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# Convergence theorems and demiclosedness principle for enriched strictly pseuocontractive mappings in real Banach spaces

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**Abstract**

In this paper, some new weak and strong convergence results of the Mann and Ishikawa iterative schemes for the class of enriched strictly psuedocontractive mappings are established in the setup of $q$-uniformly smooth Banach spaces. Further, demiclosedness principle for this class of mappings is obtained in the aforementioned space. The results obtained in this paper extend, improve, generalise and unify several well-known results currently announced in the literature.

Keywords: Accretive operators, Enriched nonlinear mapping, demicontractive mappings, Fixed point, Lipschitizian, Pseudocontractive mapping, quasi-nonexpansive mappings, Hilbert space
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## 1 Introduction

Let $\Psi$ be a Banach space, $\emptyset \neq \Delta \subset \Psi$ and $\Psi^\star$ denote the topological dual space of $\Psi$. For $q > 1$, define the mapping $J_q : \Psi \longrightarrow 2^{\Psi^\star}$ by

$$J_q(\wp) = \{\wp^\star \in \Psi^\star : \langle \wp, \wp^\star \rangle = \|\wp\|^q \ and \ \|\wp^\star\| = \|\wp\|^{q-1}, \forall \wp \in \Psi\}. \tag{1.1}$$

Then, $J_q$ is known as generalised duality map on $\Psi$, where $\langle ., . \rangle$ denotes the generalised duality pairing. In particular, $J_2$ is known as normalised duality mapping and it is often represented with $J$. It has been established (see, for instance, [16]) that $J_q(\wp) = \|\wp\|^{q-2}J(\wp)$ if $\wp \neq 0$ and that if $\Psi^\star$ is strictly convex, then $J_q$ is single-valued. In this paper, we shall denote the single-valued duality map by $j_q$. In real Hilbert spaces, $J_q$ is an identity mapping.

Let $\partial : \Delta \longrightarrow \Delta$ be a nonlinear mapping. Throughout the paper, we shall represent the set of fixed point of $\partial$, the set of natural numbers, the set of real numbers, strong and weak convergence by $F(\partial), N, R, \to$ *and* $\rightharpoonup$, respectively.

**Definition 1.1.** A mapping $\partial$ is called $(\sigma, \vartheta)$-enriched strictly pseudocontractive (see [14]) if for all $\wp, \nu \in \Delta$, there exist $\sigma \in [0, +\infty)$ and $j(\wp - nu) \in J(\wp - \nu)$ such that

$$\langle \sigma(\wp - \nu) + \partial\wp - \partial\nu, j((\sigma + 1)(\wp - \nu)) \rangle \leq (\sigma + 1)^2\|\wp - \nu\|^2 - \vartheta\|\wp - \nu - (\partial\wp - \partial\nu)\|^2, \tag{1.2}$$

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where $\vartheta = \frac{1}{2}(1-k)$ for some $k \in [0,1)$.

By setting $\sigma = \frac{1}{\gamma} - 1$ in inequality (1.2) and applying Proposition 2.5(3) (see below), it will not be difficult to see that

$$\langle \eth_\gamma \wp - \eth_\gamma \nu, j(\wp - \nu)\rangle \leq \|\wp - \nu\|^2 - \vartheta \|\wp - \nu - (\eth_\gamma \wp - \eth_\gamma \nu)\|^2, \tag{1.3}$$

where the average operator $\eth_\gamma = (1-\gamma)I + \gamma \eth$ is easily seen as being $\vartheta$-strictly pseudocontractive. If $I$ denotes the identity mapping, then inequality (1.3) is equivalently written as

$$\langle (I - \eth_\gamma)\wp - (I - \eth_\gamma)\nu, j(\wp - \nu)\rangle \geq \vartheta \|\wp - \nu - (\eth_\gamma \wp - \eth_\gamma \nu)\|^2. \tag{1.4}$$

