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Numerical solution of the Gardner equation using quartic B-spline collocation method

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Abstract

In this paper, we will consider one numerical solution to solve the nonlinear Gardner equation. The quartic B-spline (QBS) collocation method will be used to determine the unknown term in this equation. In this regard, we apply the quasilinearization technique to linearize the nonlinear terms of the equation and then, combine the QBS collocation method in space with the finite difference in time. This operation provides an efficient explicit solution with high accuracy and minimal computational effort for this problem. It is further proved that the proposed method has the order of convergence $O(k + h^2)$. Also, the method is shown to be unconditionally stable using the Von-Neumann method. Finally, the efficiency and robustness of the proposed approach for solving the nonlinear Gardner equation are demonstrated by one numerical example. These numerical computations will be compared to radial basis functions (RBFs).

Keywords: Gardner equation, Quartic B-spline collocation method, Quasilinearization technique
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1 Introduction

The nonlinear partial differential equations arise in a wide variety of physical problems such as fluid dynamics, plasma physics, solid mechanics, and quantum field theory [1, 6, 18, 29]. Real physical systems show nonlinear and disordered behaviors. Mathematical solutions of the differential equations for these systems are sometimes impossible to manage and some special methods needed to be applied to reach the analytic solutions. One of these types of equations is called the Gardner equation or the combined KdV-mKdV equation, given by [26]

$$u_t + 2\alpha u u_x - 3\beta u^2 u_x + u_{xxx} = 0, \qquad \alpha, \beta > 0, \quad \beta > \alpha, \tag{1.1}$$

where α and β are arbitrary constants, and $u(x, t)$ is the amplitude of the relevant wave mode. Equation (1.1) is completely integrable, like the KdV equation, by the inverse scattering method and is studied in [9, 26] where new kinds of solutions were obtained. The Gardner equation has been investigated in the literature because it is used to model a variety of nonlinear phenomena. This equation is widely used in various branches of physics, such as plasma physics, fluid physics, quantum field theory, hydrodynamics, and theoretical physics [3, 9, 10, 14, 27, 28]. It also describes a variety of wave phenomena in plasma and solid-state [13, 26]. Various methods for studying integrability properties and exact solutions of the Gardner equation have been reported [15, 17, 26].

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The theory of B-spline functions has possessed attention in the literature for the numerical solution of linear and nonlinear boundary value problems in science and engineering. The numerical solution of some partial differential equations can be obtained using B-spline functions of different degrees, [2, 4, 7, 8, 16, 21, 24, 25].

Our main goal in this paper is to solve the Gardner equation using the QBS method. For this purpose, assuming $\Omega_x = (0,1)$ and $\Omega_t = (0, t_{fin})$. We have considered the equation (1.1) in the dimensionless form

$$u_t + 2\alpha u u_x - 3\beta u^2 u_x + u_{xxx} = 0, \qquad \alpha, \beta > 0, \tag{1.2}$$

with the initial condition

$$u(x,0) = f(x), \qquad x \in \bar{\Omega}_x, \tag{1.3}$$

and the boundary conditions

$$u(0,t) = p(t), \qquad u(1,t) = q(t), \qquad u_x(0,t) = h(t), \qquad t \in \bar{\Omega}_t, \tag{1.4}$$

where $f(x)$, $p(t)$, $q(t)$, and $h(t)$ are piecewise known continuous functions, and t_{fin} represents the final existence time for the time evolution of the problem.

The current study aims to clarify the accuracy issues of the quartic B-splines, for solving the Gardner equation (1.2)–(1.4), which can be one of the advantages of our method.

The paper is arranged in the following manner: In the next section, the primary results of the quartic B-splines are presented. In Section 3, the QBS method is detailed for solving the Gardner equation (1.2)–(1.4). The convergence analysis of this method is discussed in Section 4. In Section 5, stability of the method is discussed. A numerical example is reported of the efficiency of the proposed method computationally in Section 6. Finally, the conclusion is made in Section 7.

2 Primary results of QBS

We divide the solution domain $\bar{\Omega}_x$ in to N-subintervals by the set of $N+1$ nodal points x_i, $0 \le i \le N$. This gives a partition $\Pi : 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$ of $\bar{\Omega}_x$, where $h = x_i - x_{i-1}$, for each $1 \le i \le N$. We define the quartic B-spline $B_i(x)$ for $i = -2, 0, \ldots, N+1$ by the following relation

$$B_i(x) = \frac{1}{h^4} \begin{cases} (x - x_{i-2})^4, & x \in [x_{i-2}, x_{i-1}), \\ (x - x_{i-2})^4 - 5(x - x_{i-1})^4, & x \in [x_{i-1}, x_i), \\ (x - x_{i-2})^4 - 5(x - x_{i-1})^4 + 10(x - x_i)^4, & x \in [x_i, x_{i+1}), \\ (x_{i+3} - x)^4 - 5(x_{i+2} - x)^4, & x \in [x_{i+1}, x_{i+2}), \\ (x_{i+3} - x)^4, & x \in [x_{i+2}, x_{i+3}], \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

