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Convergence theorems for a general class of nonexpansive mappings in Banach spaces

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Abstract

In this paper, we introduce a new iteration process for the approximation of fixed points. We show that our iteration process is faster than the existing iteration processes like the M-iteration process and the K-iteration process for contraction mappings. Also, we prove that the new iteration process is stable. Finally, we study the convergence of a new iterative scheme to a fixed point for the (α, β) -Reich-Suzuki nonexpansive type mappings in Banach space.

Keywords: Fixed point, convergence, Opial's condition, generalized nonexpansive mappings
2020 MSC: Primary 47H09; Secondary 47H10

1 Introduction and Preliminaries

Let be X be a real Banach space and K be a nonempty subset of X, and $T : K \to K$ be a mapping. A point $x \in K$ is called a fixed point of $T : K \to K$ if $x = Tx$. We denote $F(T)$ the set of a fixed points of T. A mapping $T : K \to K$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. T is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $x \in K$ and $p \in F(T)$.

Once the existence of a fixed point of some mapping is established, an algorithm to calculate the value of the fixed point is desired. Many iterative processes have been developed to approximate fixed point. The well-known Banach contraction theorem use Picard iteration process [21] for approximation of fixed point. Some of the other well-known iterative processes Mann iterative scheme [16], Ishikawa [14], Noor [17], Agarwal et al. [2], Abbas and Nazir[1], Picard-S [13], Thakur et al. [27], Ullah and Arshad [28], Hussain et al [10] and so on.

In the recent years, several generalizations of nonexpansive mappings and related fixed point have have been studied by many authors [see e.g [4], [11], [19], [25], [29]]. In 2008, Suzuki [25] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called *Condition(C)*. Let K be a nonempty convex subset of a Banach space X, a mapping $T : K \to K$ satisfies *Condition(C)* if for all $x, y \in K$, $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$ implies $\|Tx - Ty\| \leq \|x - y\|$. Suzuki [25] showed that the mapping satisfying *Condition(C)* is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. The mapping satisfying *Condition(C)* is also referred to as Suzuki generalized nonexpansive mapping. Lately, fixed point theorems for Suzuki generalized nonexpansive mappings have been studied by a number of authors (see e.g [8], [26]-[28]). In 2011, an existence theorem for a fixed point of an α-nonexpansive mapping T of a nonempty bounded, closed and convex subset K of a uniformly convex Banach space X has been established by Aoyama and Kohsaka [4] with a non-constructive argument. In

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2011, Garsia-Falset et al. [11] introduced two new classes of generalized nonexpansive mappings which are wider than those satisfying *Condition(C)*, but preserving their fixed point properties. They investigated generalized nonexpansive mappings , named as *Condition(E)*. Later in 2019, Pandey et al. [19] studied some fixed point results for a general class of nonexpansive mappings which are not necessarily continuous on their domains. They showed that there are other classes of nonexpansive type mappings which can be considered as special cases of the class of mappings satisfying the *Condition(E)*. In 2018, Amini-Harandi et al. [3] suggested a two parametric class of nonlinear mappings. They defined the class of (α, β)-nonexpansive mappings which is properly larger than the class of α-nonexpansive mappings. They also established some basic results for this class. In 2020, Ullah et al. [29] introduced a general class of generalized nonexpansive mappings which properly includes the class of Suzuki nonexpansive mappings, Reich–Suzuki type nonexpansive mappings, and generalized α-nonexpansive mappings. They also established some basic properties, existence and convergence results for this class of mappings in the context of uniformly convex Banach spaces.

Inspired and motivated by these facts, we consider this class of nonexpansive type mappings which properly contains, the Reich-Suzuki nonexpansive mappings, generalized α-nonexpansive mappings and Suzuki generalized nonexpansive mappings. Further we prove the convergence theorems of new iterative process to fixed point for the (α, β)-Reich-Suzuki nonexpansive type mappings in Banach space.

In this section, next we give some preliminaries. We recall some well-known definitions and lemmas.

A Banach space X will be said to be uniformly convex [6] if for each ε, $\varepsilon \in (0, 2]$, there corresponds a $\delta(\varepsilon) > 0$ such that the conditions $\|x\| = \|y\| = 1$, $\|x - y\| \geq \varepsilon$ imply $\frac{\|x+y\|}{2} \leq 1 - \delta(\varepsilon)$.

Recall that a Banach space X is said to satisfy *Opial's condition* [18] if, for each sequence $\{x_n\}$ in X, the condition $x_n \to x$ weakly as $n \to \infty$ and for all $y \in X$ with $y \neq x$ imply that

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|.$$

Lemma 1.1. ([25]) Let T be a mapping on a subset K of a Banach space X with Opial's condition. Assume that T satisfies *Condition(C)*. If $\{x_n\}$ converges weakly to p and $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$,then $Tp = p$. That is, $I - T$ (I is identity mapping) is demiclosed at zero.

Lemma 1.2. ([25]) Let T be a mapping on a weakly compact convex subset K of uniformly convex Banach space X. Assume that T satisfies *Condition(C)*, then T has a fixed point.

Some preliminaries to the terms asymptotic radius and asymptotic center are given, which are attributed by Edelstein [7], in the paragraphs that follow.

Let $\{x_n\}$ be a bounded sequence in a Banach space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|.$$

The asymptotic radius of $\{x_n\}$ relative to K is defined by

$$r(K, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\}.$$

The asymptotic center of $\{x_n\}$ relative to K is the set

$$A(K, \{x_n\}) = \{x \in K : r(x, \{x_n\}) = r(K, \{x_n\})\}.$$

It is known that, in uniformly convex Banach space, $A(K, \{x_n\})$ consists of exactly one-point.

Lemma 1.3. [23] Suppose that X is a uniformly convex Banach space and $0 < k \leq t_n \leq m < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequence of X such that $\limsup\limits_{n\to\infty}\|x_n\| \leq r$, $\limsup\limits_{n\to\infty}\|y_n\| \leq r$ and $\limsup\limits_{n\to\infty}\|t_n x_n + (1 - t_n)y_n\| = r$ hold for $r \geq 0$. Then $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$.

Let $\{u_n\}$ in K be a given sequence. $T : K \to K$ with the nonempty fixed point set $F(T)$ in K is said to satisfy *Condition(I)* [24] with respect to the $\{u_n\}$ if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|u_n - Tu_n\| \geq f(d(u_n, F(T)))$ for all $n \geq 1$.

2 The new iteration process

Let be X be a real Banach space and K be a nonempty subset of X, and $T : K \to K$ be a mapping. $\{a_n\}, \{b_n\}$ are real sequences in $[0, 1]$.

