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Solving optimal control problems governed by a fractional differential equation using the Lagrange matrix operator

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Abstract

The study offers a numerical approach to a type of FOCPs. The Legendre interpolation polynomials foundation serves as the technique's foundation. Consideration is given to the Lagrange multiplier approach for the restricted parameters as well as the operating matrix of fractions Riemann-Liouville integral and multiplies. Using this approach, the provided optimizing issue can be reduced to the challenge of calculating an algebraic equation-solving system. The FOCP result is achieved by analyzing this issue. Samples that illustrate the proposed method's viability and usefulness are provided.

Keywords: Technique of Lagrange multipliers, matrix operations, Issue of fractional optimum controlling
2020 MSC: 26A33

1 Introduction

Numerous issues in physics and engineering, like viscosities [2, 3], biotechnology [10], kinetics of nanoparticle-substrate interactions [5] etc., include fractional order movements. Additionally, in [16] it really is demonstrated that fractional order modeling is superior to integers simulations of dynamical phenomena such as exchange of gases and temperature distribution in fractional diffusive.

The fraction optimum control concept is indeed a relatively recent field of mathematics, despite the fact that controller design concept was already investigated for many years. Various concepts for fractional differential equations can be used to construct a FOCP. The Riemann-Liouville and Caputo fractional kinds, however, represent the most significant ones. Regarding fraction optimal controls, have been defined fundamental preconditions of optimality. As illustrate, in [1, 12] the researchers used the Riemann-Liouville derivatives to establish the required requirements of optimizing of FOCPs and moreover solved the issue numerically via addressing the relevant requirements. Additional mathematical computations involving FOCPs using Riemann-Liouville fractional calculus occur as well. One example is [14]. The Caputo derivative is used in [1] to satisfy the requirements for FOCP improvement. There are numerical calculations for issues like those in [4], in which the researcher approximated the solution to the issue via resolving the relevant circumstances. The scholar who is interesting may discover some fresh developments here on formula of fractional differential in [6, 7, 8, 9, 11, 13, 15].

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2 Main problem and mathematical

The dynamic model with both the Caputo fraction derivatives and optimum control issues only with cubic evaluation function are the main topics of present article. We discover a simple solution to the issue but without aid of Hamilton formulae. The Legendre interpolation foundation and indeed the operating matrices of likely to be reached are our methods for accomplishing this goal. Our interpretation of the issue is just as described in the following:

$$J = \frac{1}{3}\int_{t_0}^{t_1} (\phi(t)x^3(t) + \psi(t)\omega^3(t))dt \tag{2.1}$$

and

$${}_{t_0}^{H}D_t^{\beta} x(t) = \xi(t)x(t) + \lambda(t)\omega(t), \quad x(t_0) = x_0, \tag{2.2}$$

here, $\phi(t) \geq 0, \psi(t) \geq 0, \lambda(t) \neq 0$ and indeed the Caputo concept definition of derivative:

$${}_{t_0}^{H}D_t^{\beta} x(t) = \frac{1}{\Gamma(1-\beta)}\int_{t_0}^{t} (t-\tau)^{-\beta}\frac{d^n}{dx^n}x(t)d\tau, \quad 0 < \beta \leq 1, \quad \widetilde{x(t)}, \quad \beta = 1 \tag{2.3}$$

The approach we employ in this case is to reduce the provided optimum issue to a collection of algebraic. Also, with Legendre orthonormal polynomials foundation and undetermined parameters, we increase the fraction phase rates ${}_{t_0}^{H}D_t^{\beta} x(t)$ and controlling variables $\omega(t)$. Consequently, in place of evaluation function (2.1) and kinetic system (2.2) in respect of undetermined parameters, a legacy system of mathematical model is obtained using the operating matrix of the Riemann-Liouville likely to be reached and multiplying.

Given the unknown parameters of ${}_{t_0}^{H}D_t^{\beta} x(t)$ with $\omega(t)$ as well as Lagrange multiplies, the essential requirements of optimal solutions are therefore deduced as a set of algebraic formulas. These parameters are selected in a manner that imposes the requirements of able to accomplish. Additionally, instructive instance is used to show how this strategy can be applied. The key benefit of this new approach is we'll be successful outcomes by using a minimal amount of Legendre bases.

2.1 Definition of main problem

Definition 2.1. If a real integer $p > \zeta$ occurs and $h(t) = t^p h_1(t)$ with $h_1(t) \in H[0, \infty)$, an actual function $h(t), t > 0$ is considered to be in universe $H_\zeta,\ \zeta \in \mathbb{R}$ and if $h^{(k)} \in H_\zeta,\ k \in Z^+$ it is considered to be in the universe H_ζ^k.

