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Almost order-weakly compact operators on Banach lattices

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Abstract

A continuous operator T between two Banach lattices E and F is called almost order-weakly compact, whenever for each almost order bounded subset A of E, $T(A)$ is a relatively weakly compact subset of F. We show that the positive operator T from E into a Dedekind complete Banach lattice F is almost order-weakly compact iff $T(x_n) \xrightarrow{\|\cdot\|} 0$ in F for each disjoint almost order bounded sequence $\{x_n\}$ in E. In this manuscript, we study some properties of this class of operators and its relationships with the others known classes of operators.

Keywords: Almost order bounded, Weakly compact, Order weakly compact, Almost order-weakly compact.
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1 Introduction

Since order weakly compact operators play an important role in the class of positive operators, our aim in this manuscript is to introduce and study a new class of operators as almost order-weakly compact operators and we establish some of its relationships with the others known classes of operators. Under some conditions, we show that the adjoint of any almost order-weakly compact operator is so. Every compact and weakly compact operator is an almost order-weakly compact operator, but the converse in general not holds.

To state our results, we need to fix some notations and recall some definitions. Let E be a Banach lattice. A subset A is said to be almost order bounded if for any ϵ there exists $u \in E^+$ such that $A \subseteq [-u, u] + \epsilon B_E$ (B_E is the closed unit ball of E). One should observe the following useful fact, which can be easily verified using Riesz decomposition Theorem, that $A \subseteq [-u, u] + \epsilon B_E$ iff $\sup_{x\in A} \|(|x| - u)^+\| = \sup_{x\in A} \||x| - |x| \wedge u\| \leq \epsilon$. By Theorems 4.9 and 3.44 of [1], each almost order bounded subset in order continuous Banach lattice is relatively weakly compact. $A \subseteq L_1(\mu)$ is relatively weakly compact iff it is almost order bounded (see [7]). Recall that a vector $e > 0$ in vector lattice E is an order unit or a strong unit (resp, weak unit) when the ideal I_e (resp, band B_e) is equal to E; equivalently, for every $x \geq 0$ there exists $n \in \mathbb{N}$ such that $x \leq ne$ (resp, $x \wedge ne \uparrow x$ for every $x \in E^+$). Suppose that Banach lattice E is an order continuous norm with a weak unit e. It is known that E can be represented as a norm and order dense ideal in $L_1(\mu)$ for some finite measure μ (see [5]). A continuous operator T from a Banach lattice E to a Banach space X is said to be

- **order weakly compact** whenever $T[0, x]$ is a relatively weakly compact subset of X for each $x \in E^+$.
- M**-weakly compact** if $T(x_n) \xrightarrow{\|\cdot\|} 0$ holds for every norm bounded disjoint sequence $\{x_n\}$ of E.

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- b**-weakly compact** whenever T carries each b-order bounded subset of E into a relatively weakly compact subset of X.

A continuous operator T from a Banach space X to a Banach lattice E is said to be

- L**-weakly compact** whenever $y_n \xrightarrow{\|.\|} 0$ for every disjoint sequence $\{y_n\}$ in the solid hull of $T(B_X)$.
- **semicompact** whenever for each $\epsilon \geq 0$ there exists some $u \in E^+$ satisfying $\|(|Tx| - u)^+\| \leq \epsilon$ for all $x \in B_X$.

An operator $T : E \to F$ is regular if $T = T_1 - T_2$ where $T_1, T_2 : E \to F$ are positive operators. We denote by $L(E, F)$ ($L^r(E, F)$) the space of all operators (regular operators) from E into F.

An operator $T : E \to F$ between two vector lattices is said to be a lattice homomorphism (resp. a disjointness preserving) whenever $T(x \vee y) = T(x) \vee T(y)$ (resp. $x \perp y$ in E implies $T(x) \perp T(y)$ in F).

Recall that $L_b(E, F)$ is the vector space of all order bounded operators from E to F.

A Banach space X is said to be Grothendieck space whenever *weak** and *weak* convergence of sequences in X' (the norm dual of X) coincide.