In 2018, Ullah and Arshad [28] introduced the following iteration process called M-iteration process : for arbitrary $x_0 \in K$ construct a sequence $\{x_n\}$ by

$$\begin{cases} z_n = (1 - a_n)x_n + a_n T x_n, \\ y_n = T z_n, \\ x_{n+1} = T y_n. \end{cases} \tag{2.1}$$

Later, in 2018 Hussain et al. [10] introduced the following iteration process called K-iteration process : for arbitrary $x_0 \in K$ construct a sequence $\{x_n\}$ by

$$\begin{cases} z_n = (1 - b_n)x_n + b_n T x_n, \\ y_n = T((1 - a_n)T x_n + a_n T z_n), \\ x_{n+1} = T y_n. \end{cases} \tag{2.2}$$

Motivated by above, in this paper, we introduce a new iteration scheme:for arbitrary $x_0 \in K$ construct a sequence $\{x_n\}$ by

$$\begin{cases} z_n = T((1 - a_n)x_n + a_n T x_n), \\ y_n = T(T z_n), \\ x_{n+1} = T(T y_n). \end{cases} \tag{2.3}$$

Numerically we compare the speed of convergence of our new iteration process with Ullah and Arshad [28] and Hussain et al. [10] iteration processes. In order to show that our new iteration process (2.3) have a good speed of convergence comparatively to (2.1) and (2.2) , we consider the following examples.

Example 2.1. Let us define a function $T : [0, 10] \to [0, 10]$ by $T(x) = \sqrt{2x + 3}$. Then clearly T is a contraction map. Let $a_n = 0.75, b_n = 0.75$ for all n. Set the stop parameter to $\|x_n - 3\| \leq 10^{-15}$, 3 is the fixed point of T. The iterative values for initial value $x_0 = 6$ are given in Table 1. Figure 1 shows the convergence graph. The efficiency of new iteration process is clear. We can see that our new iteration process (2.3) have a good speed of convergence comparatively to (2.1) and (2.2) iteration processes.

Table 1: Sequences generated by M-iteration, K-iteration and New iteration processes for mapping T of Example 2.1.

	M-iteration	K-iteration	New iteration
x_0	6.000000000000000	6.000000000000000	6.000000000000000
x_1	3.142130383387586	3.058322910060707	3.005205080654434
x_2	3.007824545198772	3.001344515493123	3.000010706183078
x_3	3.000434476696481	3.000031120073780	3.000000022029167
x_4	3.000024136914555	3.000000720370487	3.000000000045327
x_5	3.000001340937600	3.000000016675242	3.000000000000094
x_6	3.000000074496527	3.000000000386001	3.000000000000000
x_7	3.000000004138696	3.000000000008935	3.000000000000000
x_8	3.000000000229928	3.000000000000207	3.000000000000000
x_9	3.000000000012774	3.000000000000005	3.000000000000000
x_{10}	3.000000000000710	3.000000000000000	3.000000000000000
x_{11}	3.000000000000040	3.000000000000000	3.000000000000000
x_{12}	3.000000000000002	3.000000000000000	3.000000000000000
x_{13}	3.000000000000000	3.000000000000000	3.000000000000000

3 Stability for new iterative process

In this section, we prove that the new iteration process (2.3) is stable. Before proving it, we give the following well-known definitions and lemma.

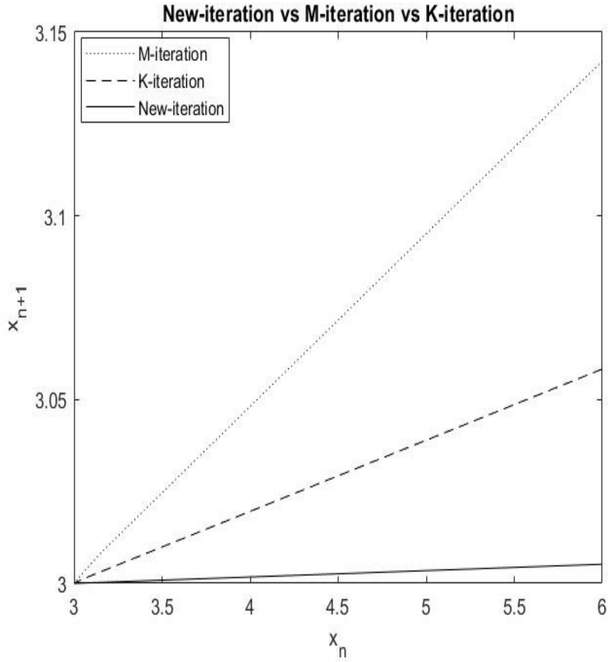

Figure 1: Convergence of M-iteration, K-iteration and New iteration processes to the fixed point 3 of the mapping defined in Example 2.1.

A point p is called fixed point of a mapping T if $Tp = p$, and $F(T)$ represents the set of all fixed points of a mapping T. Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is called contraction if there exists $\theta \in (0,1)$ such that $\|Tx - Ty\| \leq \theta\|x - y\|$, for all $x, y \in K$.

Lemma 3.1. [5] Let $\{\epsilon_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the inequality

$$s_{n+1} \leq \delta_n s_n + \epsilon_n, \tag{3.1}$$

where $\delta_n \in [0,1)$, for $n = 0, 1, 2....$ If $\lim\limits_{n\to\infty} \epsilon_n = 0$, then $\lim\limits_{n\to\infty} s_n = 0$.

Harder and Hicks [9] introduced the following concept of T-stability :

Definition 3.2. [9] Let $\{t_n\}_{n=0}^{\infty}$ be an arbitrary sequence in K. Then , an iteration procedure

$$\left\{ \quad x_{n+1} = f(T, x_n), for\ n = 0, 1, 2... \right.$$

is said to be T-stable or stable with respect to T for some function f, converging to fixed point p, if $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$ for $n = 0, 1, 2, ...$, we have $\lim\limits_{n\longrightarrow\infty} \epsilon_n = 0 \Leftrightarrow \lim\limits_{n\longrightarrow\infty} t_n = p$.

Theorem 3.3. Let K be a nonempty closed convex subset of a uniformly convex Banach space X , T be a contraction mapping with $F(T) \neq \emptyset$. For arbitrary chosen $x_0 \in K$, $\{x_n\}$ be a sequence generated by (2.3) with real sequences $\{a_n\} \in [0,1]$. Then iteration process (2.3) is (T)-stable.

