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Some Wardowski-Mizogochi-Takahashi-Type generalizations of the multi-valued version of Darbo's fixed point theorem with applications

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Abstract

In this paper, we extend the multi-valued version of Darbo's fixed point theorem using generalized Mizogochi-Takahashi mappings of the Wardowski type. The technique of measure of noncompactness is the main tool in carrying out our proofs. As an application, we investigate the existence of solutions for an integral inclusion on the space $BC(\mathbb{R}_+, E)$.

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1 Introduction

The Measure of Non-Compactness (MNC) was first introduced by Kuratowski [24] in 1930. This concept is a very useful tool in the proof of the existence of solutions for integral equations, systems of integral equations, and fractional types of these equations. For more details, we refer the reader to [1, 8, 2, 7, 6, 9, 5, 10, 26, 11, 21, 27, 19, 22, 28, 29, 18, 3, 30].

Let $\mathbb{R}$ signifies the set of all real numbers and $\mathbb{R}_+ = [0, +\infty)$. Let $(\mathcal{G}, \|\cdot\|)$ be a real Banach space. Moreover, let $\overline{B}(x, r)$ shows the closed ball with center x and radius r. Let $\overline{B}_r$ be the ball $\overline{B}(0, r)$, and $\overline{X}$ and $ConvX$ be the closure and the closed convex hull of X, respectively, for arbitrary $X \subseteq \mathcal{G}$. Furthermore, let $\mathcal{M}_{\mathcal{G}} = \{A \subseteq \mathcal{G} :\ A \text{ is nonempty and bounded }\}$ and $\mathcal{N}_{\mathcal{G}} = \{A \subseteq \mathcal{G} :\ A \text{ is relatively compact }\}$.

Definition 1.1. [4] A mapping $\mu : \mathcal{M}_{\mathcal{G}} \longrightarrow \mathbb{R}_+$ is said to be a measure of noncompactness on $\mathcal{G}$ provided that:

1° The family $ker\mu$ is nonempty and $ker\mu \subseteq \mathcal{N}_{\mathcal{G}}$;

2° $\mu(X) \leq \mu(Y)$ whenever $X \subseteq Y$;

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3° $\mu(\overline{X}) = \mu(X) = \mu(ConvX)$;

4° $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for all $\lambda \in [0,1]$;

5° If $\{X_n = \overline{X_n}\} \subseteq \mathcal{M}_{\mathcal{G}}$ such that $X_{n+1} \subset X_n$ for all $n = 1, 2, \cdots$, and if $\lim_{n\to\infty} \mu(X_n) = 0$, then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$.

The Kuratowski measure of noncompactness [4, 24] is defined by

$$\mu(X) := \inf\Big\{\sigma > 0 : X = \bigcup_{i=1}^{n} X_i, diam(X_i) \leq \sigma\Big\}$$

for bounded subset X of a metric space $\mathcal{G}$, where $diam(X) := \sup\{d(x,y) : x, y \in X\}$.

Theorem 1.2. ([1]) Let Ω be a nonempty, bounded, closed and convex (NBCC) subset of a Banach space $\mathcal{G}$. Then each continuous and compact mapping $F : \Omega \to \Omega$ has at least one fixed point.

Theorem 1.3. (Darbo[14]) Let C be a NBCC subset of a Banach space $\mathcal{G}$ and $T : C \to C$ be a continuous mapping. Assume that $\mu(TX) \leq K\mu(X)$ for any nonempty subset X of C, where $K \in [0,1)$ and μ is a MNC defined in $\mathcal{G}$. Then T has at least a fixed point in C.

Let $\mathcal{G}$ be a Banach space and let $\mathcal{P}(\mathcal{G})$ be the class of all subsets of $\mathcal{G}$. Denote

$$\mathcal{P}_p(\mathcal{G}) = \{A \subseteq \mathcal{G} \mid A \text{ is non-empty and possesses property } p\}.$$

Here, p may be the property $p =$ closed (in short cl), or $p =$ compact (in short cp), or $p =$ convex (in short cv), or $p =$ bounded (in short bd), etc. Thus, $\mathcal{P}_{cl}(\mathcal{G})$, $\mathcal{P}_{cp}(\mathcal{G})$, $\mathcal{P}_{cv}(\mathcal{G})$, $\mathcal{P}_{bd}(\mathcal{G})$, $\mathcal{P}_{cl,bd}(\mathcal{G})$, $\mathcal{P}_{cp,cv}(\mathcal{G})$ denote the classes of all closed, compact, convex, bounded, closed-bounded and compact-convex subsets of $\mathcal{G}$.

A correspondence $Q : \mathcal{G} \to \mathcal{P}_p(\mathcal{G})$ is called a multi-valued operator or multi-valued mapping on $\mathcal{G}$ into $\mathcal{G}$. A point $z \in \mathcal{G}$ is called a fixed point of Q if $z \in Qz$. Throughout this paper, unless otherwise mentioned, assume that $QA = \cup_{a\in A} Qa$ for all $A \subseteq \mathcal{G}$.

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two Banach spaces and $Q : \mathcal{G}_1 \to \mathcal{P}_p(\mathcal{G}_2)$ be a multi-valued mapping. For any subset B of $\mathcal{G}_2$ let:

$$Q^+B = \{x \in \mathcal{G}_1 | Qx \subseteq B\},$$

$$Q^-B = \{x \in \mathcal{G}_1 | Qx \cap B \neq \emptyset\},$$

$$Q^{-1}B = \{x \in \mathcal{G}_1 | \cup_x Qx = B\}.$$

A multi-valued mapping $Q : \mathcal{G}_1 \to \mathcal{P}_p(\mathcal{G}_2)$ is called upper semi-continuous (resp. lower semi-continuous and continuous) if for any open subset U of $\mathcal{G}_2$, the set Q^+U (resp. Q^-U and $Q^{-1}U$) is an open subset of $\mathcal{G}_1$.