**Remark 1.2.** If $\vartheta = 0$, then (1.2) reduces to

$$\langle \sigma(\wp - \nu) + \eth \wp - \eth \nu, j((\sigma+1)(\wp - \nu))\rangle \leq (\sigma+1)^2 \|\wp - \nu\|^2 \tag{1.5}$$

The class of mappings defined by inequality (1.5) is known as $\sigma$-enriched nonexpansive mapping. The idea of $\sigma$-enriched nonexpansive mapping was initiated by Berinde [2, 3] as a generalisation of an important class of mappings known as nonexpansive mapping. Aside being an obvious generalisation of the contraction mapping (and its close relationship with monotonicity method), nonexpansive mapping is among the first class of nonlinear mappings for which fixed point results were achieved by employing geometric properties instead of the compactness conditions. Further, this class of mappings obviously appears in applications as transition operators for initial value problems of differential inclusion, accretive operators, monotone operators, variational inequality problems and equilibrium problems. In recent times, several generalizations and extensions of this class of mappings have been considered in different directions by different researchers in the available literature; see, for example, [7, 9, 10] and the references therein.

In a real Hilbert space, inequality (1.3) is equivalent to

$$\|\eth_\gamma \wp - \eth_\sigma \nu\|^2 \leq \|\wp - \nu\|^2 + k\|\wp - \nu - (\eth_\gamma \wp - \eth_\gamma y)\|^2, \tag{1.6}$$

where $\eth_\gamma$ is as defined in inequality (1.3). If $k = 1$ in (1.6), then we have a pseudocontraction. Consequently, the class of $(\sigma, k)$-enriched strictly pseudoconyractive mappings is a subclass of the class of $\sigma$-enriched pseudocontractive mappings.

We mention in passing that the class of $(\sigma, k)$-enriched strictly pseudocontractive mappings was first studied by Berinde [1], in the setting of a real Hilbert space, as a generalization of the class of $k$-strictly pseudocontractive mappings (Recall that $\eth$ is called $k$-strictly pseudocontractive mappings if $\|\eth\wp - \eth\nu\|^2 \leq \|\wp - \nu\|^2 + k\|\wp - \nu - (\eth\wp - \eth\nu)\|^2$, for all $\wp, \nu \in \Delta$. If $k = 1$, then we have a pseudocontraction. The class of strictly pseudocontractive mappings, defined in the setup of a real Hilbert space, was introduced in 1967 by Browder and Petryshym [4] as an intermediary class of mappings between the class of nonexpansive mappings and the class of Lipshitz pseudocontractive mappings. It is of paramount important to note that while the class of Lipshitz pseudocontractive mappings are generally not continuous, the class of strictly pseudocontractive mappings inherit Lipshitz property from their definitions). It was proved in [1] that if $\Delta$ is a bounded, closed and convex subset of a real Hilbert space and $\eth : \Delta \longrightarrow \Delta$ is a $(\sigma, k)$-enriched strictly pseudocontractive mapping, then $\eth$ has a fixed point. More precisely, he proved the following theorems:

**Theorem 1.3.** [1] Let $\Delta$ be a bounded closed convex subset of a real Hilbert space and $\eth : \Delta \longrightarrow \Delta$ is a $(\sigma, k)$-enriched strictly pseudocontractive demicompact mapping. Then $F(\eth) \neq \emptyset$, and for any $\wp_0 \in \Delta$ and any fixed $0 < \gamma < 1 - k$, the Krasnoselkii iteration sequence given by

$$\wp_{n+1} = (1-\gamma)\wp_n + \gamma \eth \wp_n, n \geq 0$$

converges strongly to a fixed point of the mapping $\eth$.

**Theorem 1.4.** [1] Let $\Delta$ be a bounded closed convex subset of a real Hilbert space and $\eth : \Delta \longrightarrow \Delta$ be a $(\sigma, k)$-enriched strictly pseudocontractive demicompact mapping for some $0 \leq k < 1$. Then $F(\eth) \neq \emptyset$, and for any $\wp_0 \in \Delta$, and any control sequence $\{\alpha_n\}_n \geq 1$ such that $k < \alpha_n < 1$ and $\sum_{n=1}^{+\infty} (\alpha_n - k)(1 - \alpha_n) = +\infty$, the Krasnoselkii-Mann iteration sequence given by

$$\wp_{n+1} = (1 - \lambda\alpha_n)\wp_n + \lambda\alpha_n \eth \wp_n, n \geq 0,$$

for some $\lambda \in (0,1)$, converges weakly to a fixed point of the mapping $\eth$.

