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Differential subordinations and superordinations result for analytic univalent functions using the Darus-Faisal operator

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Abstract

In this paper, we introduce some differential subordinations and superordinations results for a subclass of analytic univalent functions in the open unit disk U using the Darus-Faisal operator $G_\lambda^m(\sigma, \delta, \tau)$. Also, we study some sandwich theorems.

Keywords: Univalent function, Subordination, Superordination, sandwich, Darus-Faisal operator
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1 Introduction

Let $B = B(U)$ the class of all functions that are analytic in U, where $U = \{z \in C : |z| < 1\}$ is the open unit disk. Let $B[a, n]$ be a subclass of the functions $f \in B$, which is given by

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \quad (a \in \mathbb{C}).$$

We also assume $A \subset B$, where A is said to be subclass of analytic and univalent functions in U, of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U). \tag{1.1}$$

Now, we suppose that f and $g \in A$, so that the function f is said to be subordinate to function g, or the function g is said to be superordinate to f, if there exists a Schwarz function w such that $f(z) = g(w(z))$, where $w(z)$ is analytic function in U with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, then one can say that $f \prec g$ or $f(z) \prec g(z)(z \in U)$ [13]. In addition, if g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ [13, 17, 18].

Definition 1.1. [17] Let $\emptyset : \mathcal{C}^3 \times U \longrightarrow \mathbb{C}$ and let $h(z)$ be univalent in U. If $p(z)$ is analytic function in U and fulfills the second-order differential subordination:

$$\emptyset(p(z), zp'(z), z^2p''(z); z) \prec h(z) \tag{1.2}$$

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then $p(z)$ is said to be a solution of the differential subordination (1.2), and the univalent function $q(z)$ say it a dominant of the solution of the differential subordination (1.2), or more simply dominant, if $p(z) \prec q(z)$ for each $p(z)$ satisfying (1.2). A dominant function $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for each dominant $q(z)$ of (1.2) is called the best dominant of (1.2).

Definition 1.2. [18] Let $p, h \in A$ and $\emptyset(r, s, t; z) : \mathbb{C}^3 \times U \to C$. If p and $\emptyset(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p satisfies the second-order differential subordination:

$$h(z) \prec \emptyset(p(z), zp'(z), z^2p''(z); z) \tag{1.3}$$

then p is said to be a differential superordination solution, (1.3). An analytic function $q(z)$, which is known a subordinat of the solutions of the differential superordination (1.3), or more simply a subordinant if $p \prec q$ for each the functions p satisfying (1.3). If $\tilde{q}$ is univalent subordinant and that satisfy $q \prec \tilde{q}$ for each the subordinats q of (1.3), then is the best subordinat.

Many authors [1, 2, 3, 10, 17, 20, 21] obtained the necessary and sufficient conditions on the functions h, p and $\emptyset$ whereby the following implication is true

$$h(z) \prec \emptyset(p(z), zp'(z), z^2p''(z); z),$$

then

$$q(z) \prec p(z) \tag{1.4}$$

Using results of other authors (see [4, 5, 6, 7, 11, 12, 15, 16, 18, 19, 22]) to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in U and $q_1(0) = q_2(0) = 1$. Also a number of authors look [2, 4, 6, 7, 8, 9] they found some differential subordination and superordination results and sandwich theorems. For $f \in A$, Darus and Faisal [14] introduced the following differential operator:

$$\begin{aligned} G^0_\lambda(\sigma, \delta, \tau)f(z) &= f(z) \\ G^1_\lambda(\sigma, \delta, \tau)f(z) &= \left[\frac{\delta - \tau + \delta - \lambda}{\sigma + \delta}\right] f(z) + \left[\frac{\tau + \lambda}{\sigma + \delta}\right] f'(z) \\ G^2_\lambda(\sigma, \delta, \tau)f(z) &= G(G^1_\lambda(\sigma, \delta, \tau)f(z)) \\ &\vdots \\ G^m_\lambda(\sigma, \delta, \tau)f(z) &= G(G^{n-1}_\lambda(\sigma, \delta, \tau)f(z)). \end{aligned} \tag{1.5}$$