Proof . It follows from (2.3), we have,

$$\begin{aligned}
\|z_n - p\| &= \|T((1-a_n)x_n + a_n T x_n) - p\| \\
&\leq \theta\|(1-a_n)(x_n - p) + a_n(Tx_n - p)\| \\
&\leq \theta[(1-a_n)\|x_n - p\| + a_n\theta\|x_n - p\|] \\
&= \theta[1 - a_n(1-\theta)]\|x_n - p\|.
\end{aligned}$$

Then we get

$$\|z_n - p\| \le \theta[1 - a_n(1-\theta)]\|x_n - p\|. \tag{3.2}$$

Similarly, using (2.3) and (3.2), we get

$$\begin{aligned}\|y_n - p\| &= \|T(Tz_n) - p\|\\ &\le \|\theta[(Tz_n - p)]\|\\ &\le \theta[\theta\|z_n - p\|]\\ &= \theta^2[\theta[1 - a_n(1-\theta)]]\|x_n - p\|.\end{aligned}$$

So we get

$$\|y_n - p\| \le \theta^3[1 - a_n(1-\theta)]\|x_n - p\|. \tag{3.3}$$

By using from (2.3) and (3.3) , we get

$$\begin{aligned}\|x_{n+1} - p\| &= \|T(Ty_n) - p\|\\ &\le \theta\|Ty_n - p\|\\ &\le \theta^2[\theta^3(1 - a_n(1-\theta)]\|x_n - p\|.\end{aligned}$$

Thus we get

$$\|x_{n+1} - p\| \le \theta^5[1 - a_n(1-\theta)]\|x_n - p\|. \tag{3.4}$$

By repeating the above process, we get

$$\begin{aligned}\|x_n - p\| &\le \theta^5[1 - a_{n-1}(1-\theta)]\|x_{n-1} - p\|\\ &.\\ &.\\ &.\\ &\le \theta^5[1 - a_1(1-\theta)]\|x_1 - p\|\\ &\le \theta^5[1 - a_0(1-\theta)]\|x_0 - p\|.\end{aligned}$$

Therefore, we obtain

$$\|x_{n+1} - p\| \le \theta^{5(n+1)}\prod_{k=0}^{n}[1 - a_k(1-\theta)]\|x_0 - p\|.$$

Now, $\theta < 1$ so $1 - \theta > 0$ and $a_n \le 1$ for $n = 0, 1, 2, ...$ Then we have $[(1 - a_{n-1}(1-\theta)] < 1$ for $n = 0, 1, 2....$ So, we know that $1 - x \le e^{-x}$ for all $x \in [0, 1]$. Hence we have

$$\|x_{n+1} - p\| \le \theta^{5(n+1)}e^{-(1-\theta)\sum_{k=0}^{n}[a_k]}\|x_0 - p\|. \tag{3.5}$$

Now we prove that the new iteration defined by (2.3) is stable with respect to (T).

Let $\{t_n\}$ be any arbitrary sequence in K. $t_{n+1} = f(T, t_n)$ is the sequence generated by (2.3) and $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$ for $n = 0, 1, 2, ...$

We have to prove that $\lim_{n\longrightarrow\infty} \epsilon_n = 0 \Leftrightarrow \lim_{n\longrightarrow\infty} t_n = p$.

Suppose $\lim_{n\longrightarrow\infty} \epsilon_n = 0$. By using (2.3) and (3.5), we get

$$\begin{aligned}\|t_{n+1} - p\| &\le \|t_{n+1} - f(T, t_n)\| + \|f(T, t_n) - p\| \\ &\le \epsilon_n + \theta^5[1 - a_n(1-\theta)]\|t_n - p\|.\end{aligned} \tag{3.6}$$

We can easily seen that all conditions of all conditions of Lemma 3.1 are fulfilled by above inequality (3.6). Hence, by Lemma 3.1, we get $\lim_{n\longrightarrow\infty} t_n = p$.

Conversely, let $\lim_{n\longrightarrow\infty} t_n = p$, we have

$$\begin{aligned}
\epsilon_n &= \|t_{n+1} - f(T, t_n)\| \\
&\leq \|t_{n+1} - p\| + \|f(T, t_n) - p\| \\
&\leq \|t_{n+1} - p\| + \theta^5[1 - a_n(1-\theta)]\|t_n - p\|.
\end{aligned}$$

This implies that $\lim_{n\longrightarrow\infty} \epsilon_n = 0$. Hence (2.3) is stable with respect to T. □

4 Generalized nonexpansive type mappings

In 1973, Kannan [15] considered a class of mappings satisfying the following condition:

$$\|Tx - Ty\| \leq \frac{1}{2}(\|Tx - x\| + \|Ty - y\|); x, y \in K. \tag{4.1}$$

A mapping satisfying (4.1) is known as Kannan nonexpansive mapping. We note that Kannan nonexpansive mappings are independent of nonexpansive mappings and need not be continuous. Gregus [12] combined nonexpansive and Kannan nonexpansive mappings as follows:

$$\|Tx - Ty\| \leq a\|x - y\| + b\|Tx - x\| + c\|Ty - y\|; x, y \in K, \tag{4.2}$$

where a, b, c are nonnegative constants such that $a + b + c = 1$. A mapping satisfying (4.2) is known as Reich nonexpansive mapping. If $a + b + c < 1$, then the mapping satisfying (4.2) is called Reich contraction mapping[22].

Lemma 4.1. (1) If T is nonexpansive then T satisfies *Condition(C)* [[25], Proposition 1],

(2) If T satisfies *Condition(C)* and has a fixed point , then T is a quasi-nonexpansive mapping [[25], Proposition 2],

(3) If T satisfies *Condition(C)*, then $\|x - Ty\| \leq 3\|Tx - x\| + \|x - y\|$ for all $x, y \in K$ [[25], Lemma 7].

In [20], authors considered the following class of nonexpansive type mappings and obtained some fixed point results for this class of mappings.

Definition 4.2. [20]. A mapping $T : K \to K$ is called a generalized α-nonexpansive mapping if there exists an $\alpha \in [0, 1)$ and for each $x, y \in K$,

$$\begin{aligned}
&\frac{1}{2}\|x - Tx\| \leq \|x - y\| \text{ implies} \\
&\|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\|.
\end{aligned} \tag{4.3}$$

Lemma 4.3. [20] Let K be nonempty subset of a Banach space X and $T : K \to K$ a generalized α-nonexpansive mapping. Then for each $x, y \in K$,

$$\|x - Ty\| \leq (\frac{3+\alpha}{1-\alpha})\|Tx - x\| + \|x - y\|.$$

In [20], authors considered the following class of nonexpansive type mappings and obtained some fixed point results for this class of mappings.