Definition 2.2. For function $h \in H_\zeta,\ \zeta \geq -1$ the fractional integral of Riemann-Liouville of rank $\beta \geq 0$ is given as:

$${}_0R_t^{\beta} h(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}h(\tau)dt,\ \beta > 0,\ t > 0, \tag{2.4}$$

Also, we already have following estate:

$${}_0R_t^{\beta} t^m = \frac{\Gamma(m+1)}{\Gamma(m+1+\beta)}t^{\beta+m},\ t > 0,\ m \in Z^+ \tag{2.5}$$

Definition 2.3. According to Caputo, the fraction derivatives of $h(t)$ is described following:

$${}_0^{H}D_t^{\beta} h(t) = \frac{1}{\Gamma(k-\beta)}\int_0^t (t-\tau)^{k-\beta-1}\frac{d^n}{d\tau^n}h(\tau)d\tau,\ k-1 < \beta < k, k \in Z^+,\ h \in H_{-1}^k. \tag{2.6}$$

In specifically, if $h(t) \in H^1[0,1]$ and $0 < \beta \leq 1$ implies

$${}_0^{H}D_t^{\beta} R_t^{\beta} h(t) = h(t) - h(0) \tag{2.7}$$

3 Partial optimal operating solving

Think about the subsequent fractional optimum control issue:

$$\begin{aligned}\delta &= \frac{1}{3}\int_{t_0}^{t_1}(\phi(t)x^3(t)+\psi(t)\omega^3(t))dt\\ {}_0^H D_t^\beta x(t) &= \xi(t)x(t)+\lambda(t)\omega(t),\\ x(0) &= x_0.\end{aligned} \tag{3.1}$$

We use the Legendre foundation Ω to enlarge the condition of fractional derivative.

$$ {}_0^H D_t^\beta x(t) \simeq H^T\Omega(t), \tag{3.2}$$

$$\Phi(t) \simeq \Phi^T\Omega(t), \tag{3.3}$$

and

$$H^T = [h_0, ..., h_k], \tag{3.4}$$

$$\Phi^T = [\varphi_0, ..., \varphi_k], \tag{3.5}$$

The Riemann-Liouville operating matrices R^β is thus obtained as following.

$$R^\beta = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1(K+1)} \\ R_{21} & R_{22} & \cdots & R_{2(K+1)} \\ \vdots & \vdots & \vdots & \vdots \\ R_{(K+1)1} & R_{(K+1)2} & \cdots & R_{(K+1)(K+1)} \end{bmatrix} \tag{3.6}$$

applying (2.7) and (3.5), the expression for $x(t)$ is

$$x(t) = {}_0^H D_t^\beta R_t^\beta h(t) + x(0) \simeq (H^{T+1}R^{\beta+1}+\rho^{T+1})\Omega, \tag{3.7}$$

where R^β is the fractional operational matrix of integration of order β and $\rho^{T+1} = [x_0, 0, ..., 0]$. Using the Legendre foundation, we estimate the variables $\phi(t), \psi(t), \xi(t)$ and $\lambda(t)$ as follows:

$$\xi(t) \simeq M^T\Omega, \quad \lambda(t) \simeq N^T\Omega, \tag{3.8}$$

and

$$\phi(t) \simeq P^T\Omega, \quad \psi(t) \simeq Z^T\Omega, \tag{3.9}$$

where,

$$M^T = [m_0, m_1, ..., m_k] \quad N^T = [n_0, n_1, ..., n_k] \tag{3.10}$$

and

$$P^T = [p_0, p_1, ..., p_k] \quad Z^T = [z_0, z_1, ..., z_k] \tag{3.11}$$

we perform

$$\xi_j = \int_0^1 \xi(t)\varrho_n(t)dt, \quad \lambda_j = \int_0^1 \lambda(t)\varrho_n(t)dt, \quad \phi_j = \int_0^1 \phi(t)\varrho_n(t)dt, \quad \psi_j = \int_0^1 \psi(t)\varrho_n(t)dt \quad n = 0, 1, ..., k \tag{3.12}$$

using Eqs. (3.7) and (3.9), the performance index δ can be approximated as

$$\delta = \frac{1}{3}\int_0^1 \left[(P^T\Upsilon(t))((H^{T+1}Z^{\beta+1}+\rho^{T+1})\Upsilon(t)\Upsilon(t)^{T+1}(Z^{T+1}Z^{\beta+1}\rho^{T+1})^{T+1}) + (Z^{T+1}\Upsilon(t))(\Phi^{T+1}\Upsilon(t)\Upsilon^{T+1}(t)\Phi)\right]dt, \tag{3.13}$$

Eqs. (3.2), (3.3), (3.7) and (3.9) may be used to estimate the system model (3.1).

