto A(rx)$ is fuzzy continuous and $A(rx) = rA(x)$ for all $r \in R$.*

Proof. Case (1): $\ell = 1$. Let $\{r_k\}$ be a sequence in R that converge to some $r \in R$, and let $t > 0$. Let $\varepsilon > 0$ be given, since $0 < \alpha < a_1$, so $\lim_{k\to\infty} \frac{2^{n-2}(a_1-\alpha)a_1^k t}{6\alpha^k} = \infty$, there is $m \in \mathbb{N}$ such that

$$N^{'}(\varphi(rx,0,0,...,0),\frac{2^{n-2}(a_1-\alpha)a_1^m t}{6\alpha^m}) > 1-\varepsilon \tag{2.19}$$

It follows from (2.13) and (2.19) that

$$N(\frac{f(a_1^m rx)}{a_1^m} - \frac{A(a_1^m rx)}{a_1^m}, \frac{t}{3}) > 1-\varepsilon \tag{2.20}$$

By the fuzzy continuity of functions $r \longmapsto f(rx)$ and $r \longmapsto \varphi(rx, 0, 0, ..., 0)$, we can find some $\jmath \in \mathbb{N}$ such that for any $k \geq \jmath$,

$$N(\frac{f(a_1^m r_k x)}{a_1^m} - \frac{f(a_1^m rx)}{a_1^m}, \frac{t}{3}) > 1-\varepsilon \tag{2.21}$$

and

$$N^{'}(\varphi(r_kx,0,0....,0)-\varphi(rx,0,0,...,0),\frac{2^{n-2}(a_1-\alpha)a_1^mt}{6\alpha^m})>1-\varepsilon \tag{2.22}$$

It follows from (2.19) and (2.22) that

$$N^{'}(\varphi(r_kx,0,0,...,0),\frac{2^{n-2}(a_1-\alpha)a_1^mt}{3\alpha^m})>1-\varepsilon \tag{2.23}$$

On the other hand,

$$\begin{aligned} N(A(r_kx)-\frac{f(a_1^mr_kx)}{a_1^m},\frac{t}{a_1^m}) &= N(\frac{A(a_1^mr_kx)}{a_1^m}-\frac{f(a_1^mr_kx)}{a_1^m},\frac{t}{a_1^m}) \\ &\geq N^{'}(\varphi(r_kx,0,0,...,0),\frac{2^{n-2}(a_1-\alpha)t}{\alpha^m}) \end{aligned} \tag{2.24}$$

It follows from (2.23) and (2.24) that

$$N(A(r_kx)-\frac{f(a_1^mr_kx)}{a_1^m},\frac{t}{3})>1-\varepsilon \tag{2.25}$$

So, it follows from (2.20), (2.21) and (2.25) that for any $k\geq \jmath$,

$$N(A(r_kx)-A(rx),t)>1-\varepsilon$$

Therefore for every choice $x\in X$, $t>0$ and $\varepsilon>0$, we can find some $\jmath\in\mathbb{N}$ such that $N(A(r_kx)-A(rx),t)>1-\varepsilon$ for every $k\geq\jmath$. This shows that $A(r_kx)\to A(rx)$.

The proof for $\ell=-1$, proceeds similarly to that in the previous case.
It is not hard to see that $A(rx)=rA(x)$ for each rational number r. Let r be a real number, then there exists a sequence $\{r_k\}$ of rational numbers such that $r_k\to r$. By the fuzzy continuous of $A(x)$, for all $x\in X$,

$$A(rx)=\lim_{k\to\infty}A(r_kx)=\lim_{k\to\infty}r_kA(x)=rA(x)$$

for each $r\in R$ (see [23]). This completes the proof of the theorem. □

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Department of Mathematics, Semnan University P. O. Box 35195-363, Semnan, Iran.

E-mail address: hkhodaei.math@yahoo.com, kamyar.mahdie@gmail.com