If f is given (1.5), then from (**??**), it can obtained

$$G^m_\lambda(\sigma, \delta, \tau)f(z) = z + \sum_{n=2}^{\infty} \left[\frac{\sigma + (\tau + \lambda)(k - 1) + \delta}{\sigma + \delta}\right]^n a_k z^k, \tag{1.6}$$

where $f \in A; \sigma, \delta, \tau, \lambda \geq 0;\ \sigma + \delta \neq 0;\ n \in N_0$. From (1.6), we note that

$$z(G^m_\lambda(\sigma, \delta, \tau)f(z))' = \left[\frac{\tau + \lambda}{\sigma + \delta}\right] G^{m+1}_\lambda(\sigma, \delta, \tau)f(z) - \left[\frac{\sigma + \delta - \lambda - \tau}{\sigma + \delta}\right] G^m_\lambda(\sigma, \delta, \tau)f(z). \tag{1.7}$$

The main object of the present investigation is to find sufficient conditions for certain normalized analytic function f to satisfy:

$$q_1(z) \prec \left[\frac{G^m_\lambda(\sigma, \delta, \tau)f(z)}{z}\right]^\Upsilon \prec q_2(z),$$

and

$$q_1(z) \prec \left[\frac{G^{m+1}_\lambda(\sigma, \delta, \tau)f(z)}{G^m_\lambda(\sigma, \delta, \tau)f(z)}\right]^\Upsilon \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. In this paper, we derive some sandwich theorems, involving the operator $G^m_\lambda(\sigma, \delta, \tau)f(z)$.

2 Preliminaries

We need the following definitions and lemmas to prove our results.

Definition 2.1. [17] Denote by Q the set of all functions q that are analytic and injective on $\bar{U}\backslash E(q)$, where $\bar{U} = U \cup \{z \in \partial U\}$, therefore

$$E(q) = \{\varepsilon \in \partial U : \lim_{z\to\varepsilon} q(z) = \infty\}$$

and are such that $q'(\varepsilon) \neq 0$ for $\varepsilon \in \partial U\backslash E(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, and $Q(0) = Q_0, Q(1) = Q_1 = \{q \in Q : q(0) = 1\}$.

Lemma 2.2. [18] Let q be a convex univalent function in U and let $\alpha \in \mathbb{C}, \beta \in \mathbb{C}\backslash\{0\}$ with

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -Re\left(\frac{\alpha}{\beta}\right)\right\}.$$

If p is analytic in U and

$$\alpha p(z) + \beta zp'(z) \prec \alpha q(z) + \beta zq'(z), \tag{2.1}$$

then $p \prec q$ and q is the best dominant of (2.1).

Lemma 2.3. [5] Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$, when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that

- $Q(z)$ is starlike univalent in U,
- $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in U$.

If p is analytic in U, with $p(0) = q(0), p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.2}$$

then $p \prec q$ and q is the best dominant of (2.2).

Lemma 2.4. [18] Let q be a convex univalent in U and let $\beta \in \mathbb{C}$, that $Re(\beta) > 0$. If $p \in B[q(0), 1] \cap Q$ and $p(z) + \beta zp'(z)$ is univalent in U, then

$$q(z) + \beta zq'(z) \prec p(z) + \beta zp'(z), \tag{2.3}$$

which implies that $q \prec p$ and q is the best subordinant of (2.3).

Lemma 2.5. [13] Let q be a convex univalent function in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that

- $Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ for $z \in U$.
- $(z) = zq'(z)\phi(q(z))$ is starlike univalent in U.

If $p \in B[q(0), 1] \cap Q$, with $p(U) \subset D, \theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \tag{2.4}$$

then $q \prec p$ and q is the best subordinant of (2.4).

3 Differential Subordination Results

Here, we introduce some differential subordination results by using the Darus-Faisal operator.

Theorem 3.1. Let q be convex univalent function in U with $q(0) = 1, 0 \neq \varepsilon \in \mathbb{C}\backslash\{0\}, \gamma > 0$ and suppose that q satisfies:

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -Re\left(\frac{\gamma}{\varepsilon}\right)\right\} \tag{3.1}$$

If $f \in A$ satisfies the subordination condition:

$$\left[\frac{\tau+\lambda}{\sigma+\delta}\right]\left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma\left(\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^m(\sigma,\delta,\tau)f(z)} - 1\right) + \left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma \prec q(z) + \frac{\varepsilon}{\gamma}zq'(z), \tag{3.2}$$

then

$$\left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma \prec q(z), \tag{3.3}$$

and q is the best dominant of (3.2).