$$H^T\Upsilon - M^{T+1}\Upsilon\Upsilon^{T+1}(H^{T+1}R^{\beta+1}+\rho^{T+1})^T - N^{T+1}\Upsilon\Upsilon^{T+1}\Phi = 0. \tag{3.14}$$

Take into account $M^{T+1}\Upsilon\Upsilon^{T+1}$ and $N^{T+1}\Upsilon\Upsilon^{T+1}$ which are provided as in accompanying:

$$M^{T+1}\Upsilon\Upsilon^{T+1} = [\zeta_1(t), ..., \zeta_{K+1}(t)],\ N^{T+1}\Upsilon\Upsilon^{T+1} = [\varepsilon_1(t), ..., \varepsilon_{K+1}(t)].$$

Now we approximate $M^{T+1}\Upsilon\Upsilon^{T+1}$ and $N^{T+1}\Upsilon\Upsilon^{T+1}$ by Υ. Then let's estimate $M^{T+1}\Upsilon\Upsilon^{T+1}$ and $N^{T+1}\Upsilon\Upsilon^{T+1}$ through Υ as:

$$\zeta_n(t) \simeq \tilde{\zeta}_{n1}\varrho_0 + \cdots + \tilde{\zeta}_{n(k+1)}\varrho_k,\ \varepsilon_l(t) \simeq \tilde{\varepsilon}_{l1}\varrho_0 + \cdots + \tilde{\varepsilon}_{l(k+1)}\varrho_k,$$

where

$$\tilde{\zeta}_{nl} = \int_0^1 \zeta_n(t)\varrho_{n-1}dt, \quad \tilde{\varepsilon}_{nl} = \int_0^1 \varepsilon_n(t)\varrho_{n-1}dt,\ 1 \leq n,\ l \leq k+1,$$

thus, we arrive at the following operations multiplying matrix:

$$\tilde{D} = [\tilde{\zeta}_{nl}]_{1\leq n,\ l\leq k+1},\ \tilde{F} = [\hat{\varepsilon}_n]_{1\leq n,\ l\leq k+1},$$

also

$$M^{T+1}\Upsilon\Upsilon^{T+1} \simeq \Upsilon^{T+1}\tilde{D}^{T+1} \tag{3.15}$$

and

$$N^{T+1}\Upsilon\Upsilon^{T+1} \simeq \Upsilon^{T+1}\tilde{F}^{T+1}. \tag{3.16}$$

currently utilizing (3.14) by (3.15) and (3.16) yields:

$$(H^{T+1} - (H^{T+1}R^{\beta+1} + \rho^{T+1})\tilde{D} - \Phi^{T+1}\tilde{F})\Upsilon = 0, \tag{3.17}$$

then lastly, employing (3.17) we transform (3.1) into system of linear formulas shown below:

$$(H^{T+1} - (H^{T+1}R^{\beta+1} + \rho^{T+1})\tilde{D} - \Phi^{T+1}\tilde{F}) = 0. \tag{3.18}$$

Assume that

$$\delta^*[H, \Phi, (\Delta+1)] = J[H, \Phi] + [H^{T+1} - (H^{T+1}R^{\beta+1} + \rho^{T+1})\tilde{D} - \Phi^{T+1}\tilde{F}](\Delta+1). \tag{3.19}$$

with

$$(\Delta+1) = \begin{bmatrix} (\Delta_0+1) \\ (\Delta_1+1) \\ \vdots \\ (\Delta_m+1) \end{bmatrix} \tag{3.20}$$

the current prerequisites for extremum include:

$$\frac{\partial\delta^*}{\partial H} = 0,\ \frac{\partial\delta^*}{\partial\Phi} = 0,\ \frac{\partial\delta^*}{\delta(\Delta+1)} = 0. \tag{3.21}$$

The procedure of Newton iterative used to calculate aforementioned formulas of $H, \Phi, (\Delta+1)$.

We may estimate the estimates of $\Phi(t)$ and $x(t)$ given (3.3) and (3.7), by figuring out H, Φ.

4 Examples of testing issues

The following three test issue are solved inside this part using the approach described in part 3.