The property of upper semi-continuity plays an essential role in the fixed point theory on multi-valued mappings. The first important result in this direction is due to Kakutani [23] which is as follows:

Theorem 1.4. Let Ω be a compact subset of a Banach space $\mathcal{G}$ and let $Q : \Omega \to \mathcal{P}_{cp,cv}(\Omega)$ be an upper semi-continuous multi-valued mapping. Then Q has at least one fixed point.

The following theorem due to Bohnenblust and Karlin is the first generalization of Theorem 1.4.

Theorem 1.5. [12] Let X be a NBCC subset of a Banach algebra $\mathcal{G}$ and let $Q : X \to \mathcal{P}_{cp,cv}(X)$ be an upper semi-continuous multi-valued operator with a relatively compact range. Then Q has a fixed point.

A multi-valued mapping $Q : X \to \mathcal{P}_{cp}(X)$ is called compact if $\overline{Q(X)}$ is a compact subset of X. If $Q : \mathcal{G}_1 \to \mathcal{P}(\mathcal{G}_2)$ be a multi-valued operator, then the graph $Gr(Q)$ is defined by $Gr(Q) = \{(x,y) \in \mathcal{G}_1 \times \mathcal{G}_2 | y \in Tx\}$. The graph $Gr(Q)$ is said to be closed whenever if $\{(x_n, y_n)\}$ be a sequence in $Gr(Q)$ such that $(x_n, y_n) \to (x, y)$, then $(x, y) \in Gr(Q)$.

Definition 1.6. A multi-valued operator $Q : \mathcal{G}_1 \to \mathcal{P}_{cl}(\mathcal{G}_2)$ is called closed if it has a closed graph in $\mathcal{G}_1 \times \mathcal{G}_2$.

The following result concerning the upper semi-continuity of multi-valued mappings in Banach spaces is a useful tool in multi-valued analysis. The details appears in Deimling [15].

Lemma 1.7. A multi-valued operator $Q : \mathcal{G}_1 \to \mathcal{P}_{cl}(\mathcal{G}_2)$ is upper semi-continuous if and only if it is closed and has compact range.

Theorem 1.8. Let X be a NBCC subset of a Banach algebra $\mathcal{G}$ and let $Q : X \to \mathcal{P}_{cp,cv}(X)$ be a compact and closed multi-valued operator. Then Q has a fixed point.

In 2010, Dhage [16] introduced multi-valued D-set-contractions and proved the existence of fixed point for such mappings as follows:

Definition 1.9. A multi-valued operator $Q : \mathcal{G} \to \mathcal{P}_{bd}(\mathcal{G})$ is called a nonlinear $\mathcal{D}$-set-Lipschitz if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(Q(A)) \le \psi(\mu(A))$ for all $A \in \mathcal{P}_{bd}(\mathcal{G})$ with $Q(A) \in \mathcal{P}_{bd}(\mathcal{G})$, where $\psi(0) = 0$. Sometimes the function ψ in the above definition is called a $\mathcal{D}$-function of Q on $\mathcal{G}$. In the special case, when $\psi(r) = kr, 0 < k < 1$, Q is called a k-set-contraction on $\mathcal{G}$. Further, if $\psi(r) < r$ for $r > 0$, then Q is called a nonlinear $\mathcal{D}$-set-contraction on $\mathcal{G}$.

Lemma 1.10. If ψ be a $\mathcal{D}$-function with $\psi(r) < r$ for all $r > 0$, then $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t \in [0, \infty)$.

Theorem 1.11. Let X be a NBCC subset of a Banach space $\mathcal{G}$ and let $Q : X \to \mathcal{P}_{cl,cv}(X)$ be a closed and nonlinear $\mathcal{D}$-set-contraction. Then Q has a fixed point.

Corollary 1.12. Let X be a NBCC subset of a Banach space $\mathcal{G}$ and let $Q : X \to \mathcal{P}_{cl,cv}(X)$ be a closed and nonlinear k-set-contraction. Then Q has a fixed point.

In this paper, we extend the multi-valued version of Darbo's fixed point theorem using generalized Mizogochi-Takahashi mappings of Wardowski type.

2 Main Results

We begin with the definition of a Mizogochi-Takahashi mapping.

Definition 2.1. The function $\beta : [0, \infty) \longrightarrow [0, 1)$ such that $\limsup_{s\to t^+} \beta(s) < 1$ for any $t > 0$, is called a Mizogochi-Takahashi mapping. We denote this class by $\mathcal{MT}$.

Denote by Γ the set of all functions $\gamma : (0, \infty) \to \mathbb{R}$ so that:

(F_1) γ is continuous and increasing;

(F_2) $\lim_{n\to\infty} t_n = 1$ iff $\lim_{n\to\infty} \gamma(t_n) = 0$ for all $\{t_n\} \subseteq (0, \infty)$;

(F_3) $\lim_{n\to\infty} t_n = 0$ iff $\lim_{n\to\infty} \gamma(t_n) = -\infty$ for all $\{t_n\} \subseteq (0, \infty)$.

Note that from (F_2), we have $\gamma(1) = 0$.

Some examples of elements in Γ is as follows:

(i) $\gamma_1(t) = \ln(t)$,

(ii) $\gamma_3(t) = -\frac{1}{\sqrt{t}} + 1$,

(iii) $\gamma_4(t) = -\frac{1}{\sqrt{t}} + t$,

(iv) $\gamma_5(t) = -\frac{1}{t} + t$.

(v) $\gamma_6(t) = -\frac{1}{t} + 1$.

Denote by Ψ the family of all mappings $\psi : [0, \infty) \longrightarrow [0, \infty)$ so that

1. $\psi(s) = 0$ iff $s = 0$;
2. ψ is nondecreasing, continuous and subadditive.