A Banach lattice E is said to be AM-space (resp. AL-space), if for $x, y \in E$ with $x \wedge y = 0$, we have $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ (resp. $\|x + y\| = \|x\| + \|y\|$). A Banach lattice E is said to be KB-space whenever every increasing norm bounded sequence of E^+ is norm convergent.

Let E be a vector lattice and $x \in E$. A net $\{x_\alpha\} \subseteq E$ is said to be order convergent to x if there is a net $\{z_\beta\}$ over different index set in E such that $z_\beta \downarrow 0$ and for every β, there exists α_0 such that $|x_\alpha - x| \leq z_\beta$ whenever $\alpha \geq \alpha_0$. We denote this convergence by $x_\alpha \xrightarrow{o} x$ and write that $\{x_\alpha\}$ is o-convergent to x. A net $\{x_\alpha\} \subseteq E$ is said to be unbounded order convergent to x if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $u \in E^+$. We denote this convergence by $x_\alpha \xrightarrow{uo} x$ and write that $\{x_\alpha\}$ is uo-convergent to x.

2 Almost order bounded operators

Let $T : E \to F$ be a continuous operator between two Banach lattices. T is said to be an almost order bounded operator whenever T maps the almost order bounded subset A of E into an almost order bounded subset of F.
The vector space of all almost order bounded operators from E to F will be denoted $L_{aob}(E, F)$.

It is obvious that if $T : E \to F$ is a semicompact operator, then it is almost order bounded. If E has an order unit and $T : E \to F$ is order bounded, then it is an almost order bounded operator and if F has an order unit and T is an almost order bounded opertor, then it is order bounded.

In the following, there is an example of almost order bounded operator whose modulus does not exist.

Example 2.1. Consider the continuous function $g : [0, 1] \to [0, 1]$ defined by $g(x) = x$ if $0 \leq x \leq \dfrac{1}{2}$ and $g(x) = \frac{1}{2}$ if $\frac{1}{2} < x \leq 1$. Now define the operator $T : C[0, 1] \to C[0, 1]$ by $Tf(x) = f(g(x)) - f(\dfrac{1}{2})$. Then T is a regular operator and therefore it is an order bounded operator. Since $C[0, 1]$ is an AM-space with order unit, T is an almost order bounded operator. Note that the modulus of T does not exist (see Exercise 9 of page 22 of [1].).

In the following, under some conditions, we show that $|T|$ is almost order bounded operator whenever T is an almost order bounded operator.

Proposition 2.2. Let $T : E \to F$ be an almost order bounded operator between two Banach lattices that F is Dedekind complete and E, F have an order unit, then the modulus of T exists and it is almost order bounded.

Proof . Let $T : E \to F$ be an almost order bounded operator. Since F has an order unit, therefore T is an order bounded operator. Because F is Dedekind complete, so by Theorem 1.18 of [1], $|T|$ exists and it is an order bounded operator. Since E has an order unit, $|T|$ is an almost order bounded operator. □

Proposition 2.3. If $T : E \to F$ is a lattice homomorphism and surjective, then T is almost order bounded.

Proof . Let $T : E \to F$ be an almost order bounded and $A \subseteq E$ be an almost order bounded set. It means that for each $\epsilon > 0$ there exists $u \in E^+$ that $\sup_{x\in A} \|(|x| - u)^+\| \leq \epsilon$. Since T is a positive operator, therefore it is a continuous operator. Hence for each $\epsilon > 0$ there exists $u \in E^+$ that $\sup_{x\in A} \|T(|x| - u)^+\| \leq \epsilon$. Since T is a lattice homomorphism, therefore $\sup_{x\in A} \|(|Tx| - Tu)^+\| = \sup_{x\in A} \|T(|x| - u)^+\| \leq \epsilon$. So the proof is complete. □

Remark 2.4. If $T : E \to F$ is an onto lattice homomorphism and F is Archimedean, then $|T|$ exists and it is an almost order bounded operator.