Definition 4.4. [20]. A mapping $T : K \to K$ is called an α- Reich-Suzuki nonexpansive mapping if there exists an $\alpha \in [0, 1)$ and for each $x, y \in K$,

$$\begin{aligned}
&\frac{1}{2}\|x - Tx\| \leq \|x - y\| \text{ implies} \\
&\|Tx - Ty\| \leq \alpha\|Tx - x\| + \alpha\|Ty - y\| + (1 - 2\alpha)\|x - y\|.
\end{aligned} \tag{4.4}$$

Lemma 4.5. [20] Let K be nonempty subset of a Banach space X and $T : K \to K$ be an α-Reich-Suzuki nonexpansive mapping.Then for each $x, y \in K$,

$$\|x - Ty\| \leq (\frac{3+\alpha}{1-\alpha})\|Tx - x\| + \|x - y\|.$$

Next, Ullah et al. [29] introduced the following concepts of the generalized (α, β)-nonexpansive type mappings in Banach space.

Definition 4.6. [29] Let K be a nonempty subset of a Banach space X, $T : K \to K$ be a generalized (α, β)-nonexpansive type mapping, for $\alpha + \beta \in (0, 1)$ and all $x, y \in K$,

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \textit{ implies } \|Tx - Ty\| \leq D(x, y), (\textit{say}$$

Condition (D)), where

$$\begin{aligned} D(x, y) &= \alpha\|Tx - y\| + \alpha\|Ty - x\| + \beta\|Tx - x\| \\ &\quad +\beta\|Ty - y\| + (1 - 2\alpha - 2\beta)\|x - y\|. \end{aligned}$$

Lemma 4.7. [29] Let $T : K \to K$ be the generalized (α, β)-nonexpansive type mappings on K. Then the following statements hold. For any $x, y \in K$,

(1) $\|Tx - T^2x\| \leq \|x - Tx\|$,

(2) Either $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$ or $\frac{1}{2}\|Tx - T^2x\| \leq \|Tx - y\|$

(3) Either $\|Tx - Ty\| \leq D(x, y)$ or $\|T^2x - Ty\| \leq E(x, y)$, where

$$\begin{aligned} E(x, y) &= \alpha\|T^2x - y\| + \alpha\|Ty - Tx\| + \beta\|T^2x - Tx\| \\ &\quad +\beta\|Ty - y\| + (1 - 2\alpha - 2\beta)\|Tx - y\|. \end{aligned}$$

Now we establish some basic properties for this class of mappings in Banach spaces.

Lemma 4.8. Let K be nonempty subset of a Banach space X and $T : K \to K$ be a generalized (α, β)-nonexpansive type mapping. Then for each $x, y \in K$,

$$\|x - Ty\| \leq (\frac{3 - \beta + \alpha^2 + \alpha\beta}{1 - \beta - \alpha^2 - \alpha\beta})\|Tx - x\| + \|x - y\|.$$

Proof . From Lemma 4.7 , for all $x, y \in K$ either

$$\begin{aligned} \|Tx - Ty\| &\leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + \beta\|Tx - x\| \\ &+ \beta\|Ty - y\| + (1 - 2\alpha - 2\beta)\|x - y\| \end{aligned}$$

or

$$\begin{aligned} \|T^2x - Ty\| &\leq \alpha\|T^2x - y\| + \alpha\|Ty - Tx\| \\ &+ \beta\|T^2x - Tx\| + \beta\|Ty - y\| + (1 - 2\alpha - 2\beta)\|Tx - y\|. \end{aligned}$$

In the first case, we have

$$\begin{aligned} \|x - Ty\| &= \|x - Tx\| + \|Tx - Ty\| \\ &\leq \|x - Tx\| + \alpha\|Tx - y\| + \alpha\|Ty - x\| \\ &+ \beta\|Tx - x\| + \beta\|Ty - y\| + (1 - 2\alpha - 2\beta)\|x - y\| \\ &\leq (1 + \alpha + \beta)\|x - Tx\| + (\alpha + \beta)\|x - Ty\| + (1 - \alpha - \beta)\|x - y\|. \end{aligned}$$

For $\alpha+\beta \in (0,1)$,

$$(1-\alpha-\beta)\|x-Ty\| \leq (1+\alpha+\beta)\|x-Tx\| + (1-\alpha-\beta)\|x-y\|.$$

This implies, for $\alpha+\beta \in (0,1)$,

$$\|x-Ty\| \leq \frac{(1+\alpha+\beta)}{(1-\alpha-\beta)}\|x-Tx\| + \|x-y\|.$$

In the other case, we have

$$\|x-Ty\| = \|x-Tx\| + \|Tx-T^2x\| + \|T^2x-Ty\|. \tag{4.5}$$

Let's find the last two statements separately in (4.5). From the second statement in the summation of (4.5), we have

$$\begin{aligned} \|T^2x-Tx\| &\leq \alpha\|T^2x-x\| + \beta\|T^2x-Tx\| \\ &+ \beta\|Tx-x\| + (1-2\alpha-2\beta)\|Tx-x\|. \end{aligned} \tag{4.6}$$

$$\|T^2x-x\| \leq \|T^2x-Tx\| + \|Tx-x\|. \tag{4.7}$$

If it is written at (4.6), (4.7), we have

$$\begin{aligned} \|T^2x-Tx\| &\leq \alpha\|T^2x-Tx\| + \alpha\|Tx-x\| + \beta\|T^2x-Tx\| \\ &\quad +\beta\|Tx)-x\| + (1-2\alpha-2\beta)\|Tx-x\|. \end{aligned}$$

This implies

$$(1-\alpha-\beta)\|T^2x-Tx\| \leq (1-\alpha-\beta)\|Tx-x\|.$$

Thus we have

$$\|T^2x-Tx\| \leq \|Tx-x\|. \tag{4.8}$$

From the third statement in the summation of (4.5), we have

$$\begin{aligned} \|T^2x-Ty\| &\leq \alpha\|T^2x-y\| + \alpha\|Tx-Ty\| + \beta\|T^2x-Tx\| \\ &\quad +\beta\|Ty-y\| + (1-2\alpha-2\beta)\|Tx-y\|. \end{aligned} \tag{4.9}$$

$$\|T^2x-y\| \leq \|T^2x-Tx\| + \|Tx-y\|. \tag{4.10}$$

If it is written at (4.9), (4.10), we have

$$\begin{aligned} \|T^2x-Ty\| &\leq \alpha\|T^2x-Tx\| + \alpha\|Tx-y\| + \alpha\|Tx-Ty\|\| + \beta\|T^2x-Tx + \beta\|Ty-x\| + \beta\|x-y\|\| \\ &\quad +(1-2\alpha-2\beta)\|Tx-y\| \\ &\leq \alpha\|x-Tx\| + \beta\|x-Tx\| + (1-2\beta-\alpha)\|Tx-y\| + \beta\|Ty-x\| + \beta\|x-y\| + \alpha\|Tx-Ty\| \\ &\leq (\alpha+\beta)\|x-Tx\| + (1-2\beta-\alpha)\|x-Tx\| + (1-2\beta-\alpha)\|x-y\| + \beta\|x-y\| \\ &\quad +\alpha(\alpha\|Tx-y\| + \alpha\|Ty-x\| + \beta\|Tx-x\| + \beta\|Ty-y\| + (1-2\alpha-2\beta)\|x-y\|) \\ &\leq (1-\beta)\|x-Tx\| + (1-\alpha-\beta)\|x-y\| + \beta\|x-Ty\| + \alpha[\alpha\|x-Tx\| + \alpha\|x-y\| \\ &\quad +\alpha\|x-Ty\| + \beta\|x-Tx\| + \beta\|x-Ty\| + \beta\|x-y\| + (1-2\alpha-2\beta)\|x-y\|]. \end{aligned}$$