Definition 2.2. A multi-valued operator $Q : \mathcal{G} \to \mathcal{P}_{bd}(\mathcal{G})$ is called a generalized Mizogochi-Takahashi mapping of Wardowski type if there exist functions $\gamma \in \Gamma$, $\psi \in \Psi$ and $\beta \in \mathcal{MT}$ such that

$$\gamma\Big(\psi\big(\mu(QA)\big)\Big) \leq \gamma\Big(\beta\big(\psi(\mu(A))\big)\Big) + \gamma\Big(\psi\big(\mu(A)\big)\Big) \tag{2.1}$$

for all $A \subseteq \mathcal{G}$ with $Q(A) \in \mathcal{P}_{bd}(\mathcal{G})$ and $\mu(QA) > 0$.

Now, we give the main result of this study regarding generalized Mizogochi-Takahashi mappings of Wardowski type.

Theorem 2.3. Let X be a NBCC subset of a Banach space $\mathcal{G}$ and let $Q : X \to \mathcal{P}_{cp,cv}(X)$ be a closed generalized Mizogochi-Takahashi mapping of Wardowski type. Then Q has a fixed point.

Proof . Define a sequence $\{X_n\}$ such that $X_0 = X$ and $X_{n+1} = \overline{Conv(Q(X_n))}$ for all $n = 0, 1, \cdots$.

If there exists an integer $N \in \mathbb{N}$ such that $\mu(X_N) = 0$, then X_N is compact and so Theorem 1.4 implies that Q has a fixed point. So, we assume that $\mu(X_N) > 0$ for each $n \in \mathbb{N}$.

It is clear that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of NBCC sets such that

$$X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq X_{n+1}.$$

On the other hand,

$$\gamma\Big(\psi\big(\mu(X_{n+1})\big)\Big) = \gamma\Big(\psi\big(\mu(QX_n)\big)\Big) \leq \gamma\Big(\beta\big(\psi(\mu(X_n))\big)\Big) + \gamma\Big(\psi\big(\mu(X_n)\big)\Big) < \gamma\Big(\psi\big(\mu(X_n)\big)\Big). \tag{2.2}$$

So, $\Big\{\gamma\Big(\psi\big(\mu(X_n)\big)\Big)\Big\}$ is a positive decreasing and bounded below sequence of real numbers. Since γ is increasing, $\Big\{\psi\big(\mu(X_n)\big)\Big\}$ is a positive decreasing and bounded below sequence of real numbers.

Thus, $\Big\{\psi\big(\mu(X_n)\big)\Big\}_{n\in\mathbb{N}}$ is a convergent sequence. Suppose that $\lim_{n\to\infty} \psi(\mu(X_n)) = r$.

Now, we show that $r = 0$. Suppose that $r > 0$. Taking the limit in (2.2),

$$\gamma(r) \leq \gamma\Big(\limsup_{n\to\infty} \beta\big(\psi(\mu(X_n))\big)\Big) + \gamma(r) < \gamma(r),$$

which is a contradiction. Therefore, we have $r = 0$ and so $\lim_{n\to\infty} \psi\big(\mu(X_n)\big) = 0$.

Since $\psi\big(\mu(X_n)\big)$ is decreasing and ψ is increasing, $\mu(X_n)$ is decreasing. Then there is some $u \geq 0$ so that $\big\{\mu(X_n)\big\}$ converges to u. Since ψ is continuous,

$$\psi(u) = \lim_{n\longrightarrow\infty} \psi\big(\mu(X_n)\big) = r = 0. \tag{2.3}$$

Therefore, $u = 0$, i.e., $\lim_{n\to\infty} \mu(X_n) = 0$. From principle (5°) of Definition 1.6 we derive that the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty. As $\mu(X_\infty) = 0$, thus from axiom 1°, X_∞ is compact. Since $Q : X_n \to \mathcal{P}_{cp,cv}(X_n)$ and $X_{n+1} \subseteq X_n$, for all $n = 0, 1, \cdots$, it is easy to show that $Q : X_\infty \to \mathcal{P}_{cp,cv}(X_\infty)$. Then in view of Theorem **??**, Q has a fixed point. □

Taking $\gamma(t) = \ln(t)$ in Theorem 2.3, we have:

Corollary 2.4. Let X be a NBCC subset of a Banach space $\mathcal{G}$ and let $Q : X \to \mathcal{P}_{cp,cv}(X)$ be a closed multi-valued mapping such that

$$\psi\big(\mu(QA)\big) \leq \beta\Big(\psi\big(\mu(A)\big)\Big)\psi\big(\mu(A)\big) \tag{2.4}$$

for all $A \in \mathcal{P}_{bd}(\mathcal{G})$ with $Q(A) \in \mathcal{P}_{bd}(\mathcal{G})$, where $\beta \in \mathcal{MT}$, $\psi \in \Psi$ and μ is an arbitrary MNC. Then Q has a fixed point.

Taking $\beta(t) = k$ in Theorem 2.3, we have:

Corollary 2.5. Let X be a NBCC subset of a Banach space $\mathcal{G}$ and let $Q : X \to \mathcal{P}_{cp,cv}(X)$ be a closed multi-valued mapping such that there exist $\tau > 0$, a function $\gamma \in \Gamma$ and $\psi \in \Psi$ so that

$$\tau + \gamma\Big(\psi\big(\mu(QA)\big)\Big) \le \gamma\Big(\psi\big(\mu(A)\big)\Big) \tag{2.5}$$

for all $A \in \mathcal{P}_{bd}(\mathcal{G})$ with $Q(A) \in \mathcal{P}_{bd}(\mathcal{G})$ and $\mu(QA) > 0$, where μ is an arbitrary MNC. Then Q has a fixed point.