Proof . Since T is a lattice homomorphism, therefore it is order bounded and disjointness preserving. Hence by Theorem 2.40 of [1], $|T|$ exists. It is obvious thst $|T|$ is a lattice homomorphism. By Proposition 2.3, $|T|$ is an almost order bounded operator. □

3 Almost order-weakly compact operators

Let $T : E \to F$ be a continuous operator between two Banach lattices. T is said to be an almost order-weakly compact operator (for short, *ao-wc* operator) whenever T maps the almost order bounded subset A of E into a relatively weakly compact subset of F.

By Theorem 3.40 of [1], T is an *ao-wc* operator iff for every almost order bounded sequence $\{x_n\}$ of E, the sequence $\{T(x_n)\}$ has a weak convergent subsequence in F.

The collection of all *ao-wc* operators between two Banach lattices E and F will be denoted by $K_{ao-wc}(E, F)$.

It is obvious that each compact and weakly compact operator are *ao-wc* and each *ao-wc* operator is an order weakly compact operator.

By Theorem 5.23 and 5.27 of [1], we have the following result.

Theorem 3.1.
1. Each continuous operator T from a Grothendieck Banach lattice E into a Banach lattice F is an *ao-wc* operator.
2. Let T be a positive operator from a Banach lattice E into a Banach lattice F and E' has order continuous norm. If F is a KB-space, then T is *ao-wc*.

In the following we have some examples of *ao-wc* operators.

Example 3.2.
1. Since $C[0, 1]$ is a Grothendieck space, therefore by Theorem 3.1(1), the continuous operator $T : C[0, 1] \to c_0$, given by
$$T(f) = (\int_0^1 f(x) \sin xdx, \int_0^1 f(x) \sin 2xdx, \cdots),$$
is an *ao-wc* operator.
2. Since c' has order continuous norm and $\mathbb{R}$ is a KB-space, therefore by Theorem 3.1(2), the functional $f : c \to \mathbb{R}$ defined by
$$f(x_1, x_2, ...) = \lim_{n\to\infty} x_n$$
is an *ao-wc* operator.

Proposition 3.3. Let E, F and G be three Banach lattices, $T : E \to F$ and $S : F \to G$ be two *ao-wc* operators. By one of the following conditions, $S \circ T$ is an *ao-wc* operator.

1. F is an AL-space.
2. F has order continuous norm with a weak unit.

Proof . Let $A \subseteq E$ be almost order bounded. By assumption, $T(A)$ is a relatively weakly compact subset of F. If F is an AL-space, then by Theorem 4.27 of [1], F is lattice isometric to some concrete $L_1(\mu)$ and if F has order continuous norm with a weak unit, then F is norm and order dense ideal in $L_1(\mu)$. Therefore $T(A)$ is an almost order bounded subset of F. So by assumption, $S(T(A))$ is a relatively weakly compact subset of G. Hence $S \circ T$ is an *ao-wc* operator. □ As following example the adjoint of *ao-wc* operator in general is not an *ao-wc* operator.

Example 3.4. Let $A \subseteq \ell^1$ be an almost order bounded set. Since ℓ^1 has order continuous norm, therefore A is relatively weakly compact. Thus the identity operator $I : \ell^1 \to \ell^1$ is an *ao-wc* operator. Since the identity operator $I : \ell^\infty \to \ell^\infty$ is not order weakly compact, therefore it is not *ao-wc*.

In the following theorem, under some conditions, we show that the adjoint of *ao-wc* operator is so.

Theorem 3.5. Let $T : E \to F$ be an *ao-wc* operator between two Banach lattices. If any of the following conditions are met, then T' is *ao-wc*.

1. E has an order unit.
2. E' is a KB-space and F' has an order unit.

Proof .