Proof . Define the function p by

$$p(z) = \left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma, \tag{3.4}$$

then the function $p(z)$ is analytic in U and $p(0) = 1$, therefore, differentiating (3.4) with respect to z and using the identity (1.7) in the resulting equation, we obtain

$$\frac{zp'(z)}{p(z)} = \gamma\left[\frac{z(G_\lambda^m(\sigma,\delta,\tau)f(z))'}{G_\lambda^m(\sigma,\delta,\tau)f(z)} - 1\right]. \tag{3.5}$$

Hence,

$$\frac{zp'(z)}{p(z)} = \gamma\left[\left[\frac{\tau+\lambda}{\sigma+\delta}\right]\left(\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^m(\sigma,\delta,\tau)f(z)} - 1\right)\right].$$

Therefore,

$$\frac{zp'(z)}{\gamma} = \left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma\left[\left[\frac{\tau+\lambda}{\sigma+\delta}\right]\left(\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^m(\sigma,\delta,\tau)f(z)} - 1\right)\right].$$

The subordination (3.2) from the hypothesis becomes

$$p(z) + \frac{\varepsilon}{\gamma}zp'(z) \prec q(z) + \frac{\varepsilon}{\gamma}zq'(z).$$

An application of lemma 2.2 with $\beta = \frac{\varepsilon}{\gamma}$ and $\alpha = 1$, we obtain (3.3). □

Putting $q(z) = \left(\frac{1+z}{1-z}\right)$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.2. Let $0 \neq \varepsilon \in \mathbb{C}\backslash\{0\}, \gamma > 0$ and

$$Re\left\{1 + \frac{2z}{1-z}\right\} > \max\left\{0, -Re\left(\frac{\gamma}{\varepsilon}\right)\right\}.$$

If $f \in A$ satisfies the subordination condition:

$$\left[\frac{\tau+\lambda}{\sigma+\delta}\right]\left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma\left(\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^m(\sigma,\delta,\tau)f(z)} - 1\right) + \left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma \prec \left(\frac{1 - z^2 + 2\frac{\varepsilon}{\gamma}z}{(1-z)^2}\right),$$

then

$$\left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma \prec \left(\frac{1+z}{1-z}\right)$$

and $q(z) = \left(\frac{1+z}{1-z}\right)$ is the best dominant.

Theorem 3.3. Let q be a convex univalent function in U with $q(0) = 1, q'(z) \neq 0 (z \in U)$ and assume that q satisfies:

$$Re\left\{1 + \frac{m}{\varepsilon}(q(z))^m + \frac{m-1}{\varepsilon}(q(z))^{m-1} - z\frac{q'(z)}{q(z)} + z\frac{q''(z)}{q'(z)}\right\} > 0, \tag{3.6}$$

where $m \in C, \varepsilon \in \mathbb{C}\backslash\{0\}$ and $z \in U$. Suppose that $z\frac{q'(z)}{q(z)}$ is starlike univalent in U. If $f \in A$ satisfies:

$$\Psi(\gamma, \tau, \delta, \lambda, \theta, k, m; z) \prec (1 + q(z))q(z)^{m-1} + \varepsilon z\frac{q'(z)}{q(z)}, \tag{3.7}$$

where,

$$\Psi(\gamma, \tau, \delta, \lambda, \theta, k, m, \varepsilon; z) = \left[\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}\right]^{\gamma m} + \left[\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}\right]^{\gamma(m-1)} + \varepsilon\gamma\left[\frac{\tau+\lambda}{\sigma+\delta}\right]\left(\frac{G_\lambda^{m+2}(\sigma,\delta,\tau)f(z)}{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)} - \frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}\right), \tag{3.8}$$

then

$$\left[\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}\right]^{\gamma} \prec q(z), \tag{3.9}$$

and q is the best dominant of (3.9).