It can be easily seen that the functions in $\{B_{-2}, B_{-1}, B_0, \ldots, B_N, B_{N+1}\}$ are linearly independent on $\bar{\Omega}_x$. If we consider $\mathcal{B}(\Pi) := \text{span}\{B_{-2}, B_{-1}, B_0, \ldots, B_N, B_{N+1}\} \subseteq C^2(\bar{\Omega}_x)$, then $\mathcal{B}(\Pi)$ is a finite-dimensional linear subspace of $C^2(\bar{\Omega}_x)$ of dimension $N+4$. The values of $B_m(x)$ and its derivatives at the nodal points x_m are given by

$$B_i(x_m) = \begin{cases} 1, & \text{if } i = m-2, \\ 11, & \text{if } i = m-1, \\ 11, & \text{if } i = m, \\ 1, & \text{if } i = m+1, \\ 0, & \text{otherwise}, \end{cases} \qquad B_i'(x_m) = \begin{cases} -\frac{4}{h}, & \text{if } i = m-2, \\ -\frac{12}{h}, & \text{if } i = m-1, \\ \frac{12}{h}, & \text{if } i = m, \\ \frac{4}{h}, & \text{if } i = m+1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

$$B_i''(x_m) = \begin{cases} \frac{12}{h^2}, & \text{if } i = m-2, \\ -\frac{12}{h^2}, & \text{if } i = m-1, \\ -\frac{12}{h^2}, & \text{if } i = m, \\ \frac{12}{h^2}, & \text{if } i = m+1, \\ 0, & \text{otherwise}, \end{cases} \qquad B_i'''(x_m) = \begin{cases} -\frac{24}{h^3}, & \text{if } i = m-2, \\ \frac{72}{h^3}, & \text{if } i = m-1, \\ -\frac{72}{h^3}, & \text{if } i = m, \\ \frac{24}{h^3}, & \text{if } i = m+1, \\ 0, & \text{otherwise}, \end{cases}$$

3 Description of the proposed numerical method

In this section, we present our method based on the QBS functions for solving the equation (1.2). A numerical solution of (1.2) will be derived by using the collocation method based on quartic B-splines. Therefore, an approximation solution $U(x,t)$ to the analytical solution $u(x,t)$ will be desired in the form of an expansion of B-splines,

$$u(x,t) \simeq U(x,t) = \sum_{i=-2}^{N+1} c_i(t) B_i(x) \in \mathcal{B}(\Pi), \tag{3.1}$$

where B_is are the quartic B-splines and c_is are unknown time-dependent quantities to be determined. It is required that approximate solutions (3.1), satisfies equation (1.2)–(1.4) at the uniform mesh (x_m, t_n) to discretize the region $\bar{\Omega}_x \times \bar{\Omega}_t$, where $x_m = mh$ and $t_n = nk$ for $0 \le m \le N$ and $1 \le n \le M$. Also, h and k are space and time steps, respectively.

By using the approximation (3.1) and the quartic B-splines and its derivatives in (2.2), the nodal value U and it's first, second, and third derivatives respect to variable x, at the nodes (x_m, t_n) are obtained as

$$\begin{aligned}
u(x_m,t_n) &\simeq U(x_m,t_n) = c^n_{m-2} + 11c^n_{m-1} + 11c^n_m + c^n_{m+1},\\
u_x(x_m,t_n) &\simeq U_x(x_m,t_n) = -\frac{4}{h}\Big(c^n_{m-2} + 3c^n_{m-1} - 3c^n_m - c^n_{m+1}\Big),\\
u_{xx}(x_m,t_n) &\simeq U_{xx}(x_m,t_n) = \frac{12}{h^2}\Big(c^n_{m-2} - c^n_{m-1} - c^n_m + c^n_{m+1}\Big),\\
u_{xxx}(x_m,t_n) &\simeq U_{xxx}(x_m,t_n) = -\frac{24}{h^3}\Big(c^n_{m-2} - 3c^n_{m-1} + 3c^n_m - c^n_{m+1}\Big).
\end{aligned} \tag{3.2}$$

For solving the equation (1.2), by using the QBS functions, at first, the time derivative is discretized in a forward finite difference approximation

$$u_t(x_m,t_n) = \frac{u(x_m,t_{n+1}) - u(x_m,t_n)}{k}. \tag{3.3}$$

On the other, to linearized the nonlinear terms uu_x and u^2u_x in equation (1.2), we use the quasilinearization technique [5], as follows

$$\begin{aligned}
u(x,t)u_x(x,t) &= u_x(x,t_n)u(x,t_{n+1}) + u(x,t_n)u_x(x,t_{n+1}) - u(x,t_n)u_x(x,t_n),\\
u^2(x,t)u_x(x,t) &= 2u(x,t_n)u_x(x,t_n)u(x,t_{n+1}) - 2u^2(x,t_n)u_x(x,t_n) + u^2(x,t_n)u_x(x,t_{n+1}).
\end{aligned} \tag{3.4}$$

So, by using (3.3) and (3.4), the nonlinear equation (1.2) can be rewritten as

$$\Gamma_1(x_m,t_n)u(x_m,t_{n+1}) + \Gamma_2(x_m,t_n)u_x(x_m,t_{n+1}) = \Gamma(x_m,t_n), \tag{3.5}$$

where,

$$\begin{aligned}
\Gamma_1(x_m,t_n) &= 1 - 6\beta k u(x_m,t_n)u_x(x_m,t_n) + 2\alpha k u_x(x_m,t_n),\\
\Gamma_2(x_m,t_n) &= 2\alpha k u(x_m,t_n) - 3\beta k u^2(x_m,t_n),\\
\Gamma(x_m,t_n) &= k\Big(2\alpha u(x_m,t_n)u_x(x_m,t_n) - 6\beta u^2(x_m,t_n)u_x(x_m,t_n) - u_{xxx}(x_m,t_n)\Big) + u(x_m,t_n).
\end{aligned}$$