1. Let E has an order unit and $T : E \to F$ be *ao-wc*. If $A \subseteq E$ is norm bounded, then A is order bounded and therefore almost order bounded. Hence by assumption, $T(A)$ is a relatively weakly compact subset of F. It means that T is a weakly compact operator. Therefore by Theorem 5.5 of [4], T' is weakly compact and hence it is an *ao-wc* operator.
2. Let $T : E \to F$ be an *ao-wc* operator. Therefore T is an order weakly compact operator. Since E' is a KB-space, by Theorem 3.3 of [2], T' also is an order weakly compact operator. Since F' has an order unit, it is clear that T' is *ao-wc*.

□ We know that each compact and weakly compact operator is an *ao-wc* operator, but by following example the converse in general not holds.

Example 3.6. The identity operator $I : \ell^1 \to \ell^1$ is an *ao-wc* operator, but is not a compact or weakly compact operator.

Corollary 3.7. Under the conditions of Theorem 3.5, an operator $T : E \to F$ is weakly compact iff it is *ao-wc*.

Proof .

Let E has an order unit and $T : E \to F$ be *ao-wc*, then it is a weakly compact operator.
Let E' be a KB-space, F' has an order unit and $T : E \to F$ is *ao-wc*. By Theorem 3.5, T' is *ao-wc*. Because F' has an order unit, T' is weakly compact. By Theorem 5.5 of [4], T is weakly compact. □

Remark 3.8. Let E be a Banach lattice with an order unit. Then a subset A of E is norm bounded iff it is order bounded iff it is almost order bounded. Therefore an operator $T : E \to F$ is weakly compact iff it is order weakly compact iff it is *ao-wc*.

Remark 3.9. Under the conditions of Theorem 3.5, if $T : E \to F$ is *ao-wc*, then by Corollary 3.7 and Theorem 5.44 of [1], there exist a reflexive Banach lattice G, a lattice homomorphism $Q : E \to G$ and a positive operator $S : G \to F$ that $T = S \circ Q$.

Note that the identity operator $I : \ell^\infty \to \ell^\infty$ is not *ao-wc*, however its adjoint $I : (\ell^\infty)' \to (\ell^\infty)'$ is *ao-wc*.

Let $T : E \to F$ be an operator between two Banach lattices. If $T' : F' \to E'$ is *ao-wc* and F' has an order unit, then T' is weakly compact and therefore T is weakly compact. It follows that T is *ao-wc*. If T is M-weakly compact or L-weakly compact, then by Theorem 5.61 of [1], T is weakly compact and therefore T is an *ao-wc* operator. Thus we have the following result.

Theorem 3.10. Let $T : E \to F$ be an operator between two Banach lattices. By one of the following conditions T is an *ao-wc* operator.

1. T is M-weakly compact.
2. T is L-weakly compact.

If $T : E \to F$ is a semicompact operator, or dominated by a semicompact opereator, then T is *ao-wc*. Let A be an almost order bounded subset of E. Then A is norm bounded. Therefore if T is a semicompact operator, $T(A)$ is an almost order bounded set in F. Since F has order continuous norm, $T(A)$ is a relatively weakly compact subset of F. Hence T is an *ao-wc* operator. If T is dominated by a semicompact operator, then by Theorem 5.72 of [1], T is a semicompact operator. Thus T is an *ao-wc* operator.

Remark 3.11. 1. An *ao-wc* operator needs not be an M-weakly or L-weakly compact operator. For instance, the identity operator $I : L_1[0,1] \to L_1[0,1]$ is *ao-wc*, but it is not a M-weakly or L-weakly compact operator.

2. Note that if F has not order continuous norm, then each semicompact operator $T : E \to F$ is not necessarily *ao-wc*. For example, the identity operator $I : \ell^\infty \to \ell^\infty$ is semicompact, but I is not *ao-wc*.

Let E and F be two normed vector lattices. Recall from [8], a continuous operator $T : E \to F$ is said to be σ-*uon*-continuous, if for each norm bounded *uo*-null sequence $\{x_n\} \subseteq E$ implies $T(x_n) \xrightarrow{\|\cdot\|} 0$.

Theorem 3.12. Let E and F be two Banach lattices that F is Dedekind complete. The positive operator $T : E \to F$ is *ao-wc* iff for each disjoint almost order bounded sequence $\{x_n\}$ in E implies $T(x_n) \xrightarrow{\|\cdot\|} 0$ in F.