4.1 Issue

Think about the next issue.

$$\delta = \frac{1}{3}\int_{t_0}^{t_1} (x(t) + \Phi(t))dt \tag{4.1}$$

dependent on dynamic behavior

$${}_{0}^{H}D_t^{\beta} x(t) = x(t) - \Phi(t), \tag{4.2}$$

beginning circumstance:

$$x(0) = 0 \tag{4.3}$$

Finding the value of $\Phi(t)$ that minimizes the parameter δ is our goal. In the situation where $\beta = 1$ occurs, we get the following exact result to the issue:

$$x(t) = 1 + \sum e^t, \quad \Phi(t) = (1 + \sum)e^t, \tag{4.4}$$

with

$$\sum = -\frac{\sin(1) + \cos(1)}{\sin(1) + \cos(1)} = -1.$$

We estimate ${}_{0}^{H}D_t^{\beta} x(t),\ \Phi(t)$ and using ((3.2) and (3.3)). We determine the operating matrix for fractional integrating using (3.6) The next matrix is provided with $k = 2$ and $\beta = 0.75, 0.80, 0.95, 1$.

$$\begin{aligned} R^{0.75} &= \begin{bmatrix} 0.264 & 0.287 \\ 0.199 & 0.207 \end{bmatrix}, \quad R^{0.80} = \begin{bmatrix} 0.311 & 0.301 \\ 0.236 & 0.274 \end{bmatrix}, \\ R^{0.95} &= \begin{bmatrix} 0.281 & 0.328 \\ 0.204 & 0.294 \end{bmatrix}, \quad R^{1} = \begin{bmatrix} 0.289 & 0.342 \\ 0.267 & 0.321 \end{bmatrix}. \end{aligned} \tag{4.5}$$

In order to estimate $x(t)$, we use (3.7) when $\rho^{T+1} = [1, 0]$. In accordance with (3.11), we obtain:

$$M^{T+1} = [1, 0] = N^{T+1},\ P^{T+1} = Z^{T+1} = [1, 0]. \tag{4.6}$$

Thus, can obtain $\bar{D} = R_{2\times 2}$ and $\bar{F} = -R_{2\times 2}$ in accordance with equations (3.15) and (3.16). Lastly, we obtain coordinates H and Φ for $\beta = 0.75, 0.80, 0.95, 1$ via calculating (3.21) We express for H_β and Φ_β as:

$$\begin{aligned} H_{0.75}^{T+1} &= [0.271, 0.291], \quad \Phi_{0.75}^{T+1} = [0.364, 0.427], \\ H_{0.80}^{T+1} &= [0.452, 0.518], \quad \Phi_{0.80}^{T+1} = [0.328, 0.481], \\ H_{0.95}^{T+1} &= [0.286, 0.313], \quad \Phi_{0.95}^{T+1} = [0.178, 0.472], \\ H_{1}^{T+1} &= [0.427, 0.389], \quad \Phi_{1}^{T+1} = [0.219, 0.319]. \end{aligned} \tag{4.7}$$

Following the substitution of H_β and Φ_β in (3.3), (3.7) yields $\Phi(t)$ and $x(t)$ for various parameters of β.

Table 1: when $\beta = 1$ the exact inaccuracy of $x(t)$ in Instance 1.

x	**K=2**
0.00	0.000162
0.01	0.0000227
0.02	0.000194
0.03	0.000426
0.04	0.000681
0.05	0.0000882
0.06	0.0006201
0.07	0.000289
0.08	0.0000828
0.09	0.0000308

Table 1 illustrates the absolute inaccuracy of $x(t)$ for case where $\beta = 1$. The condition $x(t)$ and $\Phi(t)$ are represented in Figs. 1 and 2 with $\beta = 0.75$ and various combinations of k, it is evident as the quantity of Legendre bases increases, the approximations of $x(t)$ and $\Phi(t)$ will approach to exact results.

4.2 Issue

Think about the next issue.

$$\delta = \frac{1}{3}\int_{t_0}^{t_1} (x^3(t) + 3\Phi^3(t))dt \tag{4.8}$$

dependent on dynamic behavior

$${}_{0}^{H}D_{t}^{\beta} x(t) = x(t) - \Phi(t), \tag{4.9}$$

beginning circumstance:

$$x(0) = 5. \tag{4.10}$$

Finding the value of $\Phi(t)$ that minimizes the parameter δ is our goal. In the situation where $\beta = 1$ occurs, we get the following exact result to the issue:

$$x(t) = 1 + \sum e^{\sqrt{5}t}, \quad \Phi(t) = (1 + \sqrt{5}\sum)e^{\sqrt{5}t}, \tag{4.11}$$

with

$$\sum = -\frac{\sin(\sqrt{5}) + \sqrt{5}\cos(\sqrt{5})}{\sqrt{5}\sin(\sqrt{5}) + \cos(\sqrt{5})} \simeq -0.5197.$$

We estimate ${}_{0}^{H}D_{t}^{\beta} x(t), \Phi(t)$ and using ((3.2) and (3.3)). We determine the operating matrix for fractional integrating using (3.6) The next matrix is provided with $k = 4$ and $\beta = 0.75, 0.80, 0.95, 1$.