Proof . Define the function p by

$$p(z) = \left[\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}\right]^{\gamma}, \tag{3.10}$$

then the function $p(z)$ is analytic in U and $p(0) = 1$, differentiating (3.10) with respect to z and using the identity (1.7), we get,

$$\frac{zp'(z)}{p(z)} = \gamma\left[\left[\frac{\tau+\lambda}{\sigma+\delta}\right]\left(\frac{G_\lambda^{m+2}(\sigma,\delta,\tau)f(z)}{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)} - \frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}\right)\right]$$

By setting

$$\theta(w) = (1 + w)w^{m-1} \text{ and } \phi(w) = \frac{\varepsilon}{w},\ w \neq 0.$$

We see that $\theta(w)$ is analytic in $\mathbb{C}$ and $\phi(w)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C}\backslash\{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \varepsilon z\frac{q'(z)}{q(z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = (1 + q(z))q(z)^{m-1} + \varepsilon z\frac{q'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in U, we have

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{1 + \frac{m}{\varepsilon}(q(z))^m + \frac{m-1}{\varepsilon}(q(z))^{m-1} - z\frac{q'(z)}{q(z)} + z\frac{q''(z)}{q'(z)}\right\} > 0.$$

By a straightforward computation, we obtain

$$\Psi(\gamma, \tau, \delta, \lambda, \theta, k, m, \varepsilon; z) = (1 + p(z))(p(z))^{m-1} + \varepsilon z\frac{p'(z)}{p(z)}, \tag{3.11}$$

where $\Psi(\gamma, \tau, \delta, \lambda, \theta, k, m, \varepsilon; z)$ is given by (3.8). From (3.7) and (3.11), we have

$$(1 + p(z))(p(z))^{m-1} + \varepsilon z\frac{p'(z)}{p(z)} \prec (1 + q(z))(q(z))^{m-1} + \varepsilon z\frac{q'(z)}{q(z)}. \tag{3.12}$$

Therefore, by Lemma 2.3, we get $p(z) \prec q(z)$. By using (3.10), we obtain the result. □

Putting $q(z) = \left(\frac{1+\ell z}{1+\mathrm{j}z}\right), (-1 \leq \mathrm{j} < \ell \leq 1)$ in Theorem 3.3, we obtain the following corollary:

Corollary 3.4. Let $-1 \leq \mathrm{j} < \ell \leq 1$ and

$$Re\left\{\frac{m}{\varepsilon}\left(\frac{1+\ell z}{1+\mathrm{j}z}\right)^{m} + \frac{m-1}{\varepsilon}\left(\frac{1+\ell z}{1+\mathrm{j}z}\right)^{m-1} + \frac{1+\mathrm{j}z(4+3\ell z)}{(1+\mathrm{j}z)(1+\ell z)}\right\} > 0,$$

where $\varepsilon \in \mathbb{C}\backslash\{0\}$ and $z \in U$, if $f \in A$ satisfies:

$$\Psi(\gamma,\tau,\delta,\lambda,\theta,k,m,\varepsilon;z) \prec \left[\left[1+\left(\frac{1+\ell z}{1+\mathrm{j}z}\right)\right]\left(\frac{1+\ell z}{1+\mathrm{j}z}\right)^{m-1} + \varepsilon z\frac{\ell-\mathrm{j}}{(1+\ell z)(1+\mathrm{j}z)}\right],$$

where $\Psi(\gamma,\tau,\delta,\lambda,\theta,k,m,\varepsilon;z)$ is given by (3.8), then

$$\left[\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}\right]^{\gamma} \prec \left(\frac{1+\ell z}{1+\mathrm{j}z}\right)$$

and $q(z) = \left(\frac{1+\ell z}{1+\mathrm{j}z}\right)$ is the best dominant.