Proof . Let the operator $T : E \to F$ be *ao-wc*. This means that for every $\epsilon > 0$ there exists $u \in E^+$ such that $T([-u,u] + \epsilon B_E)$ is relatively weakly compact. Let I_z be the ideal generated by $z \in [-u,u] + \epsilon B_E$ in E. Then the operator $T|_{I_z} : I_z \to F$ is weakly compact operator. Since I_z is an AM-space with order unit, therefore $T|_{I_z} : I_z \to F$ is M-weakly and hence by Remark 2.8 of [8], is σ-*uon*-continuous. It is clear that the extension of the operator $T|_{I_z}$, $T : E \to F$ is σ-*uon*-continuous. If $\{x_n\} \subseteq E$ is almost order bounded and disjoint, hence it is norm bounded and *uo*-null. So we have $T(x_n) \xrightarrow{\|\cdot\|} 0$.

Conversely, let $A \subseteq E$ be an almost order bounded set. Then for each $\epsilon > 0$ there exists $u \in E^+$ such that $A \subseteq [-u,u] + \epsilon B_E$. Let I_u be the ideal generated by u in E and $\{x_n\} \subseteq A$ be a disjoint sequence. It is clear that $\{x_n\}$ is norm bounded. By assumption, we have $T(x_n) \xrightarrow{\|\cdot\|} 0$ in F. Therefore $T : I_u \oplus E \to F$ is M-weakly compact, and so by Theorem 3.10, $T : I_u \oplus E \to F$ is an *ao-wc* operator. Thus $T : E \to F$ is *ao-wc*. □

Corollary 3.13. 1. Let $T : E \to F$ and $S : F \to G$ be two *ao-wc* operators where F and G are Dedekind complete and $\{x_n\} \subseteq E$ be a disjoint almost order bounded sequence. By Theorem 3.12, we have $T(x_n) \xrightarrow{\|\cdot\|} 0$. Since S is a continuous operator, $S(T(x_n)) \xrightarrow{\|\cdot\|} 0$. Therefore $S \circ T$ is *ao-wc* operator.

2. By Theorem 5.60 of [1], obviously that if $T : E \to F$ is an *ao-wc* operator, then for each $\epsilon > 0$ there exists some $u \in E^+$ such that $\|T((|x| - u)^+)\| < \epsilon$ holds for all $x \in A$ where A is an almost order bounded subset of E.

Recall that a Banach lattice E is said to have the dual positive Schur property if every positive w^*-null sequence in E' is norm null.

Theorem 3.14. The following statements are true.

1. Let E be a Dedekind complete Banach lattice. E has order continuous norm iff each positive operator T from E into each Banach lattice F is an *ao-wc* operator.
2. Let E be a Dedekind complete Banach lattice. E has order continuous norm iff each almost order bounded disjoint sequence $\{x_n\} \subseteq E$ is norm null.
3. If E has the property (b) and each operator $T^2 : E \to E$ is *ao-wc*, then E has order continuous norm.
4. Let $T : E \to F$ be a continuous operator between two Banach lattices E and F that F is Dedekind complete. If $|T|$ exists and it is *ao-wc*, then T is also *ao-wc*.
5. If E has the dual positive Schur propertry, F has order continuous norm and Dedekind complete, then adjoint of each positive operator $T : E \to F$ is an *ao-wc* operator.

Proof .

1. Let E has order continuous norm and $\{x_n\}$ be an almost order bounded disjoint sequence in E. Therefore $x_n \xrightarrow{uo} 0$ in E. By Proposition 3.7 of [7], $x_n \xrightarrow{\|\cdot\|} 0$. By continuity of T, it follows that $T(x_n) \xrightarrow{\|\cdot\|} 0$ in F.
Conversely, let E has no order continuous norm. By Theorem 2.7 of [3], there exists an operator T from E into ℓ^∞ such that T is not order weakly compact and therefore is not *ao-wc*.
2. Let E has order continuous norm, therefore the identity operator $I : E \to E$ is *ao-wc*. Then $x_n = I(x_n) \xrightarrow{\|\cdot\|} 0$ where $\{x_n\} \subseteq E$ is almost order bounded disjoint sequence.
Conversely, let $\{x_n\}$ be an order bounded disjoint sequence in E. Therefore $\{x_n\}$ is almost order bounded disjoint in E. Hence by assumption $x_n = I(x_n) \xrightarrow{\|\cdot\|} 0$. By Theorem 4.14 of [1], E has order continuous norm.