Now, substituting the approximate solution U for u and using equations (3.2) in (3.5), yields the following difference equation with the variables c

$$\Gamma_1(x_m,t_n)\Big(c^{n+1}_{m-2} + 11c^{n+1}_{m-1} + 11c^{n+1}_m + c^{n+1}_{m+1}\Big) - \Big(\frac{4}{h}\Big)\Gamma_2(x_m,t_n)\Big(c^{n+1}_{m-2} + 3c^{n+1}_{m-1} - 3c^{n+1}_m - c^{n+1}_{m+1}\Big) = \Gamma(x_m,t_n), \tag{3.6}$$

where,

$$\begin{aligned}
\Gamma_1(x_m,t_n) &= 1 + \frac{24}{h}\beta k\Big(c^n_{m-2} + 11c^n_{m-1} + 11c^n_m + c^n_{m+1}\Big)\Big(c^n_{m-2} + 3c^n_{m-1} - 3c^n_m - c^n_{m+1}\Big)\\
&\quad - \frac{8}{h}\alpha k\Big(c^n_{m-2} + 3c^n_{m-1} - 3c^n_m - c^n_{m+1}\Big),\\
\Gamma_2(x_m,t_n) &= 2\alpha k\Big(c^n_{m-2} + 11c^n_{m-1} + 11c^n_m + c^n_{m+1}\Big) - 3\beta k\Big(c^n_{m-2} + 11c^n_{m-1} + 11c^n_m + c^n_{m+1}\Big)^2,
\end{aligned}$$

and

$$\Gamma(x_m,t_n)=k\Bigg[-\frac{8}{h}\alpha\Big(c^n_{m-2}+11c^n_{m-1}+11c^n_m+c^n_{m+1}\Big)\Big(c^n_{m-2}+3c^n_{m-1}-3c^n_m-c^n_{m+1}\Big)$$
$$+\frac{24}{h}\beta\Big(c^n_{m-2}+11c^n_{m-1}+11c^n_m+c^n_{m+1}\Big)^2\Big(c^n_{m-2}+3c^n_{m-1}-3c^n_m-c^n_{m+1}\Big)$$
$$+\frac{24}{h^3}\Big(c^n_{m-2}-3c^n_{m-1}+3c^n_m-c^n_{m+1}\Big)\Bigg]+\Big(c^n_{m-2}+11c^n_{m-1}+11c^n_m+c^n_{m+1}\Big).$$

The system (3.6), consists of the $(N+1)$ equations in the $(N+4)$ unknown parameters. So, we still need three equations. To this end, we develop the boundary conditions, as follows

$$u(x_0=0,t_{n+1})\simeq U(x_0,t_{n+1})=c^{n+1}_{-2}+11c^{n+1}_{-1}+11c^{n+1}_0+c^{n+1}_1=p(t_{n+1}),$$
$$u(x_N=1,t_{n+1})\simeq U(x_N,t_{n+1})=c^{n+1}_{N-2}+11c^{n+1}_{N-1}+11c^{n+1}_N+c^{n+1}_{N+1}=q(t_{n+1}),$$
$$u_x(x_0=0,t_{n+1})\simeq U_x(x_0,t_{n+1})=-\frac{4}{h}\Big(c^{n+1}_{-2}+3c^{n+1}_{-1}-3c^{n+1}_0-c^{n+1}_1\Big)=h(t_{n+1}).$$

Therefore, the system (3.6) is changed to a system of $(N+4)$ linear equations in $(N+4)$ unknowns parameters, given by

$$\mathcal{A}X^{n+1}-\mathcal{B}X^{n+1}=\mathcal{F}, \tag{3.7}$$

where,

$$\mathcal{A}=\begin{pmatrix}
1 & 11 & 11 & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0\\
-\frac{4}{h} & -\frac{12}{h} & \frac{12}{h} & \frac{4}{h} & 0 & 0 & \cdots & \cdots & \cdots & 0\\
\Gamma_1(x_0,t_n) & 11\Gamma_1(x_0,t_n) & 11\Gamma_1(x_0,t_n) & \Gamma_1(x_0,t_n) & 0 & 0 & \cdots & \cdots & \cdots & 0\\
0 & \Gamma_1(x_1,t_n) & 11\Gamma_1(x_1,t_n) & 11\Gamma_1(x_1,t_n) & \Gamma_1(x_1,t_n) & 0 & \cdots & \cdots & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & \cdots & \cdots & 0 & \Gamma_1(x_N,t_n) & 11\Gamma_1(x_N,t_n) & 11\Gamma_1(x_N,t_n) & \Gamma_1(x_N,t_n)\\
0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 11 & 11 & 1
\end{pmatrix},$$

$$\mathcal{B}=\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0\\
\frac{4}{h}\Gamma_2(x_0,t_n) & \frac{12}{h}\Gamma_2(x_0,t_n) & -\frac{12}{h}\Gamma_2(x_0,t_n) & -\frac{4}{h}\Gamma_2(x_0,t_n) & 0 & 0 & \cdots & \cdots & \cdots & 0\\
0 & \frac{4}{h}\Gamma_2(x_1,t_n) & \frac{12}{h}\Gamma_2(x_1,t_n) & -\frac{12}{h}\Gamma_2(x_1,t_n) & -\frac{4}{h}\Gamma_2(x_1,t_n) & 0 & \cdots & \cdots & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & \cdots & \cdots & 0 & \frac{4}{h}\Gamma_2(x_N,t_n) & \frac{12}{h}\Gamma_2(x_N,t_n) & -\frac{12}{h}\Gamma_2(x_N,t_n) & -\frac{4}{h}\Gamma_2(x_N,t_n)\\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\mathcal{F}=\begin{pmatrix} p(t_{n+1})\\ h(t_{n+1})\\ \Gamma(x_0,t_n)\\ \Gamma(x_1,t_n)\\ \vdots\\ \Gamma(x_N,t_n)\\ q(t_{n+1})\end{pmatrix},\qquad X^{n+1}=\begin{pmatrix} c_{-2}^{n+1}\\ c_{-1}^{n+1}\\ c_{0}^{n+1}\\ c_{1}^{n+1}\\ \vdots\\ c_{N}^{n+1}\\ c_{N+1}^{n+1}\end{pmatrix}.$$