4 Differential Superordination Results

Theorem 4.1. Let q be convex univalent function in U with $q(0)=1, \gamma > 0$ and $Re\{\varepsilon\} > 0$. Let $f \in A$ satisfies

$$\left[\frac{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}{z}\right]^{\gamma} \in B[q(0),1] \cap Q$$

and

$$\left[\frac{\tau+\lambda}{\sigma+\delta}\right]\left[\frac{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}{z}\right]^{\gamma}\left(\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^{m}(\sigma,\delta,\tau)f(z)} - 1\right) + \left[\frac{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}{z}\right]^{\gamma}$$

be univalent in U. If

$$q(z) + \frac{\varepsilon}{\gamma}zq'(z) \prec \left[\frac{\tau+\lambda}{\sigma+\delta}\right]\left[\frac{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}{z}\right]^{\gamma}\left(\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^{m}(\sigma,\delta,\tau)f(z)} - 1\right) + \left[\frac{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}{z}\right]^{\gamma} \tag{4.1}$$

then

$$q(z) \prec \left[\frac{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}{z}\right]^{\gamma}, \tag{4.2}$$

and q is the best subordinant of (4.1).

Proof . Define the function p by

$$p(z) = \left[\frac{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}{z}\right]^{\gamma}. \tag{4.3}$$

Differentiating (4.3) with respect to z, we get

$$\frac{zp'(z)}{p(z)} = \gamma\left[\frac{z(G_\lambda^{m}(\sigma,\delta,\tau)f(z))'}{G_\lambda^{m}(\sigma,\delta,\tau)f(z)} - 1\right]. \tag{4.4}$$

After some computations and using (1.7), from (4.4), we obtain

$$\left[\frac{\tau+\lambda}{\sigma+\delta}\right]\left[\frac{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}{z}\right]^{\gamma}\left(\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^{m}(\sigma,\delta,\tau)f(z)} - 1\right) + \left[\frac{G_\lambda^{m}(\sigma,\delta,\tau)f(z)}{z}\right]^{\gamma} = p(z) + \frac{\varepsilon}{\gamma}zp'(z)$$

and now, by using Lemma 2.4, we get the desired result. □

Putting $q(z) = \left(\frac{1+z}{1-z}\right)$ in Theorem 4.1, we obtain the following corollary:

Corollary 4.2. Let $\gamma > 0$ and $Re\{\varepsilon\} > 0$. If $f \in A$ satisfies

$$\left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma \in B[q(0),1] \cap Q$$

and

$$\left[\frac{\tau+\lambda}{\sigma+\delta}\right]\left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma\left(\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^m(\sigma,\delta,\tau)f(z)}-1\right)+\left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma$$

be univalent in U. If

$$\left(\frac{1-z^2+2\frac{\varepsilon}{\gamma}z}{(1-z)^2}\right) \prec \left[\frac{\tau+\lambda}{\sigma+\delta}\right]\left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma\left(\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^m(\sigma,\delta,\tau)f(z)}-1\right)+\left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma$$

then

$$\left(\frac{1+z}{1-z}\right) \prec \left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma,$$

and $q(z) = \left(\frac{1+z}{1-z}\right)$ is the best subordinant.

Theorem 4.3. Let q be convex univalent function in U with $q(0) = 1, q'(z) \neq 0$ and assume that q satisfies:

$$Re\left\{\frac{m}{\varepsilon}(q(z))^m q'(z) + \frac{m-1}{\varepsilon}(q(z))^{m-1}q'(z)\right\} > 0, \tag{4.5}$$

where $m \in \mathbb{C}, \varepsilon \in \mathbb{C}\backslash\{0\}$ and $z \in U$. Suppose that $z(q'(z))/(q(z))$ is starlike univalent in U. Let $f \in A$ satisfies:

$$\left[\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^m(\sigma,\delta,\tau)f(z)}\right]^\gamma \in B[q(0),1] \cap Q,$$

and $\Psi(\gamma,\tau,\delta,\lambda,\theta,k,m,\varepsilon;z)$ is univalent function in U, where $\Psi(\gamma,\tau,\delta,\lambda,\theta,k,m,\varepsilon;z)$ is given by (3.8). If

$$(1+q(z))(q(z))^{m-1} + \varepsilon z\frac{q'(z)}{q(z)} \prec \Psi(\gamma,\tau,\delta,\lambda,\theta,k,m,\varepsilon;z), \tag{4.6}$$

then

$$q(z) \prec \left[\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^m(\sigma,\delta,\tau)f(z)}\right]^\gamma, \tag{4.7}$$

and q is the best subordinant of (4.6).