From the above study, it becomes necessary to ask the following question:

**Question 1.1.** 1. Is the result of Theorem 1.3 still valid in the setting of $q$-uniformly smooth Banach spaces?

2. Can the results established in [13] be extended to those of the class of enriched strictly pseudocontractive mappings?

Inspired and motivated by the results in [1, 13], in this paper, our aim is to study the convergence problems of Ishikawa [8] and Mann [12] iterative schemes for the class of enriched strictly psuedocontractive mappings under a more general setting. The results presented in this paper not only extend and improve the main results of [4, 6, 7] but also give an affirmative answer to Question 1.1

## 2 Preliminaries

The following theorems, definitions, proposition and lemma will be helpful in the proofs of our main results. Throughout this section, $\Psi$ and $\Delta$ will denote a real $q$-uniformly smooth Banach space and a nonempty, closed and convex subset of $\Psi$, respectively.

Observe, from (1.4), that

$$\|\wp-\nu\| \geq \vartheta\|\wp-\nu-(\eth_\gamma\wp-\eth_\gamma\nu)\| \geq \vartheta\|\eth_\gamma\wp-\eth_\gamma\nu\|-\vartheta\|\wp-nu\|.$$

Thus,

$$\|\eth\wp-\eth\nu\| \leq \vartheta\|\wp-\nu\|, \forall\wp,\nu\in\Delta,$$

where $\vartheta=\dfrac{1+\lambda}{\lambda}$. Again, since (from (1.4))

$$\|\wp-\nu\| \geq \vartheta\|\wp-\nu-(\eth_\gamma\wp-\eth_\gamma\nu)\|,$$

it follows that

$$\begin{aligned}\langle\wp-\eth\wp-(\nu-\eth\nu),j_q(\wp-\nu)\rangle &= \|\wp-\nu\|^{q-2}\langle\wp-\eth\wp-(\nu-\eth\nu),j(\wp-\nu)\rangle\\ &\geq \vartheta\|\wp-\nu\|^{q-2}\|\wp-\nu-(\eth_\gamma\wp-\eth_\gamma\nu)\|^2\\ &\geq \vartheta^{q-1}\|\wp-\nu-(\eth_\gamma\wp-\eth_\gamma\nu)\|^q.\end{aligned}$$

Let $\Psi$ be as described above. The modulus of smoothness of $\Psi$ is the function $\rho_\Psi:[0,\infty)\longrightarrow[0\infty)$ given by

$$\rho_\Psi(\bar\tau)=\sup\Big\{\frac{1}{2}(\|\wp+\nu\|+\|\wp-\nu\|)-1:\|\wp\|\leq 1,\|\nu\|\leq\bar\tau\Big\}.$$

Here, we note that:

($a$) $\Psi$ is called uniformly smooth if and only if $\lim_{\bar\tau\to 0}\dfrac{\rho_\Psi(\bar\tau)}{\bar\tau}=0$;

($b$) $\Psi$ is called $q$-uniformly smooth (or to have a modulus of of smoothness of power type $q>1$) if there exists a constant $c>0$ such that $\rho_\Psi(\bar\tau)\leq c\bar\tau^q$. Examples of $q$-uniformly smooth spaces include Hilbert spaces, $L_p(or\ \ell_p)$ spaces, $1<p<\infty$ and Sobolev spaces, $W_m^p, 1<p<\infty$. While Hilbert spaces are 2-uniformly smooth,

$$L_p(or\ \ell_p)\ or\ W_m^p\ is\ \begin{cases}p-uniformly\ smooth\ if\ 1<p<2\\ \\ 2-uniformly\ smooth\ if\ p\geq 2.\end{cases}$$