Equation (3.7) is a linear system, which can be easily and efficiently solved. To solve this system, we need the initial vector $X^0 = \begin{pmatrix} c_{-2}^0 & c_{-1}^0 & c_0^0 & c_1^0 & \cdots & c_N^0 & c_{N+1}^0 \end{pmatrix}^T$, which can be obtained from the procedure of Subsection 3.1.

3.1 The initial vector X^0

We can determine the initial vector X^0 by using the initial and boundary conditions (1.3) and (1.4), as the following expression

$$\begin{aligned}
&u(x_m,t_0)\simeq U(x_m,t_0)=c_{m-2}^0+11c_{m-1}^0+11c_m^0+c_{m+1}^0=f(x_m), && 0\le m\le N,\\
&u(x_0=0,t_0)\simeq U(x_0,t_0)=c_{-2}^0+11c_{-1}^0+11c_0^0+c_1^0=p(t_0),\\
&u(x_N=1,t_0)\simeq U(x_N,t_0)=c_{N-2}^0+11c_{N-1}^0+11c_N^0+c_{N+1}^0=q(t_0),\\
&u_x(x_0=0,t_0)\simeq U_x(x_0,t_0)=-\frac{4}{h}\Big(c_{-2}^0+3c_{-1}^0-3c_0^0-c_1^0\Big)=h(t_0).
\end{aligned}$$

This system is the form

$$AX^0=B, \tag{3.8}$$

where

$$A=\begin{pmatrix}
1 & 11 & 11 & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0\\
-\frac{4}{h} & -\frac{12}{h} & \frac{12}{h} & \frac{4}{h} & 0 & 0 & \cdots & \cdots & \cdots & 0\\
1 & 11 & 11 & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0\\
0 & 1 & 11 & 11 & 1 & 0 & \cdots & \cdots & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 11 & 11 & 1\\
0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 11 & 11 & 1
\end{pmatrix},\qquad B=\begin{pmatrix} p(t_0)\\ h(t_0)\\ f(x_0)\\ f(x_1)\\ \vdots\\ \vdots\\ f(x_N)\\ q(t_0)\end{pmatrix}.$$

From equation (3.8), the initial vector X^0 can be successively calculated.

4 Convergence analysis

In this section, we analyze the convergence of our proposed scheme. First, we need to recall a theorem.

Theorem 4.1. [11] Suppose that $u(x)\in C^5(\bar{\Omega}_x)$ and for all $x\in\bar{\Omega}_x$, $|u^{(5)}(x)|\le\mathcal{L}$. Also, assume that

$$\Pi: 0=x_0<x_1<\cdots<x_{N-1}<x_N=1,$$

be the partition of $\bar{\Omega}_x$ with step size h. If $\mathcal{S}_\Pi(x)$ be the unique spline function interpolate $u(x)$ at nodes $x_0,\ldots,x_N\in\Pi$, then, there exist constants $\lambda_j\le 2$ such that

$$\left\|u^{(j)}-\mathcal{S}_\Pi^{(j)}\right\|_\infty\le\lambda_j\mathcal{L}h^{5-j},\qquad j=0,1,2,3,4. \tag{4.1}$$

Now, we state and prove the following convergence theorem.

Theorem 4.2. The collocation approximation $U(x,t)$ from the space $\mathcal{B}(\Pi)$ for the solution $u(x,t)$ of the problem (1.2)-(1.4) satisfy the following error estimate

$$\Big\|u-U\Big\|_\infty \le \gamma h^2,$$

for sufficiently small h (i.e. for sufficiently large N) where γ is a positive constant.

Proof . Let $u(x,t)$ be the exact solution of the problem (1.2)-(1.4). Also, we set

$$U(x,t)=\sum_{i=-2}^{N+1} c_i(t)B_i(x),$$

to be B-spline collocation approximation to $u(x,t)$. Due to round off errors in computations, we assume that $\tilde{U}(x,t)$ be the computed spline for $U(x,t)$ so that

$$\tilde{U}(x,t)=\sum_{i=-2}^{N+1} \tilde{c}_i(t)B_i(x).$$

To estimate the error $\|u-U\|_\infty$, it is needed to estimate the errors $\|u-\tilde{U}\|_\infty$ and $\|\tilde{U}-U\|_\infty$, separately. Following (3.7) for $\tilde{U}$ we have

$$\mathcal{A}\tilde{X}^{n+1}-\mathcal{B}\tilde{X}^{n+1}=\tilde{\mathcal{F}}, \tag{4.2}$$

where

$$\tilde{X}^{n+1}=\Big[\tilde{c}_{-2}^{n+1},\tilde{c}_{-1}^{n+1},\tilde{c}_{0}^{n+1},\tilde{c}_{1}^{n+1},\dots,\tilde{c}_{N}^{n+1},\tilde{c}_{N+1}^{n+1}\Big]^T,$$
$$\tilde{\mathcal{F}}=\Big[p(t_{n+1}),h(t_{n+1}),\tilde{\Gamma}(x_0,t_n),\tilde{\Gamma}(x_1,t_n),\dots,\tilde{\Gamma}(x_N,t_n),q(t_{n+1})\Big]^T.$$