Proof . Define the function p by

$$p(z) = \left[\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^m(\sigma,\delta,\tau)f(z)}\right]^\gamma. \tag{4.8}$$

Differentiating (4.8) with respect to z, we get

$$\frac{zp'(z)}{p(z)} = \gamma\left[\left[\frac{\tau+\lambda}{\sigma+\delta}\right]\left(\frac{G_\lambda^{m+2}(\sigma,\delta,\tau)f(z)}{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)} - \frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^m(\sigma,\delta,\tau)f(z)}\right)\right].$$

By setting

$$\theta(w) = (1+w)w^{m-1} \text{ and } \phi(w) = \frac{\varepsilon}{w},\ w \neq 0,$$

we see that $\theta(w)$ is analytic function in $\mathbb{C}$ and $\phi(w)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C}\backslash\{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \varepsilon z\frac{q'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent function in U,

$$Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = Re\left\{\frac{m}{\varepsilon}(q(z))^m q'(z) + \frac{m-1}{\varepsilon}(q(z))^{m-1}q'(z)\right\} > 0.$$

By a straightforward computation, we obtain

$$\Psi(\gamma,\tau,\delta,\lambda,\theta,k,m,\varepsilon;z)=(1+p(z))(p(z))^{m-1}+\varepsilon z\frac{p'(z)}{p(z)}, \tag{4.9}$$

where $\Psi(\gamma,\tau,\delta,\lambda,\theta,k,m,\varepsilon;z)$ is given by (3.8). From (4.6) and (4.9), we have

$$(1+q(z))(q(z))^{m-1}+\varepsilon z\frac{q'(z)}{q(z)}\prec(1+p(z))(p(z))^{m-1}+\varepsilon z\frac{p'(z)}{p(z)}. \tag{4.10}$$

Therefore, by Lemma 2.5, we get $q(z)\prec p(z)$. □

5 Sandwich Results

Theorem 5.1. Let q_1 be a convex univalent function in U with $q_1(0)=1,\gamma>0$ and $Re\{\varepsilon\}>0$ and q_2 be univalent function U, with $q_2(0)=1$ satisfies (3.1). Let $f\in A$ satisfies:

$$\left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma\in B[1,1]\cap Q,$$

and

$$\left[\frac{\tau+\lambda}{\sigma+\delta}\right]\left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma\left(\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^m(\sigma,\delta,\tau)f(z)}-1\right)+\left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma$$

be univalent in U. If

$$q_1(z)+\frac{\varepsilon}{\gamma}zq_1'(z)\prec\left[\frac{\tau+\lambda}{\sigma+\delta}\right]\left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma\left(\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^m(\sigma,\delta,\tau)f(z)}-1\right)+\left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma\prec q_2(z)+\frac{\varepsilon}{\gamma}zq_2'(z),$$

then

$$q_1(z)\prec\left[\frac{G_\lambda^m(\sigma,\delta,\tau)f(z)}{z}\right]^\gamma\prec q_2(z),$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

Theorem 5.2. Let q_1 be a convex univalent in U with $q_1(0)=1$, and satisfies (4.5). Let q_2 be univalent function in U with $q_2(0)=1$ satisfies (3.6). Let $f\in A$ satisfies:

$$\left[\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^m(\sigma,\delta,\tau)f(z)}\right]^\gamma\in B[1,1]\cap Q,$$

and $\Psi(\gamma,\tau,\delta,\lambda,\theta,k,m,\varepsilon;z)$ is univalent in U, where $\Psi(\gamma,\tau,\delta,\lambda,\theta,k,m,\varepsilon;z)$ is given by (3.8). If

$$(1+q_1(z))(q_1(z))^{m-1}+\varepsilon z\frac{q_1'(z)}{q_1(z)}\prec\Psi(\gamma,\tau,\delta,\lambda,\theta,k,m,\varepsilon;z)\prec(1+q_2(z))(q_2(z))^{m-1}+\varepsilon z\frac{q_2'(z)}{q_2(z)}$$

then

$$q_1(z)\prec\left[\frac{G_\lambda^{m+1}(\sigma,\delta,\tau)f(z)}{G_\lambda^m(\sigma,\delta,\tau)f(z)}\right]^\gamma\prec q_2(z)$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

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