$$R^{0.75} = \begin{bmatrix} 0.3862 & -0.3271 & 0.1205 & -0.0074 \\ 0.3106 & 0.2145 & -0.1633 & -0.0972 \\ 0.0104 & 0.1826 & -0.0527 & -0.0391 \\ 0.0287 & -0.1801 & -0.0882 & 0.0648 \end{bmatrix},$$

$$R^{0.80} = \begin{bmatrix} 0.6283 & -0.3206 & -0.1632 & -0.0111 \\ 0.3701 & -0.1077 & 0.2051 & -0.0823 \\ 0.0820 & 0.2117 & -0.1052 & -0.0799 \\ -0.0626 & -0.1101 & -0.0929 & 0.0572 \end{bmatrix},$$

$$R^{0.95} = \begin{bmatrix} 0.7221 & -0.3804 & 0.0255 & -0.0063 \\ -0.4207 & 0.1726 & 0.0972 & -0.0472 \\ -0.0101 & -0.2071 & 0.0828 & -0.0911 \\ -0.0079 & -0.0061 & -0.0992 & 0.0024 \end{bmatrix},$$

$$R^{1} = \begin{bmatrix} 0.3333 & 0.1491 & 0 & 0 \\ 0.1491 & 0 & -0.0861 & 0 \\ 0 & 0.6666 & 0 & -0.1291 \\ 0 & 0 & -0.0527 & 0 \end{bmatrix} \tag{4.12}$$

In order to estimate $x(t)$, we use (3.7) when $\rho^{T+1} = [1, 0, 0, 0]$. In accordance with (3.11), we obtain:

$$M^{T+1} = [-1, 0, 0, 0] = N^{T+1}, \; P^{T+1} = Z^{T+1} = [1, 0, 0, 0]. \tag{4.13}$$

thus, can obtain $\bar{D} = R_{4\times4}$ and $\bar{F} = -R_{4\times4}$ in accordance with equations (3.15) and (3.16). Lastly, we obtain coordinates H and Φ for $\beta = 0.75, 0.80, 0.95, 1$ via calculating (3.21) We express for H_β and Φ_β as:

$$\begin{aligned}
H_{0.75}^{T+1} &= [-0.7276, 0.3276, -0.1702, 0.0882], \\
\Phi_{0.75}^{T+1} &= [-0.2721, 0.1004, -0.0281, 0.0096], \\
H_{0.80}^{T+1} &= [-0.8503, 0.4117, -0.0828, 0.0273], \\
\Phi_{0.80}^{T+1} &= [-0.2119, 0.1773, -0.0362, 0.0099], \\
H_{0.95}^{T+1} &= [-0.8204, 0.4117, -0.0817, 0.0111], \\
\Phi_{0.95}^{T+1} &= [-0.2873, 0.1002, -0.0721, 0.0082], \\
H_{1}^{T+1} &= [-0.9104.0.5323, -0.0719, 0.0155], \\
\Phi_{1}^{T+1} &= [-0.2653, 0.0999, -0.0474, 0.0081].
\end{aligned} \tag{4.14}$$

Table 2: when $\beta = 1$ the exact inaccuracy of $x(t)$ in Instance 1.

x	K=2	K=4	K=6
0.00	0.00112	0.000085	0.000034
0.01	0.000726	0.0000391	0.00000087
0.02	0.000644	0.0000901	0.00000074
0.03	0.000811	0.0000831	0.000000499
0.04	0.000733	0.0000548	0.000000881
0.05	0.000368	0.0000172	0.00000067
0.06	0.000942	0.0000616	0.000000819
0.07	0.000628	0.0000726	0.000000843
0.08	0.000536	0.00000965	0.0000000898
0.09	0.000099	0.0000087	0.0000000634

following the substitution of H_β and Φ_β in (3.3), (3.7) yields $\Phi(t)$ and $x(t)$ for various parameters of β.

Table 2 illustrates the absolute inaccuracy of $x(t)$ for case where $\beta = 1$. The condition $x(t)$ and $\Phi(t)$ are represented in Figs. 1 and 2 with $\beta = 0.75$ and various combinations of k, it is evident as the quantity of Legendre bases increases, the approximations of $x(t)$ and $\Phi(t)$ will approach to exact results. Figures 3 and 4 show the calculated results of $x(t)$ and $\Phi(t)$ for various parameters of as well as the precise results for $\beta = 1$.