3. By contradiction, assume that E has no order continuous norm, it follows from the proof of Theorem 2 of [11], that E contains a closed order copy of c_0 and there exists a positive projection $P : E \to c_0$. Let $i : c_0 \to E$ be the canonical injection. Obviously that $T = i \circ P : E \to E$ is not b-weakly compact. Since E has the property (b), therefore T is not order weakly compact, and so T^2 is not *ao-wc*.
4. Let $0 \leq T \leq S$ and S be an *ao-wc* operator. If $\{x_n\}$ is an almost order bounded and disjoint sequence in E, then by Theorem 3.12, $S(x_n) \xrightarrow{\|\cdot\|} 0$. Therefore $T(x_n) \xrightarrow{\|\cdot\|} 0$. We have $-|T| \leq T \leq |T|$, and so $0 \leq T + |T| \leq 2|T|$. It follows that T is an *ao-wc* operator whenever $|T|$ is *ao-wc*.
5. Let $\{f_n\}$ be an almost order bounded disjoint sequence in F'. Then $f_n \xrightarrow{uo} 0$ in F'. Without loss of generality, assume that $0 \leq f_n$. Note that $0 \leq T' f_n$. Now since F has order continuous norm, by Theorem 2.1 from [6], $f_n \xrightarrow{w^*} 0$ in F'. Since T' is w^*-to-w^* continuous, hence $T' f_n \xrightarrow{w^*} 0$ in E'. Since E has the dual positive Schur property, hence $T' f_n \xrightarrow{\|\cdot\|} 0$ in E'.

□

Proposition 3.15. Suppose E has an order unit. Then $T : E \to F$ is σ-*uon*-continuous iff it is an *ao-wc* operator.

Proof . Since $T : E \to F$ is an *ao-wc* operator, T is order weakly compact. Let $\{x_n\} \subseteq E$ be a norm bounded disjoint sequence. Since E has an order unit, then $\{x_n\}$ is order bounded disjoint sequence. By assumption and Theorem 5.57 of [1], $T(x_n) \xrightarrow{\|\cdot\|} 0$. So T is M-weakly compact and therefore by remark 2.8 of [8], T is σ-*uon*-continuous. □
By Remark 3.11, we know that the class of *ao-wc* operators different with the class of semicompact operators. In the following, under some conditions, we establish the relationship between them.

Theorem 3.16. Let $T : E \to F$ be an *ao-wc* operator between two Banach lattices. Then T is semicompact operator.

Proof . Let $T : E \to F$ be *ao-wc* and A be an almost order bounded subset of E. Without loss of generality we assume that for each ϵ there exists $u \in E^+$ such that $A = [-u, u] + \epsilon B_E$. Let $p(x) = \|x\|$. Then $\lim p(T(x_n)) = 0$ holds for each disjoint sequence $\{x_n\}$ in A. By Theorem 4.36 of [1], there exists some $v \in E^+$ satisfying $\|T(|x| - v)^+\| \leq \epsilon$ for all $x \in A$. Put $w = Tv \in F^+$, and note that

$$\begin{aligned} (|Tx| - w)^+ &= (|Tx| - Tv)^+ \\ &\leq (T|x| - Tv)^+ \\ &= (T(|x| - v))^+ \\ &\leq T((|x| - v)^+). \end{aligned}$$

Therefore T is a semicompact operator. □

By Theorems 3.10 and 3.16, we have the following result.