If ψ be the identity mapping in Corollary 2.5, we derive the following Wardowski type result for multi-valued mappings:

Corollary 2.6. Let X be a NBCC subset of a Banach space $\mathcal{G}$ and let $Q : X \to \mathcal{P}_{cp,cv}(X)$ be a closed multi-valued mapping such that there exist $\tau > 0$ and a function $\gamma \in \Gamma$ so that

$$\tau + \gamma\big(\mu(QA)\big) \le \gamma\big(\mu(A)\big) \tag{2.6}$$

for all $A \in \mathcal{P}_{bd}(\mathcal{G})$ with $Q(A) \in \mathcal{P}_{bd}(\mathcal{G})$ and $\mu(QA) > 0$, where μ is an arbitrary MNC. Then Q has a fixed point.

Taking $\gamma(t) = -\frac{1}{t} + 1$ and ψ the identity function in Theorem 1.4, we have:

Corollary 2.7. Let X be a NBCC subset of a Banach space $\mathcal{G}$ and let $Q : X \to \mathcal{P}_{cp,cv}(X)$ be a closed multi-valued mapping such that

$$\mu\big(Q(X)\big) \le \frac{\beta\big(\mu(X)\big)\mu(X)}{\beta\big(\mu(X)\big) + \mu(X) - \beta\big(\mu(X)\big)\mu(X)} \tag{2.7}$$

for all $A \in \mathcal{P}_{bd}(\mathcal{G})$ with $Q(A) \in \mathcal{P}_{bd}(\mathcal{G})$ and $\mu(QA) > 0$, where $\beta \in \mathcal{MT}$ and μ is an arbitrary MNC. Then Q has a fixed point.

Taking $\gamma(t) = -\frac{1}{t} + 1$, $\beta(t) = \frac{2}{3}$ and ψ the identity function in Theorem 1.4, we have:

Corollary 2.8. Let X be a NBCC subset of a Banach space $\mathcal{G}$ and let $Q : X \to \mathcal{P}_{cp,cv}(X)$ be a closed multi-valued mapping such that

$$\mu\big(QA)\big) \le \frac{\mu(A)}{1 + \frac{1}{2}\mu(A)} \tag{2.8}$$

for all $A \in \mathcal{P}_{bd}(\mathcal{G})$ with $Q(A) \in \mathcal{P}_{bd}(\mathcal{G})$ and $\mu(QA) > 0$, where μ is an arbitrary MNC. Then Q has a fixed point.

3 n-tuplet fixed point

In [17], Erturk and Karakaya studied the existence and uniqueness of fixed points of the operator $F : X^n \to X$ (n-tuplet fixed point), where n is an arbitrary positive integer and X is a partially ordered complete metric space. On the other hand, in [31], some results on the existence of n-tuplet fixed points for multivalued contraction mappings have been proved via a measure of noncompactness. As an application, the existence of solutions for a system of integral inclusions was studied.

Definition 3.1. [31, 13] Let X be a nonempty set and $P : X^n \to \mathcal{P}(X)$ be a multivalued mapping. An element $(x_1, x_2, ..., x_n) \in X$ is called an n-tuplet fixed point of P if

$$\begin{aligned} &x_1 \in P(x_1, x_2, ..., x_{n-1}, x_n);\\ &x_2 \in P(x_2, x_3, ..., x_n, x_1);\\ &\vdots\\ &x_n \in P(x_n, x_1, ..., x_{n-2}, x_{n-1}). \end{aligned} \tag{3.1}$$

Theorem 3.2. [5] If $\mu_1, \mu_2, \ldots, \mu_n$ be measures of noncompactness in Banach spaces $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n$, respectively, and if $f : [0,\infty)^n \longrightarrow [0,\infty)$ be a convex function so that $f(x_1, \ldots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, \ldots, n$, then

$$\tilde{\mu}(X) = f\Big(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n)\Big),$$

defines a measure of noncompactness in $\mathcal{G}_1 \times \mathcal{G}_2 \times \ldots \times \mathcal{G}_n$ where X_i denotes the natural projection of X into $\mathcal{G}_i$, for $i = 1, 2, \ldots, n$.

In the following, we assume that the function $\psi \in \Psi$ is always convex.

Theorem 3.3. Let X be a NBCC subset of a Banach space $\mathcal{G}$ and $P : X^n \to \mathcal{P}_{cp,cv}(X)$ be a continuous function such that

$$\begin{aligned}\gamma\Big(\psi\big(\mu(P(X_1 \times X_2 \times ... \times X_n))\big)\Big) \quad &\le \gamma\Big(\beta\big(\psi(\tfrac{1}{n}\textstyle\sum_{i=1}^n \mu(X_i))\big)\Big) \\ &+\gamma\big(\psi(\tfrac{1}{n}\textstyle\sum_{i=1}^n \mu(X_i))\big)\end{aligned} \tag{3.2}$$

for all subsets $X_1, X_2, ..., X_n$ of X, where $\gamma \in \Gamma$, $\beta \in \mathcal{MT}$, $\psi \in \Psi$ is a convex function and μ is an arbitrary MNC. Then P has at least an n-tuplet fixed point.

Proof . We define the mapping $\widetilde{P} : X^n \to \mathcal{P}_{cp,cv}(X^n)$ by

$$\widetilde{P}(x_1, x_2, ..., x_n) = P(x_1, x_2, ..., x_n) \times P(x_2, x_3, ..., x_n, x_1) \times ... \times P(x_n, x_1, ..., x_{n-1}).$$