**Theorem 2.1.** [16] Let $q>1$ and $\Psi$ be as described above. Then the following are equivalent:

1. $\Psi$ is $q$-uniformly smooth.
2. There exists a constant $c_q\geq 0$ such that

$$\|\wp+\nu\|^q \leq \|\wp\|^q+q\langle\nu,j_q(\wp)\rangle+c_q\|\nu\|^q, \quad \forall\wp,\nu\in\Psi. \tag{2.1}$$

3. There exists a constant $c_q \geq 0$ such that
$$\|(1-\Im)\wp + \Im\nu\|^q \geq (1-\Im)\|\wp\|^q + \Im\|\nu\|^q - \omega_q(\Im)d_q\|\wp-\nu\|^q, \quad \wp,\nu \in \Psi, \tag{2.2}$$
where $\omega_q(\Im) = \Im^q(1-\Im) + \Im(1-\Im)^q$.

In addition, it was shown in [17] (Remark 5) that if $\Psi$ is $q$-uniformly smooth with $q > 1$, then for all $\wp, \nu \in \Psi$, there exists a constant $\vartheta_\star$ such that
$$\|j_q(\wp) - j_q(\nu)\| \leq \vartheta_\star \|\wp - \nu\|^{q-1}. \tag{2.3}$$

Also, note that $\Psi$ is given Frechet differentiable norm if for all $\wp \in \bar{U} = \{\wp \in \Psi : \|\wp\| = 1\}$,
$$\lim_{\Im\to\infty} \frac{\|\wp+\nu\| - \|\wp-\nu\|}{\Im}$$
exists and is attained uniformly in $\nu \in \bar{U}$. In this case, there exists an increasing function $b : [0,\infty) \longrightarrow [0,\infty)$ with $\lim_{\Im\to\infty} b(\Im) = 0$ such that
$$\frac{1}{2}\|\wp\|^2 + \langle h, j(\wp)\rangle \leq \frac{1}{2}\|\wp+h\|^2 \leq \|\wp\|^2 + \langle h, j(\wp)\rangle + b(\|h\|), \quad \forall \wp, h \in \Psi. \tag{2.4}$$

**Definition 2.2.** Let $\Psi$ be as described above. A mapping $\eth : \Psi \longrightarrow \Psi$ is known as asymptotically regular on $\Psi$ if for each $\hbar \in \Psi$, $\eth^{n+1}\wp - \eth^n\wp \to 0$ strongly as $n \to \infty$.

According to Krasnolskij [11], if $\eth$ is nonexpansive (which is generally known to be non-asymptotically regular) and
$$\eth_\gamma = (1-\gamma)I + \gamma\eth, \quad \gamma \in [0,1],$$
where $I$ is the identity mapping on $\Psi$, $\eth_\gamma$ is the average operator associated with $\eth$, then for any $\beta \in (0,1)$, we have

1. $\eth_\gamma$ is nonexpansive;
2. $Fix(\eth) = Fix(\eth_\gamma)$;
3. $\eth_\gamma$ is asymptotically regular.

It is easy to see, from (1) and (3), that the average operator is richer than the original operator, and, in addition, both operators share the same set of fixed points (see (2)). Hence, the important of $\eth_\gamma$ cannot be over-emphasized as it helps to obtain the approximate fixed point of the original operator $\eth$ through its inclusion in an iterative scheme that seeks the approximate fixed point of $\eth$. It is on this note that Berinde introduced the concept enriched nonexpansive and strictly pseudocontractive mappings in real Hilbert spaces, which were later extended to a real Banach space by Saleem et al [14]; see [1, 2, 3, 14] for more details.