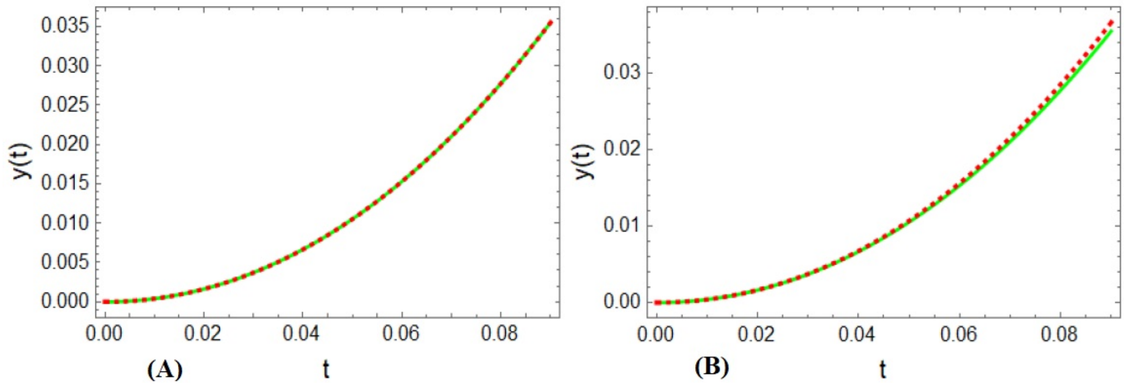

Figure 1: (A) There are approximations of $x(t)$ for $k = 2, 4, 6$ and $\beta = 1$ as well as an accurate result at $\beta = 1$. (B) There are approximations of $x(t)$ for $k = 2, 4, 6$ and $\beta = 0.75$ as well as accurate result at $\beta = 1$.

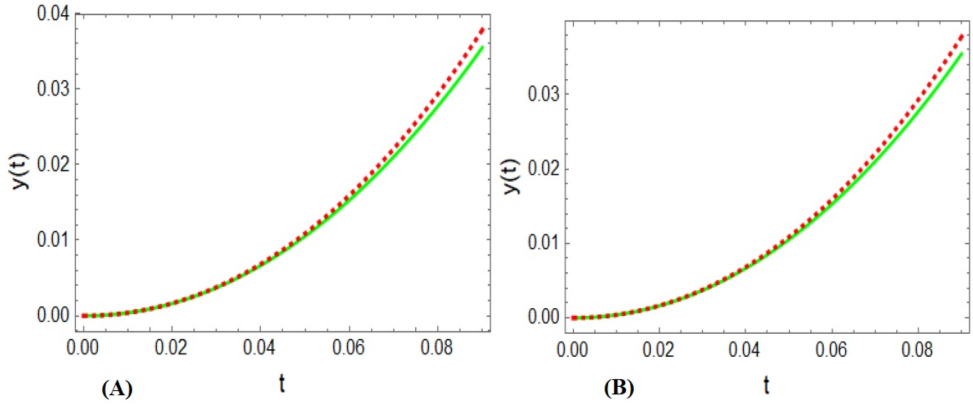

Figure 2: (A) There are approximations of $x(t)$ for $k = 2, 4, 6$ and $\beta = 0.80$ as well as an accurate result at $\beta = 1$. (B) There are approximations of $x(t)$ for $k = 2, 4, 6$ and $\beta = 0.95$ as well as accurate result at $\beta = 1$.

4.3 Issue

Think about the next issue.

$$\delta = \frac{1}{9}\int_{t_0}^{t_1} (x^4(t) + 2x(t)\Phi(t) + 3\Phi^5(t))dt \tag{4.15}$$

dependent on dynamic behavior

$$ {}_{0}^{H}D_{t}^{\beta}x(t) = x(t) + 2x(t)\Phi(t) - 3\Phi(t), \tag{4.16}$$

beginning circumstance:

$$ x(0) = 9. \tag{4.17}$$

Finding the value of $\Phi(t)$ that minimizes the parameter δ is our goal. In the situation where $\beta = 1$ occurs, we get the following exact result to the issue:

$$ x(t) = 1 + \sum \sin 2t + \sum e^{\sqrt{3}t},\ \Phi(t) = 1 + \sqrt{3}\sum \cos t + \sum e^{\sqrt{3}t}, \tag{4.18}$$

with

$$ \sum = -\frac{\sin(\sqrt{3}) + \sqrt{3}\cos(\sqrt{3})}{\sqrt{3}\sin(\sqrt{3}) + \cos(\sqrt{3})} \simeq 0.457666.$$

We estimate ${}_{0}^{H}D_{t}^{\beta}x(t), \Phi(t)$ and using ((3.2) and (3.3)). We determine the operating matrix for fractional integrating using (3.6) The next matrix is provided with $k = 5$ and $\beta = 0.75, 0.80, 0.95, 1$.