It is clear that $\widetilde{P}$ is continuous. Also, in view of Theorem 3.2, $\widetilde{\mu}(A) = \frac{1}{n}\sum_{i=1}^n \mu(A_i)$, for all $A \subseteq \mathcal{P}_{bd}(X^n)$ is a (MNC) on X^n, where $A_1, A_2, ...A_n$ denote the natural projections of A into X. We show that $\widetilde{P}$ satisfies all the conditions of Theorem 2.3. Let $A \subseteq \mathcal{P}_{bd}(X^n)$. From (3.2) we have

$$\begin{aligned}\gamma\{\psi[\widetilde{\mu}(\widetilde{P}(A))]\} &\le \gamma\{\psi[\widetilde{\mu}\big(P(A_1 \times A_2...A_n) \times ... \times P(A_n \times A_1...A_{n-1})\big)]\} \\ &= \gamma\{\psi[\frac{1}{n}\big(\mu(P(A_1 \times A_2...A_n)) + ... + \mu(P(A_n \times A_1...A_{n-1}))\big)]\} \\ &\le \gamma\{\frac{1}{n}[\psi\big(\mu(P(A_1 \times A_2...A_n))\big) + ... + \psi\big(\mu(P(A_n \times A_1...A_{n-1}))\big)]\} \\ &\le \gamma\{\frac{1}{n}[\gamma^{-1}\big(\gamma\big(\beta\big(\psi(\frac{1}{n}\sum_{i=1}^n \mu(X_i))\big)\big) + \gamma\big(\psi(\frac{1}{n}\sum_{i=1}^n \mu(X_i))\big)\big) + ... \\ &+ \gamma^{-1}\big(\gamma\big(\beta\big(\psi(\frac{1}{n}\sum_{i=1}^n \mu(X_i))\big)\big) + \gamma\big(\psi(\frac{1}{n}\sum_{i=1}^n \mu(X_i))\big)\big)]\} \\ &= \gamma\{\gamma^{-1}\big(\gamma\big(\beta\big(\psi(\frac{1}{n}\sum_{i=1}^n \mu(X_i))\big)\big) + \gamma\big(\psi(\frac{1}{n}\sum_{i=1}^n \mu(X_i))\big)\big)\} \\ &= \gamma\big(\beta\big(\psi(\frac{1}{n}\sum_{i=1}^n \mu(X_i))\big)\big) + \gamma\big(\psi(\frac{1}{n}\sum_{i=1}^n \mu(X_i))\big) \\ &= \gamma\big(\beta(\psi(\widetilde{\mu}(X)))\big) + \gamma\big(\psi(\widetilde{\mu}(X))\big).\end{aligned}$$

Now, from Theorem 2.3 we deduce that $\widetilde{P}$ has at least a fixed point which implies that P has at least an n-tuplet fixed point. $\square$

Theorem 3.4. Let X be a NBCC subset of a Banach space $\mathcal{G}$ and $P : X^n \to \mathcal{P}_{cp,cv}(X)$ be a continuous function such that

$$\begin{aligned}\gamma\big(\psi\big(\mu(P(X_1 \times X_2 \times ... \times X_n))\big)\big) \le \quad &\gamma\big(\beta\big(\psi(\max\{\mu(X_1), ..., \mu(X_n)\})\big)\big) \\ &+\gamma\big(\psi(\max\{\mu(X_1), ..., \mu(X_n)\})\big)\end{aligned} \tag{3.3}$$

for all subsets $X_1, X_2, ..., X_n$ of X, where $\gamma \in \Gamma$, $\beta \in \mathcal{MT}$, $\psi \in \Psi$ and μ is an arbitrary MNC. Then P has at least an n-tuplet fixed point.

Proof . We define the mapping $\widetilde{P}: X^n \to \mathcal{P}_{cp,cv}(X^n)$ by

$$\widetilde{P}(x_1, x_2, ..., x_n) = P(x_1, x_2, ..., x_n) \times P(x_2, x_3, ..., x_n, x_1) \times ... \times P(x_n, x_1, ..., x_{n-1}).$$

It is clear that $\widetilde{P}$ is continuous. Also, in view of Theorem 3.2, $\widetilde{\mu}(A) = \max\{\mu(X_1), ..., \mu(X_n)\}$, for all $A \subseteq \mathcal{P}_{bd}(X^n)$ is a (MNC) on X^n, where $A_1, A_2, ...A_n$ denote the natural projections of A into X. We show that $\widetilde{P}$ satisfies all the conditions of Theorem 2.3. Let $A \subseteq \mathcal{P}_{bd}(X^n)$. From (3.3) we have

$$\begin{aligned}
\gamma\big[\psi\big(\widetilde{\mu}\big(\widetilde{P}(A)\big)\big)\big] &\leq \gamma\big[\psi\big(\widetilde{\mu}\big(P(A_1 \times A_2...A_n) \times ... \times P(A_n \times A_1...A_{n-1})\big)\big)\big] \\
&= \gamma\big[\psi\big(\max\{\mu(P(A_1 \times A_2...A_n)), ..., \mu(P(A_n \times A_1...A_{n-1}))\}\big)\big] \\
&= \gamma\big[\max\{\psi\big(\mu(P(A_1 \times A_2...A_n))\big), ..., \psi\big(\mu(P(A_n \times A_1...A_{n-1}))\big)\}\big] \\
&\leq \gamma\big[\max\{\gamma^{-1}\big[\gamma\big(\beta\big(\psi(\max\{\mu(X_1), ..., \mu(X_n)\})\big)\big) + \gamma\big(\psi(\max\{\mu(X_1), ..., \mu(X_n)\})\big)\big], \\
&..., \gamma^{-1}\big[\gamma\big(\beta\big(\psi(\max\{\mu(X_1), ..., \mu(X_n)\})\big)\big) + \gamma\big(\psi(\max\{\mu(X_1), ..., \mu(X_n)\})\big)\big]\}\big] \\
&= \gamma\big[\gamma^{-1}\big[\gamma\big(\beta\big(\psi(\max\{\mu(X_1), ..., \mu(X_n)\})\big)\big) + \gamma\big(\psi(\max\{\mu(X_1), ..., \mu(X_n)\})\big)\big]\big] \\
&= \gamma\big(\beta\big(\psi(\max\{\mu(X_1), ..., \mu(X_n)\})\big)\big) + \gamma\big(\psi(\max\{\mu(X_1), ..., \mu(X_n)\})\big) \\
&= \gamma\big(\beta(\psi(\widetilde{\mu}(X)))\big) + \gamma\big(\psi(\widetilde{\mu}(X))\big).
\end{aligned}$$