Thus we have

$$\begin{aligned} \|T^2x-Ty\| &\leq (1-\beta)\|x-Tx\| + (1-\alpha-\beta)\|x-y\| + \beta\|x-Ty\| \\ &\quad +\alpha(\alpha+\beta)\|x-Tx\| + \alpha(\alpha+\beta)\|x-Ty\| + \alpha(1-\alpha-\beta)\|x-y\|. \end{aligned} \tag{4.11}$$

If it is written at (4.5), (4.11), we have

$$\begin{aligned} \|x-Ty\| &= 2\|x-Tx\| + (1-\beta)\|x-Tx\| + (1-\alpha-\beta)\|x-y\| + \beta\|x-Ty\| \\ &\quad +\alpha(\alpha+\beta)\|x-Tx\| + \alpha(\alpha+\beta)\|x-Ty\| + \alpha(1-\alpha-\beta)\|x-y\|. \end{aligned}$$

This implies, for $\beta+\alpha^2+\alpha\beta\in(0,1)$,

$$(1-\beta-\alpha^2-\alpha\beta)\|x-Ty\| = (3-\beta+\alpha(\alpha+\beta))\|x-Tx\|+(1-\beta-\alpha^2-\alpha\beta)\|x-y\|.$$

Thus we have, for $\beta+\alpha^2+\alpha\beta\in(0,1)$,

$$\|x-Ty\|\leq (\frac{3-\beta+\alpha(\alpha+\beta)}{1-\beta-\alpha^2-\alpha\beta})\|Tx-x\|+\|x-y\|.$$

Therefore in both cases, we get the desired result. □

The mapping in Lemma 4.8 can be said (α,β)-Reich-Suzuki nonexpansive type mapping satisfying *Condition (D)* with

$$L=\frac{3-\beta+\alpha(\alpha+\beta)}{1-\beta-\alpha^2-\alpha\beta}$$

for $\beta+\alpha^2+\alpha\beta\in(0,1)$. We notice that (α,β)-Reich-Suzuki nonexpansive type nonexpansive mappings are more general than the Suzuki nonexpansive , α-nonexpansive type mappings and their generalized nonexpansive mappings. Throughout this paper, we consider (α,β)-Reich-Suzuki nonexpansive type mappings satisfying *Condition (D)* with $L=\frac{3-\beta+\alpha(\alpha+\beta)}{1-\beta-\alpha^2-\alpha\beta}$ for $\beta+\alpha^2+\alpha\beta\in(0,1)$. Now we prove the following basic properties and results for (α,β)-Reich-Suzuki nonexpansive type nonexpansive mappings.

Lemma 4.9. Let K be a closed subset of a Banach space X. Let $T:K\to K$ be an (α,β)-Reich-Suzuki nonexpansive type mapping. Then $F(T)$ is closed. Moreover, if X strictly convex and K is convex then $F(T)$ is also convex.

Proof . Let $\{x_n\}\subset F(T)$ be a sequence such that $x_n\to z$ as $n\to\infty$. Then using Lemma 4.8, we get

$$\begin{aligned}\limsup_{n\to\infty}\|x_n-T(z)\| &\leq \limsup_{n\to\infty}(\frac{3-\beta+\alpha(\alpha+\beta)}{1-\beta-\alpha^2-\alpha\beta})\|x_n-Tx_n\|+\|x_n-z\|\\ &= \limsup_{n\to\infty}\|x_n-z\|=0.\end{aligned}$$

That is $\{x_n\}\to Tz$ as $n\to\infty$. This implies $Tz=z$. Therefore $F(T)$ is closed. Next, we assume that X strictly convex and K is convex. For, we fix, $\delta\in(0,1)$ and $s,t\in F(T)$ with $s\neq t$ and $z:=\delta s+(1-\delta)t\in K$. Then we have

$$\|Ts-Tz\|\leq\|s-Tz\|\leq(\frac{3-\beta+\alpha(\alpha+\beta)}{1-\beta-\alpha^2-\alpha\beta})\|s-Ts\|+\|s-z\|=\|s-z\|.$$

Similarly

$$\|Tt-Tz\|\leq\|t-Tz\|\leq(\frac{3-\beta+\alpha(\alpha+\beta)}{1-\beta-\alpha^2-\alpha\beta})\|t-Tt\|+\|t-z\|=\|t-z\|.$$

Since X strictly convex, there exists $\eta\in(0,1)$ such that T$z:=\eta s+(1-\eta)t$. Then we have

$$(1-\eta)\|s-t\|=\|Ts-Tz\|\leq\|s-z\|=(1-\delta)\|s-t\|,$$

$$\eta\|s-t\|=\|Tt-Tz\|\leq\|t-z\|=\delta\|s-t\|.$$

We have $1-\eta<1-\delta$ and $\eta\leq\delta$ implies $\eta=\delta$. Therefore $Tz=z$, that is , $z\in F(T)$. □

Lemma 4.10. Let X be a uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of X. Let $T:K\to K$ be an (α,β)-Reich-Suzuki nonexpansive type mapping. If $\{x_n\}$ weakly converges $z\in K$ and $\lim_{n\to\infty}\|x_n-Tx_n\|=0$, then $Tz=z$.That is, $I-T$ is demiclosed at zero.

Proof . By Lemma 4.8, we have

$$\begin{aligned}\|x_n-Tz\| &\leq ((\frac{3-\beta+\alpha(\alpha+\beta)}{1-\beta-\alpha^2-\alpha\beta})\|Tx_n-x_n\|+\|x_n-z\|)\\ &\leq L\|Tx_n-x_n\|+\|x_n-z\|\end{aligned}$$

for $n\in\mathbb{N}$ and hence

$$\liminf_{n\to\infty}\|x_n-Tz\|<\liminf_{n\to\infty}\|x_n-z\|. \tag{4.12}$$

Let $Tz \neq z$. Since $x_n \to z$ (weakly), by the Opial's property, we obtain

$$\liminf_{n\to\infty}\|x_n - z\| < \liminf_{n\to\infty}\|x_n - Tz\|.$$

which is a contradiction to (4.12). Thus, $Tz = z$.This completes the proof. □

We recall that (α, β)-Reich-Suzuki nonexpansive type mappings are more general than the Suzuki's nonexpansive mappings, generalized α-nonexpansive mappings and α- Reich-Suzuki type mappings. The next example can show these facts.