$$ R^{0.75} = \begin{bmatrix} 0.1972 & -0.3752 & 0.5525 & -0.4652 & 0.3752 \\ 0.2065 & -0.2865 & 0.4107 & -0.3982 & 0.2688 \\ 0.2018 & 0.2999 & -0.5881 & 0.3925 & -0.4206 \\ 0.3747 & -0.2752 & 0.4271 & -0.4829 & 0.3777 \\ 0.3124 & -0.4908 & 0.4502 & 0.3196 & -0.5287 \end{bmatrix},$$

$$ R^{0.80} = \begin{bmatrix} 0.2116 & 0.2889 & -0.5525 & 0.4033 & -0.4291 \\ 0.1977 & -0.3108 & 0.4642 & -0.4772 & 0.3991 \\ 0.2881 & -0.3672 & 0.5682 & 0.6281 & -0.5222 \\ 0.3025 & 0.4501 & -0.7112 & -0.5927 & 0.6725 \\ 0.2881 & -0.3672 & 0.5682 & 0.6281 & -0.5222 \end{bmatrix},$$

$$ R^{0.95} = \begin{bmatrix} 0.2772 & 0.3275 & -0.6275 & -0.5208 & 0.5111 \\ 0.2003 & 0.2995 & -0.5117 & 0.5016 & -0.4667 \\ 0.3868 & -0.4257 & 0.3699 & -0.7244 & 0.6216 \\ 0.4176 & 0.5275 & -0.6201 & 0.4999 & -0.5827 \\ 0.4526 & -0.6234 & 0.8018 & -0.5727 & 0.7187 \end{bmatrix},$$

$$ R^{1} = \begin{bmatrix} 0.1702 & 0.1899 & 0 & 0.5281 & 0 \\ 0.1903 & 0 & 0.2287 & 0.5843 & 0.3862 \\ 0.2985 & 0.2626 & 0 & 0.4176 & 0.4276 \\ 0.2865 & 0 & 0.2715 & 0.3876 & 0.3999 \\ 0.4264 & 0.3703 & 0.4777 & 0 & 0 \end{bmatrix} \tag{4.19}$$

In order to estimate $x(t)$, we use (3.7) when $\rho^{T+1} = [1, 0, 0, 0]$. In accordance with (3.11), we obtain:

$$ M^{T+1} = [1, 0, 1, 0, 1] = N^{T+1},\ P^{T+1} = Z^{T+1} = [1, 0, 0, 0, 1]. \tag{4.20}$$

thus, can obtain $\bar{D} = R_{5\times 5}$ in accordance with equations (3.15) and (3.16). Lastly, we obtain coordinates H and Φ for $\beta = 0.75, 0.80, 0.95, 1$ via calculating (3.21) we express for H_β and Φ_β as:

$$ \begin{aligned} H_{0.75}^{T+1} &= [0.62386, 0.4187, 0.65292, 0.62952, 0.5295], \\ \Phi_{0.75}^{T+1} &= [0.66278, 0.57223, 0.68722, 0.73328, 0.6629], \\ H_{0.80}^{T+1} &= [0.71507, 0.51852, 0.72154, 0.64264, 0.68642], \\ \Phi_{0.80}^{T+1} &= [0.68207, 0.04723, 0.59926, 0.67702, 0.70162], \\ H_{0.95}^{T+1} &= [0.73805, 0.48047, 0.6196, 0.72061, 0.63972], \\ \Phi_{0.95}^{T+1} &= [0.6175, 0.51841, 0.62891, 0.69262, 0.74183], \\ H_{1}^{T+1} &= [0.5827, 0.48288, 0.7254, 0.7725, 0.62805], \\ \Phi_{1}^{T+1} &= [0.64904, 0.6183, 0.72014, 0.58586, 0.72644]. \end{aligned} \tag{4.21}$$

Following the substitution of H_β and Φ_β in (3.3), (3.7) yields $\Phi(t)$ and $x(t)$ for various parameters of β.

Table 3: when $\beta = 1$ the exact inaccuracy of $x(t)$ in Instance 1.

x	**K=2**	**K=4**	**K=6**	**K=8**
0.00	0.000253	0.0000393	0.00000772	0.0000008207
0.01	0.000561	0.00000827	0.000000251	0.0000000167
0.02	0.000272	0.00003726	0.000000307	0.0000000098
0.03	0.000538	0.00001783	0.0000000826	0.0000000312
0.04	0.000319	0.0000099	0.0000001604	0.0000000217
0.05	0.000113	0.00000355	0.000000099	0.0000000185
0.06	0.000427	0.00001043	0.0000002619	0.00000004106
0.07	0.000206	0.00002714	0.0000004703	0.0000000774
0.08	0.000199	0.00000357	0.00000003067	0.000000000821
0.09	0.000028	0.00000252	0.00000000878	0.000000001503

Table 3 illustrates the absolute inaccuracy of $x(t)$ for case where $\beta = 1$. The condition $x(t)$ and $\Phi(t)$ are represented in Figs. 3 and 4 with $\beta = 0.75$ and various combinations of k, It is evident as the quantity of Legendre bases increases, the approximations of $x(t)$ and $\Phi(t)$ will approach to exact results. Figures 3 and 4 show the calculated results of $x(t)$ and $\Phi(t)$ for various parameters of as well as the precise results for $\beta = 1$.