Now, from Theorem 2.3, we deduce that $\widetilde{P}$ has at least a fixed point which implies that P has at least an n-tuplet fixed point. □

4 Application to solvability of integral inclusions

Let $\mathbb{R}_+ = [0, +\infty)$, $|.|$ be the Euclidean norm on $\mathbb{R}^n := \mathcal{G}$, $H = \{(\rho, \varrho) \in \mathbb{R}_+ \times \mathbb{R}_+ : \varrho \leq \rho\}$ and $U : H \times \mathcal{G} \to 2^{\mathcal{G}}$ be a multi-valued map. For each $x \in C(\mathbb{R}_+, \mathcal{G})$, which consists of all continuous functions on $\mathbb{R}_+$ with values in $\mathcal{G}$, the set of L^1-selections $S_{U,x}$ of the multivalued map U is defined by

$$S_{U,x} := \Big\{f_x \in L^1(\mathbb{R}_+, \mathcal{G}) : f_x(\rho, \varrho) \in U(\rho, \varrho, x(\varrho)) \ \text{ a.e., for all } \ \rho \geq 0\Big\}.$$

Remark 4.1. $S_{U,x}$ may be empty. It is nonempty if and only if the function $Y : J \to \mathbb{R}$ defined by

$$Y(\varrho) = \inf\{|v| : v \in U(\rho, \varrho, x(\varrho))\}$$

belongs to $L^1(J, \mathbb{R})$ where $J = [0, T]$ with $T > 0$ and $\rho \in \mathbb{R}_+$ is fixed (see, [25]).

We will prove the existence of at least one solution in $C(\mathbb{R}_+, \mathcal{G})$ for the multivalued integral inclusion

$$x(\rho) \in f(\rho, x(\rho)) \int_0^\rho U(\rho, \varrho, x(\varrho))d\varrho, \ \rho \geq 0, \tag{4.1}$$

where $f : \mathbb{R}_+ \times \mathcal{G} \to \mathcal{G}$ is a single-valued map and $U : H \times \mathcal{G} \to 2^{\mathcal{G}}$ is a multi-valued mapping.

To derive the existence of solutions we need some notations and preliminaries. By $BC := BC(\mathbb{R}_+, \mathcal{G})$ we mean the Banach algebra consisting of all bounded and continuous functions defined on $\mathbb{R}_+$ with the norm

$$\|x\|_0 = \sup\{|x(\rho)| : \rho \geq 0\}.$$

Let $L^1(\mathbb{R}_+, \mathcal{G})$ be the Banach space of all measurable functions $x : \mathbb{R}_+ \to \mathcal{G}$ which are Lebesgue integrable with norm

$$\|x\|_1 = \int_0^\infty |x(\rho)|dt.$$

We denote by $\mathcal{P}_{nbcc}(\mathcal{G})$ the set of all NBCC subsets of $\mathcal{G}$. A multivalued map $G : \mathcal{G} \to 2^{\mathcal{G}}$ is said to be convex (closed) if $G(x)$ is convex (closed) for all $x \in C(\mathbb{R}_+, \mathcal{G})$. G is bounded if $G(B) = \cup_{x \in B} G(x)$ is bounded in $\mathcal{G}$ for any bounded subset B of $\mathcal{G}$ (i.e., $\sup_{x \in B}\Big\{\sup\{|y| : y \in G(x)\}\Big\} < \infty$). For the multivalued mapping $U : H \times \mathcal{G} \to 2^{\mathcal{G}}$, by $\|U(\rho, \varrho, x)\|$ we mean the $\sup\{|y| : y \in U(\rho, \varrho, x)\}$. A multivalued map $U : H \times \mathcal{G} \to 2^{\mathcal{G}}$ is said to be L^1-Caratheodory if

(i) $(\rho, \varrho, x) \to U(\rho, \varrho, x)$ is a measurable multivalued map with respect to ϱ for each $\rho \in \mathbb{R}_+$ and $x \in C(\mathbb{R}_+, \mathcal{G})$;

(ii) $(\rho, \varrho, x) \to U(\rho, \varrho, x)$ is an u.s.c. multivalued map with respect to x for each $(\rho, \varrho) \in H$.

Throughout this paper, we always assume that the multivalued map U has nonempty closed values. In the following theorem, we need to add the following hypothesis to the functions f and U.

(h1) $f : \mathbb{R}_+ \times \mathcal{G} \to \mathcal{G}$ is continuous and maps bounded sets into bounded sets, that is, there exists a function $C : \mathbb{R}_+ \to \mathbb{R}_+$ such that $|f(\rho, x)| \leq C(q)$, for all $\rho \geq 0$ and all $x \in C(\mathbb{R}_+, \mathcal{G})$ with $|x| \leq q$.

(h2) $U : H \times \mathcal{G} \to \mathcal{P}_{nbcc}(\mathcal{G})$ is L^1-Caratheodory and the set $S_{U,x}$ is nonempty for each fixed $x \in C(\mathbb{R}_+, \mathcal{G})$.

(h3) There exists a function $\gamma \in \Gamma$ such that $\gamma\big(|f(\rho, x) - f(\rho, y)|\big) \leq \gamma\Big(\beta\big(\psi(|x - y|)\big)\Big) + \gamma\big(\psi(|x - y|)\big)$ for any $\rho \geq 0$ and $x, y \in \mathcal{G}$.