By subtracting (3.7) and (4.2) we have

$$\Big(\mathcal{A}-\mathcal{B}\Big)\Big(X^{n+1}-\tilde{X}^{n+1}\Big)=\Big(\mathcal{F}-\tilde{\mathcal{F}}\Big). \tag{4.3}$$

On the other hand

$$\mathcal{F}-\tilde{\mathcal{F}}=\Big[0,0,\Gamma(x_0,t_n)-\tilde{\Gamma}(x_0,t_n),\Gamma(x_1,t_n)-\tilde{\Gamma}(x_1,t_n),\dots,\Gamma(x_N,t_n)-\tilde{\Gamma}(x_N,t_n),0\Big]^T, \tag{4.4}$$

such that for every $0\le m\le N$,

$$\Gamma(x_m,t_n)=k\Bigg[\phi\Big(x_m,t_n,(c_{m-2}^n+11c_{m-1}^n+11c_m^n+c_{m+1}^n),(-\frac{4}{h})(c_{m-2}^n+3c_{m-1}^n-3c_m^n-c_{m+1}^n)\Big)$$
$$+\frac{24}{h^3}\Big(c_{m-2}^n-3c_{m-1}^n+3c_m^n-c_{m+1}^n\Big)\Bigg]+\Big(c_{m-2}^n+11c_{m-1}^n+11c_m^n+c_{m+1}^n\Big),$$

$$\tilde{\Gamma}(x_m,t_n)=k\Bigg[\phi\Big(x_m,t_n,(\tilde{c}_{m-2}^n+11\tilde{c}_{m-1}^n+11\tilde{c}_m^n+\tilde{c}_{m+1}^n),(-\frac{4}{h})(\tilde{c}_{m-2}^n+3\tilde{c}_{m-1}^n-3\tilde{c}_m^n-\tilde{c}_{m+1}^n)\Big)$$
$$+\frac{24}{h^3}\Big(\tilde{c}_{m-2}^n-3\tilde{c}_{m-1}^n+3\tilde{c}_m^n-\tilde{c}_{m+1}^n\Big)\Bigg]+\Big(\tilde{c}_{m-2}^n+11\tilde{c}_{m-1}^n+11\tilde{c}_m^n+\tilde{c}_{m+1}^n\Big),$$

where

$$\phi\Big(x_m,t_n,(c_{m-2}^n+11c_{m-1}^n+11c_m^n+c_{m+1}^n),(-\frac{4}{h})(c_{m-2}^n+3c_{m-1}^n-3c_m^n-c_{m+1}^n)\Big)=$$
$$-\frac{8}{h}\alpha\Big(c_{m-2}^n+11c_{m-1}^n+11c_m^n+c_{m+1}^n\Big)\Big(c_{m-2}^n+3c_{m-1}^n-3c_m^n-c_{m+1}^n\Big)$$
$$+\frac{24}{h}\beta\Big(c_{m-2}^n+11c_{m-1}^n+11c_m^n+c_{m+1}^n\Big)^2\Big(c_{m-2}^n+3c_{m-1}^n-3c_m^n-c_{m+1}^n\Big),$$

and

$$\begin{aligned}&\phi\Big(x_m,t_n,(\tilde{c}^n_{m-2}+11\tilde{c}^n_{m-1}+11\tilde{c}^n_m+\tilde{c}^n_{m+1}),(-\frac{4}{h})(\tilde{c}^n_{m-2}+3\tilde{c}^n_{m-1}-3\tilde{c}^n_m-\tilde{c}^n_{m+1})\Big)=\\&-\frac{8}{h}\alpha\Big(\tilde{c}^n_{m-2}+11\tilde{c}^n_{m-1}+11\tilde{c}^n_m+\tilde{c}^n_{m+1}\Big)\Big(\tilde{c}^n_{m-2}+3\tilde{c}^n_{m-1}-3\tilde{c}^n_m-\tilde{c}^n_{m+1}\Big)\\&+\frac{24}{h}\beta\Big(\tilde{c}^n_{m-2}+11\tilde{c}^n_{m-1}+11\tilde{c}^n_m+\tilde{c}^n_{m+1}\Big)^2\Big(\tilde{c}^n_{m-2}+3\tilde{c}^n_{m-1}-3\tilde{c}^n_m-\tilde{c}^n_{m+1}\Big).\end{aligned}$$

So,

$$\Big|\mathcal{F}(x_m)-\tilde{\mathcal{F}}(x_m)\Big|=\Big|k\Big[\phi(x_m,U(x_m),U'(x_m))-\phi(x_m,\tilde{U}(x_m),\tilde{U}'(x_m))\Big]-k[U'''(x_m)-\tilde{U}'''(x_m)]+[U(x_m)-\tilde{U}(x_m)]\Big|.$$