Example 4.11. Define a mapping $T : [0, 2] \to [0, 2]$ by

$$Tx = \begin{cases} 0, & 0 \leq x \leq \frac{17}{10} \\ 0.8, & x > \frac{17}{10} \end{cases}$$

Here T is an (α, β)-Reich-Suzuki nonexpansive type mapping, but T does not satisfy *Condition(C)*. Also T does not satisfy generalized $\alpha-$nonexpansive mapping.

To verify that T is an (α, β)-Reich-Suzuki nonexpansive type mapping, consider the following cases: for $\alpha = \frac{1}{4}, \beta = \frac{1}{4}$, we get $\beta + \alpha^2 + \alpha\beta = 0.375 < 1$,

Case I: If $x \in [0, \frac{17}{10}]$ and $y \in [0, \frac{17}{10}]$, then

$$\begin{aligned} \|Tx - Ty\| = 0 \quad &\leq \quad \frac{1}{4}\|Tx - y\| + \frac{1}{4}\|Ty - x\| + \frac{1}{4}\|Tx - x\| \\ &\quad + \frac{1}{4}\|Ty - y\| + (1 - 2(\frac{1}{4}) - 2(\frac{1}{4})\|x - y\| \\ &= \quad \frac{1}{4}\|Tx - y\| + \frac{1}{4}\|Ty - x\| + \frac{1}{4}\|Tx - x\| + \frac{1}{4}\|Ty - y\|. \end{aligned}$$

Case II: If $x \in (\frac{17}{10}, 2]$ and $y \in (\frac{17}{10}, 2]$, then

$$\begin{aligned} \|Tx - Ty\| = 0 \quad &\leq \quad \frac{1}{4}\|Tx - y\| + \frac{1}{4}\|Ty - x\| + \frac{1}{4}\|Tx - x\| + \frac{1}{4}\|Ty - y\| \\ &= \quad \frac{1}{4}\|x - 0.8\| + \frac{1}{4}\|y - 0.8\| + \frac{1}{4}\|x - 0.8\| + \frac{1}{4}\|y - 0.8\| \\ &= \quad \frac{1}{2}\|x - 0.8\| + \frac{1}{2}\|y - 0.8\| \\ &= \quad \frac{1}{2}(x - 0.8) + \frac{1}{2}(y - 0.8) = \frac{1}{2}(x + y - 1.6). \end{aligned}$$

Then for $x \in (\frac{17}{10}, 2]$ and $y \in (\frac{17}{10}, 2]$, we get

$$\frac{1}{4}\|Tx - y\| + \frac{1}{4}\|Ty - x\| + \frac{1}{4}\|Tx - x\| + \frac{1}{4}\|Ty - y\| \geq 0.$$

Case III: If $x \in [0, \frac{17}{10}]$ and $y \in (\frac{17}{10}, 2]$, then

$$\begin{aligned} \frac{1}{4}\|Tx - y\| + \frac{1}{4}\|Ty - x\| \quad + \quad & \frac{1}{4}\|Tx - x\| + \frac{1}{4}\|Ty - y\| \\ = \quad & \frac{1}{4}\|x - 0.8\| + \frac{1}{4}\|y - 0\| + \frac{1}{4}\|x\| + \frac{1}{4}\|y - 0.8\| \\ = \quad & \frac{1}{4}(\|x - 0.8\| + \|x\|) + \frac{1}{4}(\|y - 0.8\| + \|y\|) \end{aligned}$$

Now, we have two cases for $x \in [0, \frac{17}{10}]$, $y \in (\frac{17}{10}, 2]$.

Case III(i): If $x \in [0, 0.8]$ and $y \in (\frac{17}{10}, 2]$, then

$$\begin{aligned}\frac{1}{4}\|Tx-y\|+\frac{1}{4}\|Ty-x\| &+ \frac{1}{4}\|Tx-x\|+\frac{1}{4}\|Ty-y\| \\ &= \frac{1}{4}\|x-0.8\|+\frac{1}{4}\|y-0\|+\frac{1}{4}\|x\|+\frac{1}{4}\|y-0.8\| \\ &= \frac{1}{4}(0.8)+\frac{1}{4}(2y-0.8) \\ &= \frac{y}{2} \geq 0.8 = \|Tx-Ty\|.\end{aligned}$$

Case III(ii): If $x \in (0.8, \frac{17}{10}]$ and $y \in (\frac{17}{10}, 2]$, then

$$\begin{aligned}\frac{1}{4}\|Tx-y\|+\frac{1}{4}\|Ty-x\| &+ \frac{1}{4}\|Tx-x\|+\frac{1}{4}\|Ty-y\| \\ &= \frac{1}{4}\|x-0.8\|+\frac{1}{4}\|y-0\|+\frac{1}{4}\|x\|+\frac{1}{4}\|y-0.8\| \\ &= \frac{1}{4}(2x-0.8)+\frac{1}{4}(2y-0.8) \\ &= \frac{x}{2}+\frac{y}{2}-0.4\end{aligned}$$

If we take $x = 0.81$ and $y = 1.71$, then we have $\frac{1}{4}\|Tx-y\|+\frac{1}{4}\|Ty-x\|+\frac{1}{4}\|Tx-x\|+\frac{1}{4}\|Ty-y\| = 1.26-0.4 = 0.86 > 0.8 = \|Tx-Ty\|$.

Thus T is a $(\frac{1}{4}, \frac{1}{4})$-Reich-Suzuki nonexpansive type mapping on $[0, 2]$.

Now we take $x = 1.72, y = 1.2$ then

$$\frac{1}{2}\|x-Tx\| = \frac{1}{2}\|1.72-0.8\| = 0.46 < 0.52 = \|x-y\|.$$

$$\|Tx-Ty\| = \|0.8-0\| = 0.8 > 0.52 = \|x-y\|.$$

Thus T does not satisfy Suzuki's *Condition(C)*.

Also we take $x = 1.72, y = 1.2$ and $\alpha = \frac{1}{4}$ then we get

$$\|Tx-Ty\| = \|0.8-0\| = 0.8$$

$$\begin{aligned}\alpha\|Tx-y\|+\alpha\|Ty-x\| &+ (1-2\alpha)\|x-y\| \\ &= \frac{1}{4}\|0.8-1.2\|+\frac{1}{4}\|0-1.72\|+\frac{1}{2}\|1.72-1.2\| = 0.79\end{aligned}$$

Then we have

$$\|Tx-Ty\| = 0.8 > 0.79 = \frac{1}{4}\|Tx-y\|+\frac{1}{4}\|Ty-x\|+(1-\frac{1}{2})\|x-y\|.$$

Thus T does not satisfy generalized $\frac{1}{4}$-nonexpansive mapping.