(h4) There exist a bounded function $\alpha \in L^1(\mathbb{R}_+, \mathbb{R}_+)$, a bounded function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ and a nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|U(0, \varrho, x)\| \leq \alpha(\varrho)\phi(|x(\varrho)|)$ for any $\varrho \in \mathbb{R}_+$ and $x \in C(\mathbb{R}_+, \mathcal{G})$. Moreover,

$$|u_x(\rho, \varrho) - u_x(\rho', \varrho)| \leq |\beta(\rho) - \beta(\rho')|\alpha(\varrho)\phi(|x(\varrho)|)$$

for any fixed $x \in C(\mathbb{R}_+, \mathcal{G})$ and for all $u_x \in S_{U,x}$ and $(\rho, \varrho), (\rho', \varrho) \in H$ and

$$\|U(\rho, \varrho, x) - U(\rho, \varrho, y)\| \leq \beta(\rho)\alpha(\varrho)\gamma^{-1}\Big\{\gamma\Big(\beta\big(\psi(|x - y|)\big)\Big) + \gamma\big(\psi(|x - y|)\big)\Big\}$$

for any $x, y \in \mathcal{G}$ with $x \neq y$, $(\rho, \varrho) \in H$ and γ given in $(h3)$.

(h5)

$$[C(q)\beta(\rho) + \phi(q)|\beta(\rho) - \beta(0)| + \phi(q)] \int_0^\rho \alpha(\varrho)d\varrho \leq \frac{1}{2}$$

for each $q \geq 0$ and $\rho \geq 0$.

(h6) There exists $r > 0$ such that

$$2C(r)\phi(r)\|\beta\| \int_0^\infty \alpha(\varrho)d\varrho + C(r)\phi(r) \int_0^\infty \alpha(\varrho)d\varrho \leq r.$$

Theorem 4.2. Assume that conditions $(h1) - (h5)$ are satisfied. Then (4.2) has at least one solution $x \in BC(\mathbb{R}_+, \mathcal{G})$.

Proof . Let us define the multivalued map Q on the space $BC(\mathbb{R}_+, \mathcal{G})$ by the formula

$$(Qx)(\rho) = \Big\{f(\rho, x(\rho)) \int_0^\rho u_x(\rho, \varrho)d\varrho : u_x \in S_{U,x},\ \ \rho \geq 0\Big\}. \tag{4.2}$$

We will show that Q has a fixed point.

Step 1. Set $\overline{B}_r = \Big\{x \in BC(\mathbb{R}_+, \mathcal{G}) : \|x\| \leq r\Big\}$.

We will prove that $Q : \overline{B}_r \to \mathcal{P}_{nbcc}(\overline{B}_r)$. Fix an element $x \in \overline{B}_r$. First, note that for any $y \in Qx$, there exists $u_x \in S_{U,x}$ such that $y(\rho) = f(\rho, x(\rho)) \int_0^\rho u_x(\rho, \varrho)d\varrho$, for all $\rho \geq 0$. It is easy to see that $y(\rho)$ is continuous. Applying our assumptions we have:

$$\begin{aligned}
|y(\rho)| &\leq |f(\rho, x)| \int_0^\rho |u_x(\rho, \varrho)|d\varrho \leq |f(\rho, x)| \int_0^\rho \Big[|u_x(\rho, \varrho) - u_x(0, \varrho)| + |u_x(0, \varrho)|\Big]d\varrho \\
&\leq C(\|x\|_0) \int_0^\rho \Big[|u_x(\rho, \varrho) - u_x(0, \varrho)| + |u_x(0, \varrho)|\Big]d\varrho \\
&\leq C(\|x\|_0)|\beta(\rho) - \beta(0)| \int_0^\rho \alpha(\varrho)\phi(|x(\varrho)|)d\varrho + C(\|x\|_0) \int_0^\rho \alpha(\varrho)\phi(|x(\varrho)|)d\varrho \\
&\leq C(\|x\|_0)\phi(\|x\|_0)|\beta(\rho) - \beta(0)| \int_0^\rho \alpha(\varrho)d\varrho + C(\|x\|_0)\phi(\|x\|_0) \int_0^\rho \alpha(\varrho)d\varrho \\
&\leq 2C(r)\phi(r)\|\beta\| \int_0^\infty \alpha(\varrho)d\varrho + C(r)\phi(r) \int_0^\infty \alpha(\varrho)d\varrho \\
&\leq r.
\end{aligned}$$

This implies that $Qx \subseteq \overline{B}_r$. Therefore Q maps $\overline{B}_r$ into $\mathcal{P}_{bd}(\overline{B}_r)$. Now, we show that Qx is convex. Let $h_1, h_2 \in Qx$. Thus, there exist $u_x, v_x \in S_{U,x}$ such that

$$h_1(\rho) = f(\rho, x(\rho)) \int_0^\rho u_x(\rho, \varrho) d\varrho, \quad h_2(\rho) = f(\rho, x(\rho)) \int_0^\rho v_x(\rho, \varrho) d\varrho$$

for each $\rho \geq 0$. Let $0 \leq k \leq 1$. Then

$$\begin{aligned}\Big(kh_1 + (1-k)h_2\Big)(\rho) &= kf(\rho, x(\rho)) \int_0^\rho u_x(\rho, \varrho) d\varrho + (1-k) f(\rho, x(\rho)) \int_0^\rho v_x(\rho, \varrho) d\varrho \\ &= f(\rho, x(\rho)) \int_0^\rho \Big(ku_x(\rho, \varrho) + (1-k) v_x(\rho, \varrho)\Big) d\varrho,\end{aligned}$$

for each $\rho \geq 0$. Since $S_{U,x}$ is convex (because U has convex values), $ku_x(\rho, \varrho) + (1-k)v_x(\rho, \varrho) \in S_{U,x}$. Therefore, $kh_1 + (1-k)h_2 \in Qx$. Thus, Qx is convex. Obviously, Qx is closed. Hence, we derive the claim of Step 1.

Step 2. Q has closed graph: The proof of this step is identical to the proof of Theorem 1. in [20].