By following Theorem 4.1 and [19] (page 218) we obtain

$$\begin{aligned}\Big\|\mathcal{F}-\tilde{\mathcal{F}}\Big\|_\infty&\le k\mathcal{M}\bigg(\Big|U(x_m)-\tilde{U}(x_m)\Big|+\Big|U'(x_m)-\tilde{U}'(x_m)\Big|\bigg)+k\lambda_3\mathcal{L}h^2+\lambda_0\mathcal{L}h^5\\&\le k\mathcal{M}\lambda_0\mathcal{L}h^5+k\mathcal{M}\lambda_1\mathcal{L}h^4+k\lambda_3\mathcal{L}h^2+\lambda_0\mathcal{L}h^5\\&=h^2\Big(k\mathcal{M}\lambda_0\mathcal{L}h^3+k\mathcal{M}\lambda_1\mathcal{L}h^2+k\lambda_3\mathcal{L}+\lambda_0\mathcal{L}h^3\Big)\end{aligned}\tag{4.5}$$

where $\|\phi'\|_\infty\le\mathcal{M}$. Therefore, we can rewrite (4.5) as follows

$$\Big\|\mathcal{F}-\tilde{\mathcal{F}}\Big\|_\infty\le w^*h^2,\tag{4.6}$$

where $w^*=k\mathcal{M}\lambda_0\mathcal{L}h^3+k\mathcal{M}\lambda_1\mathcal{L}h^2+k\lambda_3\mathcal{L}+\lambda_0\mathcal{L}h^3$. Also, it is obvious that the matrix $(\mathcal{A}-\mathcal{B})$ in (4.3) is a nonsingular matrix, thus we have

$$X^{n+1}-\tilde{X}^{n+1}=\Big(\mathcal{A}-\mathcal{B}\Big)^{-1}\Big(\mathcal{F}-\tilde{\mathcal{F}}\Big).$$

Taking the infinity norm, then using (4.6), one can deduce that

$$\Big\|X^{n+1}-\tilde{X}^{n+1}\Big\|_\infty=\Big\|(\mathcal{A}-\mathcal{B})^{-1}\Big\|_\infty\Big\|\mathcal{F}-\tilde{\mathcal{F}}\Big\|_\infty\le\overline{w}h^2,$$

where $\overline{w}=w^*\Big\|(\mathcal{A}-\mathcal{B})^{-1}\Big\|_\infty$. Now, we compute $\Big\|u-U\Big\|_\infty$ as the following

$$\Big\|u-U\Big\|_\infty\le\Big\|u-\tilde{U}\Big\|_\infty+\Big\|\tilde{U}-U\Big\|_\infty.\tag{4.7}$$

From Theorem 4.1 we have

$$\Big\|u-\tilde{U}\Big\|_\infty\le\lambda_0\mathcal{L}h^5.\tag{4.8}$$

Also,

$$U(x)-\tilde{U}(x)=\sum_{i=-2}^{N+1}(c_i-\tilde{c}_i)B_i(x),$$

thus,

$$\Big|U(x_m)-\tilde{U}(x_m)\Big|\le\max_{-2\le i\le N+1}\Big|c_i-\tilde{c}_i\Big|\,\Big|\sum_{i=-2}^{N+1}B_i(x_m)\Big|,\qquad 0\le m\le N.$$

By using the values of $B_i(x_m)$'s given in Section 2, one can easily see that $\Big|\sum_{i=-2}^{N+1}B_i(x_m)\Big|\le 35$, $0\le m\le N$ (see, [20, Lemma 2.1]), therefore

$$\Big\|U-\tilde{U}\Big\|_\infty\le 35\overline{w}h^2.\tag{4.9}$$

So, by substituting (4.8) and (4.9) in (4.7), we obtain

$$\Big\|u-U\Big\|_\infty \le \lambda_0\mathcal{L}h^5+35\overline{w}h^2 = h^2\Big(\lambda_0\mathcal{L}h^3+35\overline{w}\Big).$$

Setting $\gamma=\lambda_0\mathcal{L}h^3+35\overline{w}$, we have

$$\Big\|u-U\Big\|_\infty \le \gamma h^2.$$

This completes the proof. □

Theorem 4.3. Let $u(x,t)$ be the solution of the initial boundary value problem (1.2)-(1.4). Also, suppose that $U(x)$ is the collocation approximation of the solution $u(x)$ after the temporal discretization stage. Then, the error estimate of the discrete scheme is given by

$$\Big\|u-U\Big\|_\infty \le \vartheta(k+h^2),$$

where ϑ is some finite constant.

Proof . The time discretization process (3.3) that we use to discretize the system (1.2)-(1.4) in the time variable is of the one order convergence (see, [22]). So, according to Theorem 4.2, we have

$$\Big\|u-U\Big\|_\infty \le \vartheta(k+h^2),$$

where ϑ is some finite constant. □

Remark 4.4. According to Theorem 4.3, the order of convergence of our process is $O(k+h^2)$.

5 Stability analysis

Von-Neumann stability method [23] is used for the stability of scheme developed in Section 3. Being applicable to only linear schemes, the nonlinear term uu_x and u^2u_x is linearized by taking u as a locally constant value δ. The linearized form of proposed scheme is given as

$$c_{m-2}^{n+1}+11c_{m-1}^{n+1}+11c_{m}^{n+1}+c_{m+1}^{n+1}=\theta_2c_{m-2}^{n}+\theta_3c_{m-1}^{n}+\theta_4c_{m}^{n}+\theta_5c_{m+1}^{n}, \tag{5.1}$$

where

$$\theta_2=1+\frac{4}{h}\theta_1k+\frac{24}{h^3}k, \qquad \theta_3=11+\frac{12}{h}\theta_1k-\frac{72}{h^3}k,$$
$$\theta_4=11-\frac{12}{h}\theta_1k+\frac{72}{h^3}k, \qquad \theta_5=1-\frac{4}{h}\theta_1k-\frac{24}{h^3}k,$$
$$\theta_1=2\alpha\delta-3\beta\delta^2.$$