Also we take $x = 1.72, y = 1.2$ and $\alpha = \frac{1}{4}$, then we get

$$\begin{aligned}\alpha\|Tx-x\|+\alpha\|Ty-y\| &+ (1-2\alpha)\|x-y\| \\ &= \frac{1}{4}\|0.8-1.72\|+\frac{1}{4}\|0-1.2\|+\frac{1}{2}\|1.72-1.2\| = 0.79\end{aligned}$$

Then we have

$$\|Tx-Ty\| = 0.8 > 0.79 = \frac{1}{4}\|Tx-x\|+\frac{1}{4}\|Ty-y\|+(1-\frac{1}{2})\|x-y\|.$$

Thus T does not satisfy $\frac{1}{4}-$ Reich-Suzuki type mapping.

5 Convergence of new iteration process for generalized nonexpansive mappings

In this section, we prove that our new iterative process (2.3) converges to a fixed point for (α, β)-Reich-Suzuki nonexpansive type mappings in uniformly convex Banach space.

Lemma 5.1. Let K be a nonempty closed convex subset of a uniformly convex Banach space X , $T : K \to K$ be an (α, β)-Reich-Suzuki nonexpansive type mapping with $F(T) \neq \emptyset$. For arbitrary chosen $x_0 \in K$, let $\{x_n\}$ be a sequence generated by (2.3) with $\{a_n\}$ a real sequence in $[0, 1]$, then $\lim_{n\to\infty} \|x_n - p\|$ exits for each $p \in F(T)$.

Proof . For any $p \in F(T)$, and $x \in K$, since T satisfies *Condition(D)*, $\frac{1}{2}\|p - Tp\| = 0 \leq \|p - x\|$ implies that

$$\begin{aligned} \|Tp - Tx\| &\leq \alpha\|Tp - x\| + \alpha\|Tx - p\| + \beta\|Tp - p\| + \beta\|Tx - x\| + (1 - 2\alpha - 2\beta)\|p - x\| \\ &\leq \alpha\|Tp - x\| + \alpha\|Tp - Tx\| + \beta\|Tx - Tp\| + \beta\|Tp - x\| + (1 - 2\alpha - 2\beta)\|p - x\| \end{aligned}$$

$$\begin{aligned} (1 - \alpha - \beta)\|Tp - Tx\| &\leq \alpha\|Tp - x\| + \beta\|Tp - x\| + (1 - 2\alpha - 2\beta)\|p - x\| \\ &\leq (1 - \alpha - \beta)\|p - x\|. \end{aligned}$$

Thus, $\|Tp - Tx\| \leq \|p - x\|$. Then we show that T is a quasi-nonexpansive mapping. Now, using (2.3), we have,

$$\begin{aligned} \|z_n - p\| &= \|T((1 - a_n)x_n + a_nTx_n) - p\| \\ &\leq \|(1 - a_n)(x_n - p) + a_n(Tx_n - p)\| \\ &\leq (1 - a_n)\|x_n - p\| + a_n\|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{5.1}$$

Using (2.3) and (5.1), we get

$$\begin{aligned} \|y_n - p\| &= \|T(Tz_n) - p\| \\ &\leq \|Tz_n - p\| \leq \|z_n - p\|. \end{aligned} \tag{5.2}$$

By using (2.3), (5.1) and (5.2) , we get

$$\begin{aligned} \|x_{n+1} - p\| &= \|T(Ty_n) - p\| \\ &\leq \|Ty_n - p\| \leq \|y_n - p\| \leq \|z_n - p\| \leq \|x_n - p\|. \end{aligned} \tag{5.3}$$

Thus, $\{\|x_n - p\|\}$ is bounded and non-increasing, which implies that $\lim_{n\to\infty} \|x_n - p\|$ exits for each $p \in F(T)$. □

Theorem 5.2. Let K be a nonempty closed convex subset of a uniformly convex Banach space X. $T : K \to K$ be an (α, β)-Reich-Suzuki nonexpansive type mapping with $F(T) \neq \emptyset$. For arbitrary chosen $x_0 \in K$, let $\{x_n\}$ be a sequence in K defined by (2.3) with $\{a_n\}$ a real sequence in $[0, 1]$, then $F(T) \neq \emptyset$ *if and only if* $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof . Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. By Lemma 5.1, $\lim_{n\to\infty} \|x_n - p\|$ exits and $\{x_n\}$ is bounded. Put $\lim_{n\to\infty} \|x_n - p\| = r$. From (5.1) and (5.2), we have

$$\limsup_{n\to\infty} \|z_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq r$$

and

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq r$$

and also we have

$$\limsup_{n\to\infty} \|Tx_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq r.$$

By Lemma 5.1, we have $r = \liminf_{n\to\infty} \|x_{n+1} - p\| \leq \liminf_{n\to\infty} \|z_n - p\|$. Thus we have $r = \liminf_{n\to\infty} \|z_n - p\|$. Now,

$$\begin{aligned} r &= \lim_{n\to\infty} \|z_n - p\| \\ &= \lim_{n\to\infty} \|T((1-c_n)x_n + c_nTx_n) - p\| \\ &\leq \lim_{n\to\infty} \|(1-c_n)x_n + c_nTx_n - p\| \\ &\leq \lim_{n\to\infty} \|(1-c_n)(x_n - p) + c_n(Tx_n - p)\| \\ &\leq \lim_{n\to\infty} (1-c_n)\|x_n - p\| + c_n\|Tx_n - p\| \\ &\leq \lim_{n\to\infty} [(1-c_n)\|x_n - p\| + c_n\|x_n - p\|] \\ &\leq \lim_{n\to\infty} \|x_n - p\| = r. \end{aligned}$$

Hence we have

$$\lim_{n\to\infty} \|(1-c_n)(x_n - p) + c_n(Tx_n - p)\| = r.$$

Thus by Lemma 1.3 we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Conversely, suppose that $\{x_n\}$ is bounded $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Let $p \in A(K, \{x_n\})$. By Lemma 4.8, for $L = \frac{3-\beta+\alpha(\alpha+\beta)}{1-\beta-\alpha^2-\alpha\beta}$, we have,

$$\begin{aligned} r(Tp, \{x_n\}) = \limsup_{n\to\infty} \|x_n - Tp\| &\leq \limsup_{n\to\infty}(L\|Tx_n - x_n\| + \|x_n - p\|) \\ &\leq \limsup_{n\to\infty}\|x_n - p\| = r(p, \{x_n\}) \end{aligned}$$

This implies that for $Tp = p \in A(K, \{x_n\})$. Since X is uniformly Banach space, $A(K, \{x_n\})$ is singleton, hence $Tp = p$. This completes the proof. □

Next, we prove the following theorems of fixed points of (α, β)-Reich-Suzuki nonexpansive type mappings. In the next result, we prove our strong convergence theorem as follows.