Corollary 3.17.
1. Each operator $T : E \to F$ that it is *ao-wc* is an almost order bounded operator.
2. Let F be a Banach lattice with order continuous norm. Then $T : E \to F$ is *ao-wc* iff it is a semicompact operator.

If T is *ao-wc*, in general $|T|$ is not exist, see the following example.

Example 3.18. The operator $T : L_1[0, 1] \to c_0$ defined by

$$T(f) = (\int_0^1 f(x) \sin x dx, \int_0^1 f(x) \sin 2x dx, \cdots),$$

is an *ao-wc* operator. Note that by Exercise 10 of page 289 of [1], its modulus does not exist.

In the following theorem, under some conditions, we show that $|T|$ exists and is *ao-wc* whenever T is *ao-wc*.

Recall that a Banach lattice E is said to have the property (P) if there exists a positive contractive projection $P : E'' \to E$ where E is identified with a sublattice of its topological bidual E''.

Theorem 3.19. Let $T : E \to F$ be an *ao-wc* operator. By one of the following conditions, the modulus of T exists and it is an *ao-wc* operator.

1. E is an AL-space and F has the propery (P).
2. E and F have an order unit.
3. F is Archimedean Dedekind complete and T is an order bounded operator that preserves disjointness.

Proof .

1. By Theorem 1.7 of [10], we have $L^r(E,F) = L(E,F)$. Therefore $|T|$ exists. Since E has order continuous norm, by Theorem 3.14, $|T| : E \to F$ is an *ao-wc* operator.
2. Since E has an order unit, T is a weakly compact operator. Since F has an order unit, by Theorem 2.3 of [9], the modulus of T exists and it is a weakly compact operator. It is obvious that $|T|$ is an *ao-wc* operator.
3. By Theorem 2.40 of [1], $|T|$ exists and for all x, we have $|T|(|x|) = |T(|x|)| = |T(x)|$. If $\{x_n\} \subseteq E$ is an almost order bounded disjoint sequence, then by assumption $T(x_n) \xrightarrow{\|\cdot\|} 0$. For each n, we have $|T|(|x_n|) = |T(|x_n|)| = |T(x_n)| \xrightarrow{\|\cdot\|} 0$ in F. The inequality $|(|T|(x_n))| \leq |T||x_n|$, implies that
$$|T|(x_n) \xrightarrow{\|\cdot\|} 0.$$
Hence $|T|$ is an *ao-wc* operator.

□

Theorem 3.20. Let E and F have an order unit with F Dedekind complete. Then $K_{ao-wc}(E,F) \cap L_b(E,F)$ is a band in $L_b(E,F)$.

Proof . It is obvious that if T and $S \in K_{ao-wc}(E,F) \cap L_b(E,F)$ and $\alpha \in \mathbb{R}$, then $T+S$ and $\alpha T \in K_{ao-wc}(E,F) \cap L_b(E,F)$.

Let $|S| \leq |T|$ where $T \in K_{ao-wc}(E,F) \cap L_b(E,F)$, $S \in L_b(E,F)$ and $\{x_n\} \subseteq E$ be almost order bounded disjoint sequence. Without loss of generality, assume that $x_n \geq 0$ for all n. By Theorem 3.19, $|T|(x_n) \xrightarrow{\|\cdot\|} 0$. The inequality $|S(x_n)| \leq |S|(x_n) \leq |T|(x_n)$ implies that $S(x_n) \xrightarrow{\|\cdot\|} 0$. Therefore $S \in K_{ao-wc}(E,F) \cap L_b(E,F)$ and so $K_{ao-wc}(E,F) \cap L_b(E,F)$ is an ideal of $L_b(E,F)$.

Now let $0 \leq T_\alpha \uparrow T$ in $L_b(E,F)$ with $\{T_\alpha\} \subseteq K_{ao-wc}(E,F) \cap L_b(E,F)$. Since T is positive, therefore T is order bounded and since E has an order unit, then by Example 3.2, T is *ao-wc*. Hence $T \in K_{ao-wc}(E,F) \cap L_b(E,F)$. □

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