Substitution of $c_m^n=\xi^ne^{im\varphi}$, $i=\sqrt{-1}$, into equation (5.1) leads to

$$\xi\Big[1+11e^{i\varphi}+11e^{2i\varphi}+e^{3i\varphi}\Big]=\theta_2+\theta_3e^{i\varphi}+\theta_4e^{2i\varphi}+\theta_5e^{3i\varphi}. \tag{5.2}$$

Simplifying equation (5.2), we get

$$\xi=\frac{\mathcal{X}_1+i\mathcal{Y}_1}{\mathcal{X}_2-i\mathcal{Y}_2},$$

where

$$\mathcal{X}_1=\theta_2+\theta_3\cos(\varphi)+\theta_4\cos(2\varphi)+\theta_5\cos(3\varphi), \qquad \mathcal{Y}_1=\theta_3\sin(\varphi)+\theta_5\sin(3\varphi)+\theta_4\sin(2\varphi),$$
$$\mathcal{X}_2=1+11\cos(\varphi)+11\cos(2\varphi)+\cos(3\varphi), \qquad \mathcal{Y}_2=11\sin(\varphi)+\sin(3\varphi)+11\sin(2\varphi).$$

A method is stable if $|\xi|\le 1$. Since,

$$\mathcal{X}_1^2+\mathcal{Y}_1^2-\mathcal{X}_2^2-\mathcal{Y}_2^2=\frac{128\,k^2\,(\cos(x)-1)\,\big(6\,\cos(x)-6\,\beta\,\delta^2\,h^2+4\,\alpha\,\delta\,h^2-3\,\beta\,\delta^2\,h^2\,\cos(x)+2\,\alpha\,\delta\,h^2\,\cos(x)-6\big)^2}{h^6}\le 0,$$

$|\xi|\le 1$ is provided. Hence, the scheme is unconditionally stable.

6 Numerical test

In this section, we demonstrate the effectiveness of the proposed method for solving the Gardner equation (1.2)–(1.4). To show the ability, rigidity, and proficiency of the presented method, we use the root-mean-square error (RMSE) and the infinity-norm of absolute error to reveal the accuracy of the method, using following formulas

$$\text{RMSE} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N-1}\Big(u(x_i,t)-U(x_i,t)\Big)^2},$$
$$L_\infty = ||u(x,t)-U(x,t)||_\infty = \max_{1\le i\le N-1}|u(x_i,t)-U(x_i,t)|.$$

Numerical results are compared with the RBF method (Multiquadrics-RBF) [12]. The proposed method is written in the MATLAB R2015b and is tested on a personal computer with Intel(R) Core(TM)2 Duo CPU and 4GB RAM. In the following numerical example, we take $\alpha = 1$, $\beta = 3$, $t_{fin} = 1$, $h = 0.1$, and $k = 0.0001$.

Example 6.1. We consider the Gardner equation (1.2)–(1.4) satisfying,

$$u_t + 2uu_x - 9u^2u_x + u_{xxx} = 0, \qquad x \in \Omega_x, \quad t \in \Omega_t.$$

The exact solution is given as [26]

$$u(x,t) = \frac{1}{9}\left[1-\tanh\Big(\frac{1}{3\sqrt{6}}(x-\frac{2}{27}t)\Big)\right], \qquad x \in \bar{\Omega}_x, \quad t \in \bar{\Omega}_t,$$

Tables 1 and 2, compare the RMSE and the infinity-norm for QBS and RBF methods at the different values of time $t = 0.1$, 0.5, 0.8, and 1. Figures 1 and 2, show the difference between the $u(x,t)$ and $U(x,t)$ in QBS and RBF methods at $x = 0.2, 0.7$, respectively.

Table 1: Comparison among the exact and numerical solutions for $u(x,t)$ at times $t = 0.1, 0.5$.

	$t=0.1$			$t=0.5$		
x	$u(x,t)$	QBS	RBF	$u(x,t)$	QBS	RBF
0.1	0.109711	0.109722	0.099315	0.110159	0.110163	0.088315
0.2	0.108200	0.108213	0.094104	0.108647	0.068081	0.068086
0.3	0.106689	0.106704	0.092479	0.107137	0.107146	0.053876
0.4	0.105181	0.105197	0.091064	0.105627	0.105640	0.045539
0.5	0.103674	0.103690	0.087693	0.104120	0.104134	0.041860
0.6	0.102170	0.102185	0.082503	0.102616	0.102630	0.042436
0.7	0.100670	0.100681	0.077840	0.101114	0.101127	0.047992
0.8	0.099173	0.099178	0.077041	0.099616	0.099626	0.059368
0.9	0.097681	0.097676	0.082809	0.098122	0.098128	0.076247
RMSE	–	$1.262832e-05$	$1.693915e-02$	–	$1.043002e-05$	$4.829083e-02$
L_∞	–	$1.634052e-05$	$2.282962e-02$	–	$1.399932e-05$	$6.225995e-02$

Table 2: Comparison among the exact and numerical solutions for $u(x,t)$ at times $t = 0.8, 1$.