**Definition 2.3.** $\eth$ is said to be demiclosed at a point $\eth$ if whenever $\{\wp_n\}_{n\geq1}$ is a sequence in $D(\eth)$ (domain of $\eth$) such that $\{\wp_n\}_{n\geq1}$ converges weakly to $\wp \in D(\eth)$ and $\eth\wp_n$ converges strongly to $\eth$, $\eth\wp = \eth$. Further, $\eth$ is called demicompact if whenever $\{\wp_n\}_{n\geq1}$ is a bounded sequence in $D(\eth)$ such that $\{\wp_n - \eth\wp_n\}_{n\geq1}$ converges strongly to $\eth$, $\{\wp_n\}_{n\geq1}$ has a subsequence which converges strongly.

**Theorem 2.4.** (see [5]) Let $\Psi$ be a uniformly convex Banach space, $\emptyset \neq \Delta \subset \Psi$ be closed and convex and $\eth : \Delta \longrightarrow \Psi$ be a nonexpansive mapping. Then, $(I - \eth)$ is demiclosed at zero.

The following proposition provides some fundamental properties of duality mapping:

**Proposition 2.5.** [17] Let $\Psi$ be a real Banach space. For $1 < q < \infty$, the duality $J_q : \Psi \longrightarrow 2^{\Psi^\star}$ has the following fundamental properties:

1. $J_q(\wp) \neq \emptyset$ for all $\wp \in \Psi$ and $D(J_q)$(: *the dormain of* $J_q$) $= \Psi$;
2. $J_q(\wp) = \|\wp\|^{q-1}J_2(\wp), \quad \forall \wp \in \Psi (\wp \neq 0)$;
3. $J_q(\alpha\wp) = \alpha^{q-1}J_q(\wp) \quad \alpha \in [0,\infty)$;
4. $J_q(-\wp) = -J_q(\wp)$;

5. $J_q$ is bounded, that is, for any bounded subset $A \subset \Psi$, $J_q(A)$ is a bounded subset in $\Psi^\star$;
6. $J_q$ can be equivalently defined as the subdifferential of the functional $\Psi(\wp) = q^{-1}.\|\wp\|^q$ (see ), that is,

$$J_q(\wp) = \partial\Psi(\wp) = \{f \in \Psi^\star : \Psi(\nu - \Psi(\wp)) \geq \langle f, \nu - \wp\rangle, \forall \nu \in \Psi\};$$

7. $\Psi$ is uniformly smooth Banach space (equivalently, $\Psi^\star$ is uniformly convex Banach space) if and only if $J_q$ is single-valued and uniformly continuous on any bounded subset of $\Psi$.

**Lemma 2.6.** [15] Let $\{\bar{a_n}\}_{n=1}^{\infty}$ and $\{\bar{b_n}\}_{n=1}^{\infty}$ be squences of nonnegative real numbers such that $\sum_{n=1}^{\infty} \bar{b_n} < \infty$ and

$$\bar{a_{n+1}} \leq \bar{a_n} + \bar{b_n}, \quad n \geq 1.$$

Then, $\lim_{n\to\infty} \bar{a_n}$ exists.

## 3 Main Results

Throughout this section, $\vartheta_\star, c_q, d_q$ *and* $w_q(\Im)$ are the constant appearing inequalities (2.1)-(2.3). In the sequel, we state the following definition.

**Definition 3.1.** Let $\Psi$ be a real $q$-uniformly smooth Banach space. A mapping $\eth$ with domain $D(\Psi)$ and range $R(\Psi)$ in $\Psi$ is known as $(\sigma, \vartheta)$-enriched strictly pseudocontractive in the sense Browder and Petryshyn [4] if there exist $\sigma \in [0, \infty)$ and $j_q(\wp - \nu) \in J_q(\wp - \nu)$ such that for all $\wp, \nu \in D(\Psi)$, the following inequality holds:

$$\langle \sigma(\wp - \nu) + \eth\wp - \eth\nu, j_q((\sigma + 1)(\wp - \nu))\rangle \leq (\sigma + 1)^q\|\wp - \nu\|^q - \vartheta^{q-1}\|\wp - \eth\wp - (\nu - \eth\nu)\|^q. \tag{3.1}$$

**Remark 3.2.** The following inequalities are immediate consequences of inequality (3.1).

1. If $q = 2$ and Proposition 2.5(3) is employed, then inequality (3.1) reduces to

$$\langle \sigma(\wp - \nu) + \eth\wp - \eth\nu, j((\sigma + 1)(\wp - \nu))\rangle \leq (\sigma + 1)^2\|\wp - \nu\|^2 - \vartheta\|\wp - \eth\wp - (\nu - \eth\nu)\|^2,$$

a definition considered for a study conducted in [14].