Step 3. Q satisfies the contractive condition (2.1): Fix a bounded set $D \subseteq \overline{B}_r$. Let $q = \sup_{x \in D} \|x\|_0$. Let us choose functions $x, y \in D$ with $x \neq y$ and take $(\rho, \varrho) \in H$. Then, for any $h_1 \in Qx$ and $h_2 \in Qy$ there exist functions $u_x \in S_{U,x}$ and $v_y \in S_{U,y}$ such that

$$h_1(\rho) = f(\rho, x(\rho)) \int_0^\rho u_x(\rho, \varrho) d\varrho, \quad h_2(\rho) = f(\rho, y(\rho)) \int_0^\rho v_y(\rho, \varrho) d\varrho.$$

In view of our assumptions we have

$$\begin{aligned}|h_1(\rho) - h_2(\rho)| \leq & \Big| f\big(\rho, x(\rho)\big) \int_0^\rho u_x(\rho, \varrho) d\varrho - f\big(\rho, x(\rho)\big) \int_0^\rho v_y(\rho, \varrho) d\varrho \Big| \\ & + \Big| f\big(\rho, x(\rho)\big) \int_0^\rho v_y(\rho, \varrho) d\varrho - f\big(\rho, y(\rho)\big) \int_0^\rho v_y(\rho, \varrho) d\varrho \Big| \\ \leq & |f\big(\rho, x(\rho)\big)| \int_0^\rho |u_x(\rho, \varrho) d\varrho - v_y(\rho, \varrho)| d\varrho + |f\big(\rho, x(\rho)\big) - f\big(\rho, y(\rho)\big)| \int_0^\rho |v_y(\rho, \varrho)| d\varrho \\ \leq & C(q) \gamma^{-1} \Big\{ \gamma \Big(\beta \big(\psi(\|x - y\|_0) \big) \Big) + \gamma \big(\psi(\|x - y\|_0) \big) \Big\} \int_0^\rho \alpha(\varrho) \beta(\rho) d\varrho \\ & + \gamma^{-1} \Big\{ \gamma \Big(\beta \big(\psi(\|x - y\|_0) \big) \Big) + \gamma \big(\psi(\|x - y\|_0) \big) \Big\} \int_0^\rho [|v_y(\rho, \varrho) - v_y(0, \varrho)| + |v_y(0, \varrho)|] d\varrho \\ \leq & C(q) \gamma^{-1} \Big\{ \gamma \Big(\beta \big(\psi(\|x - y\|_0) \big) \Big) + \gamma \big(\psi(\|x - y\|_0) \big) \Big\} \int_0^\rho \alpha(\varrho) \beta(\rho) d\varrho \\ & + \gamma^{-1} \Big\{ \gamma \Big(\beta \big(\psi(\|x - y\|_0) \big) \Big) + \gamma \big(\psi(\|x - y\|_0) \big) \Big\} \int_0^\rho [\alpha(\varrho) |\beta(\rho) - \beta(0)| \phi(|y(\varrho)|) + \alpha(\varrho) \phi(|y(\varrho)|)] d\varrho \\ \leq & \gamma^{-1} \Big\{ \gamma \Big(\beta \big(\psi(\|x - y\|_0) \big) \Big) + \gamma \big(\psi(\|x - y\|_0) \big) \Big\} [C(q) \beta(\rho) + \phi(q) |\beta(\rho) - \beta(0)| + \phi(q)] \int_0^\rho \alpha(\varrho) d\varrho \\ \leq & \frac{1}{2} \gamma^{-1} \Big\{ \gamma \Big(\beta \big(\psi(\|x - y\|_0) \big) \Big) + \gamma \big(\psi(\|x - y\|_0) \big) \Big\}.\end{aligned}$$

If $h_1, h_2 \in Qx$, then for any $h \in Qy$, it is easy to see

$$\begin{aligned}|h_1(\rho) - h_2(\rho)| &\leq |h_1(\rho) - h(\rho)| + |h(\rho) - h_2(\rho)| \\ &\leq \gamma^{-1} \Big\{ \gamma \Big(\beta \big(\psi(\|x - y\|_0) \big) \Big) + \gamma \big(\psi(\|x - y\|_0) \big) \Big\}.\end{aligned}$$

Therefore, for any bounded $D \subseteq \overline{B}_r$, we have

$$diamQ(D) \leq \gamma^{-1} \Big\{ \gamma \Big(\beta \big(\psi(diamD) \big) \Big) + \gamma \big(\psi(diamD) \big) \Big\}.$$

For any given $\varepsilon > 0$, there exist a finite number of subsets $D_1, D_2, ..., D_n$ of $\overline{B}_r$ such that

$$D \subseteq \bigcup_{i=1}^{n} D_i, \ \ \mathrm{diam} D_i \leq \mu(D) + \varepsilon,$$

where μ denotes the Kuratowski's measure of noncompactness. Since

$$QD \subseteq \bigcup_{i=1}^{n} QD_i$$

and

$$\begin{aligned} diamQ(D_i) &\leq \gamma^{-1}\Big\{\gamma\Big(\beta\big(\psi(diamD_i)\big)\Big) + \gamma\big(\psi(diamD_i)\big)\Big\} \\ &\leq \gamma^{-1}\Big\{\gamma\Big(\beta\big(\psi(\mu(D)+\varepsilon)\big)\Big) + \gamma\big(\psi(\mu(D)+\varepsilon)\big)\Big\}, \end{aligned}$$

this implies that

$$\mu(Q(D)) \leq \gamma^{-1}\Big\{\gamma\Big(\beta\big(\psi(\mu(D)+\varepsilon)\big)\Big) + \gamma\big(\psi(\mu(D)+\varepsilon)\big)\Big\}.$$

Taking $\varepsilon \to 0$, we have

$$\mu(Q(D)) \leq \gamma^{-1}\Big\{\gamma\Big(\beta\big(\psi(\mu(D))\big)\Big) + \gamma\big(\psi(\mu(D))\big)\Big\},$$

and so

$$\gamma(\psi(\mu\big(Q(D)\big))) \leq \gamma\Big(\beta\big(\psi(\mu(D))\big)\Big) + \gamma\big(\psi(\mu(D))\big).$$

Thus, Q satisfies the contractive condition (2.1). $\square$

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