	$t = 0.8$			$t = 1$		
x	$u(x,t)$	QBS	RBF	$u(x,t)$	QBS	RBF
0.1	0.110495	0.110497	0.082581	0.110719	0.110735	0.080184
0.2	0.108983	0.108989	0.055607	0.109207	0.109223	0.050408
0.3	0.107472	0.107481	0.037616	0.107696	0.107712	0.031032
0.4	0.105963	0.105974	0.029442	0.106186	0.106201	0.023240
0.5	0.104455	0.104467	0.028814	0.104678	0.104690	0.024165
0.6	0.102950	0.102962	0.033549	0.103172	0.103179	0.030742
0.7	0.101447	0.101459	0.042943	0.101670	0.101668	0.041629
0.8	0.099949	0.099958	0.057151	0.100170	0.100157	0.056764
0.9	0.098454	0.098460	0.075765	0.098675	0.098647	0.075825
RMSE	–	$9.379007e-06$	$5.837684e-02$	–	$1.535156e-05$	$6.226015e-02$
L_∞	–	$1.251616e-05$	$7.652068e-02$	–	$2.821895e-05$	$8.294669e-02$

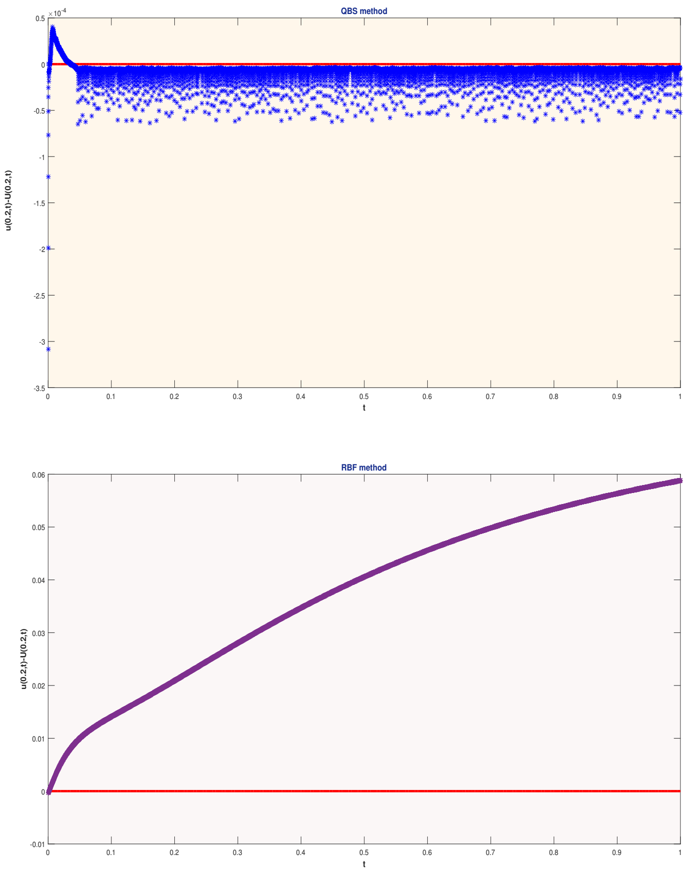
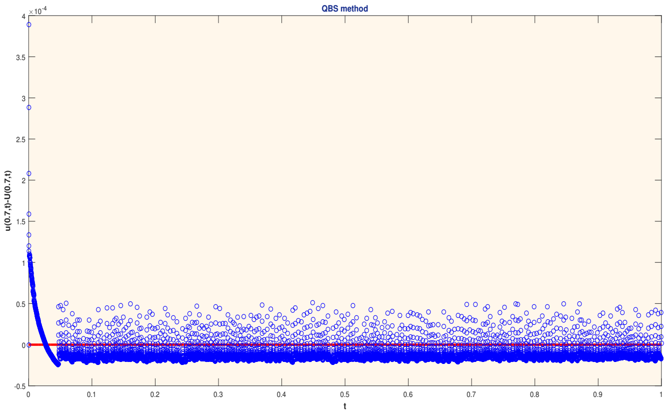


Figure 1: Difference between the $u(0.2,t)$ and $U(0.2,t)$ in QBS and RBF methods.



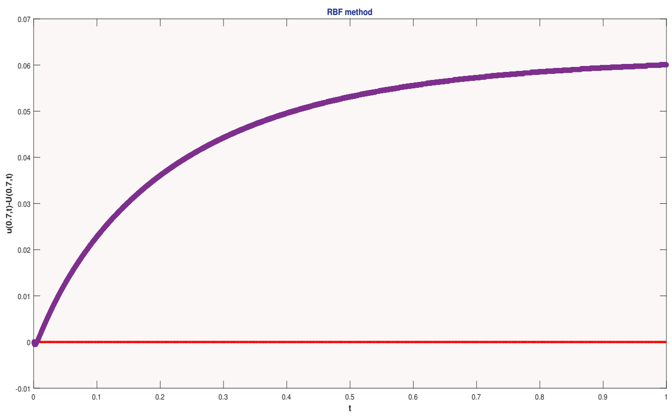


Figure 2: Difference between the $u(0.7, t)$ and $U(0.7, t)$ in QBS and RBF methods.

CPU time consumption in Matlab for the QBS method is 338.568102 and for the RBF method is 1283.362879 seconds. These computational results show that our proposed method (QBS) is effective and accurate in comparison with RBF. Also, the QBS method is superior to the RBF method due to the smaller CPU time.

7 Conclusion

In this paper, we used the QBS method to solve the Gardner equation (1.2)–(1.4). The convergence analysis of the proposed method has been discussed and shown that the order of convergence of our process is $O(k + h^2)$. The obtained results show the accuracy of this method and its stability compared to accurate solutions. We also compared this method with the RBF method to show the effectiveness of the QBS method. The results of this comparison also confirm the accuracy of the method. Compared to the execution time of the program, the QBS method has a better speed with low-storage space. Given that the Gardner equation is nonlinear, the QBS method can be considered a suitable method for solving such equations.

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