Theorem 5.3. Let T, K and X be as in Theorem 5.2. The sequence $\{x_n\}$ generated by iteration process (2.3) converges to a point $F(T)$ if and only if $\lim_{n\to\infty} dist(x_n, F(T)) = 0$, where $\lim_{n\to\infty} dist(x_n, F(T)) = inf\,\{\|x_n - p\| : p \in F(T)\}$.

Proof . Necessity is obvious. Conversely, assume that $\lim_{n\to\infty} inf dist(x_n, F(T)) = 0$ and $p \in F(T)$. By Lemma 5.1, we have $\lim_{n\to\infty} \|x_n - p\|$ exists, for all $p \in F(T)$, then by assumption, $\lim_{n\to\infty} dist(x_n, F(T)) = 0$. Now it is enough to show that $\{x_n\}$ is a Cauchy sequence in K. Since $\lim_{n\to\infty} dist(x_n, F(T)) = 0$, for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ $dist(x_n, F(T)) < \frac{\epsilon}{2}$ and $inf\,\{\|x_n - p\| : p \in F(T)\} < \frac{\epsilon}{2}$. In particular, $inf\,\{\|x_{n_0} - p\| : p \in F(T)\} < \frac{\epsilon}{2}$. Therefore, there exists $p \in F(T)$ such that $\|x_{n_0} - p\| < \frac{\epsilon}{2}$. Now for $n, m \geq n_0$,

$$\begin{aligned} \|x_m - x_n\| &\leq \|x_m - p\| + \|x_n - p\| \\ &\leq \|x_{n_0} - p\| + \|x_{n_0} - p\| \\ &\leq 2\|x_{n_0} - p\| < \epsilon. \end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence in K. Since K is closed, there is a point $p \in K$ such that $\lim_{n\to\infty} x_n = p$. Now $\lim_{n\to\infty} dist(x_n, F(T)) = 0$ gives that $dist(p, F(T)) = 0$, that is $p \in F(T)$. The proof is completed. □

Theorem 5.4. Let X be a real uniformly convex Banach space and K be a nonempty compact convex subset of X and let T and $\{x_n\}$ be as in Theorem 5.2. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof . $F(T) \neq \emptyset$, so by Theorem 5.2, we have $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow p$ as $k \to \infty$ for some $p \in K$. Because the map T satisfies Lemma 4.8 one can find some real constant L, such that

$$\|x_{n_k} - Tp\| \quad \leq \quad L\|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - p\| \text{ for all } k = 0, 1, 2, \dots.$$

Letting $k \to \infty$, we get $x_{n_k} \longrightarrow Tp$. Thus $Tp = p$, i.e. $p \in F(T)$. Also, $\lim_{n\to\infty} \|x_n - p\|$ exists for every $p \in F(T)$, so $\{x_n\}$ converges strongly to a fixed point of T. □

Theorem 5.5. Let the conditions of Theorem 5.2 be satisfied. Also if T satisfies *Condition(I)* and $F(T) \neq \emptyset$, then $\{x_n\}$ defined by (2.3) converges strongly to a fixed point of T.

Proof . By Lemma 5.1, we have $\lim_{n\to\infty} \|x_n - p\|$ exists and so $\lim_{n\to\infty} dist(x_n, p)$ exists for all $p \in F(T)$. Also by Theorem 5.2, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. It follows from *Condition(I)* that $0 \leq \lim_{n\to\infty} f(dist(x_n, F(T))) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0$. That is, $\lim_{n\to\infty} f(dist(x_n, F(T))) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, we have $\lim_{n\to\infty} dist(x_n, F(T)) = 0$. All the conditions of Theorem 5.3 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a fixed point of T. The proof is completed. □

Finally, we prove the weak convergence of the iterative scheme (2.3) for (α, β)-Reich-Suzuki nonexpansive type mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 5.6. Let X be a real uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of X. Let $T : K \to K$ be an (α, β)-Reich-Suzuki nonexpansive type mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in K defined by (2.3). Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof . Since $F(T) \neq \emptyset$, it follows from Theorem 5.2 that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. For , let q_1, q_2 be weak limit of subsequence $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed with respect to zero by Lemma 4.10, therefore we obtain $T(q_1) = q_1$. Again in the same manner, we can $T(q_2) = q_2$. Next we prove the uniqueness. By Lemma 5.1, $\lim_{n\to\infty} \|x_n - q_1\|$ and $\lim_{n\to\infty} \|x_n - q_2\|$ exist. For suppose that $q_1 \neq q_2$, then by the Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - q_1\| &= \lim_{j\to\infty} \|x_{n_j} - q_1\| < \lim_{j\to\infty} \|x_{n_j} - q_2\| = \lim_{n\to\infty} \|x_n - q_2\| \\
&= \lim_{k\to\infty} \|x_{n_k} - q_2\| < \lim_{k\to\infty} \|x_{n_k} - q_1\| = \lim_{n\to\infty} \|x_n - q_1\|,
\end{aligned}$$

which is a contradiction; hence $q_1 = q_2$. Consequently, $\{x_n\}$ converges weakly to a fixed point of T. This completes the proof. □

6 Conclusions

The purpose of this paper is to introduce a new iteration process for approximation of fixed points. Table 1 and Figure 1 are shown that our new iteration process is faster than M-iteration process and K-iteration process. Also we prove that the new iteration process is stable. We remark that there exist the (α, β)-Reich-Suzuki nonexpansive type mappings satisfying *Condition (D)* with $L = \frac{3-\beta+\alpha(\alpha+\beta)}{1-\beta-\alpha^2-\alpha\beta}$ for $\beta + \alpha^2 + \alpha\beta \in (0, 1)$ that are not generalized $\alpha-$nonexpansive mappings, generalized α-Reich-Suzuki type mappings and Suzuki generalized nonexpansive mappings, as in Example 4.1 of this paper. Further, we study the convergence of a new iterative scheme to fixed point for the (α, β)-Reich-Suzuki nonexpansive type mappings in Banach space.

Next, we give some open problems related to the (α, β)-Reich-Suzuki nonexpansive type mappings for future studies.

7 Open problems

Could it be used some other iteration process for finding a common element of the set of solutions of an equilibrium problem? Is it possible to develop an modified iteration process for general semi-group contractions?

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