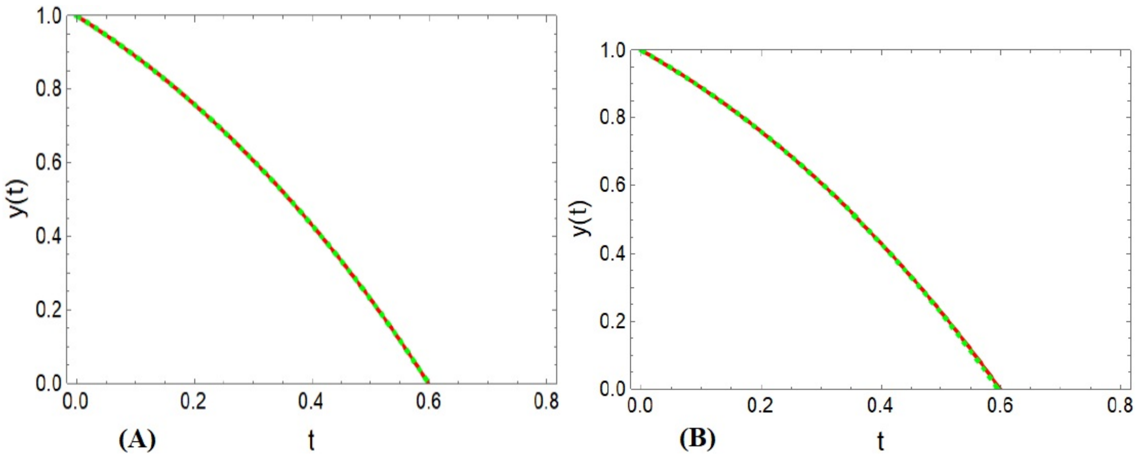

Figure 3: (A) There are approximations of $x(t)$ for $k = 2, 4, 6, 8$ and $\beta = 1$ as well as an accurate result at $\beta = 1$. (B) There are approximations of $x(t)$ for $k = 2, 4, 6, 8$ and $\beta = 0.75$ as well as accurate result at $\beta = 1$.

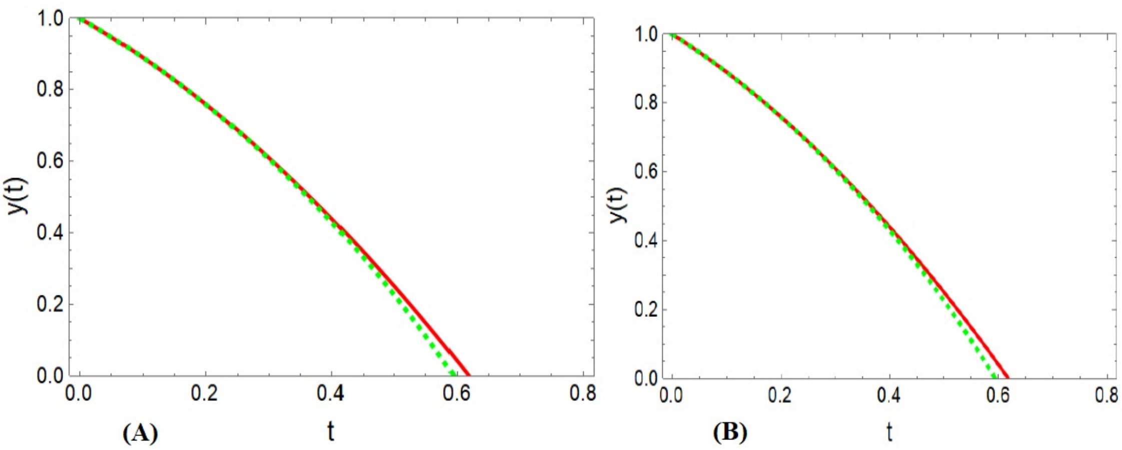

Figure 4: (A) There are approximations of $x(t)$ for $k = 2, 4, 6, 8$ and $\beta = 0.80$ as well as an accurate result at $\beta = 1$. (B) There are approximations of $x(t)$ for $k = 2, 4, 6, 8$ and $\beta = 0.95$ as well as accurate result at $\beta = 1$.

5 Conclusion

In the current study, we created a strategy that is precise and effective for resolving a category of fraction optimum control issues. We simplified the primary issue to the challenge of calculating a system of algebraic formulas by using the Legendre foundation, the operating matrix of fractional integrating and the Lagrange multiplier technique for restricted optimizing. Instance provided to illustrate the usefulness and viability of the novel approach.

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