2. If $\sigma = 0$, then inequality (3.1) reduces to

$$\langle \eth\wp - \eth\nu, j_q(\wp - \nu)\rangle \leq \|\wp - \nu\|^q - \vartheta^{q-1}\|\wp - \eth\wp - (\nu - \eth\nu)\|^q,$$

a definition considered for a study carried out in [13].

3. If $\Psi$ is a real Hilbert space, then inequality (3.1) reduces to

$$\langle \sigma(\wp - \nu) + \eth\wp - \eth\nu, (\sigma + 1)(\wp - \nu)\rangle \leq (\sigma + 1)^2\|\wp - \nu\|^2 - \vartheta\|\wp - \eth\wp - (\nu - \eth\nu)\|^2,$$

a definition considered for a study conducted in [1].

4. If $\sigma = 0$ *and* $q = 2$, then inequality (3.1) reduces to

$$\langle \eth\wp - \eth\nu, j(\wp - \nu)\rangle \leq \|\wp - \nu\|^2 - \vartheta\|\wp - \eth\wp - (\nu - \eth\nu)\|^2,$$

a definition considered for a study conducted in [6, 7].

5. If $q = 2, \vartheta = 0$ and Proposition 2.5(3) is employed, then inequality (3.1) reduces to

$$\langle \sigma(\wp - \nu) + \eth\wp - \eth\nu, j((\sigma + 1)(\wp - \nu))\rangle \leq (\sigma + 1)^2\|\wp - \nu\|^2,$$

a definition considered for a study conducted in [2, 3].

Let $\gamma = \dfrac{1}{\sigma + 1}$, then it is clear that $\gamma \in (0, 1]$, and as a consequence, inequality (3.1) becomes

$$\Big\langle \frac{(1-\gamma)}{\gamma}(\wp - \nu) + \eth\wp - \eth\nu, j_q(\frac{1}{\gamma}(\wp - \nu))\Big\rangle \leq \frac{1}{\gamma^q}\|\wp - \nu\|^q - \vartheta^{q-1}\|\wp - \eth\wp - (\nu - \eth\nu)\|^q,$$

which, on applying Proposition 2.5(3) and simplifying, yields

$$\langle \eth_\gamma \wp - \eth_\gamma \nu, j_q(\wp - \nu)\rangle \leq \|\wp - \nu\|^q - \vartheta^{q-1}\|\wp - \eth_\gamma \wp - (\nu - \eth_\gamma \nu)\|^q, \tag{3.2}$$

where $\eth_\gamma = (1-\gamma)I + \gamma\eth$. It is not hard to see that inequality (3.2) is equivalent to

$$\langle (I - \eth_\gamma)\wp - (I - \eth_\gamma)\nu, j_q(\wp - \nu)\rangle \geq \vartheta^{q-1}\|\wp - \eth_\gamma \wp - (\nu - \eth_\gamma \nu)\|^q, \tag{3.3}$$

where $I$ is the identity mapping on $\Delta$. Observe that the average operator $\eth_\gamma$ in both (3.2) and (3.3) is strictly pseudocontractive.

**Lemma 3.3.** Let $\Psi$ be a real $q$-uniformly smooth Banach space and $\emptyset \neq \Delta \subset \Psi$ be convex. Let $\eth : \Delta \longrightarrow \Delta$ be an enriched strictly pseudocontractive mapping. Let $\{\delta_n\}_{n=0}^\infty$ and $\{\tau_n\}_{n=0}^\infty$ be real sequences in $[0,1]$. Define $\eth_n : \Delta \longrightarrow \Delta$ by

$$\eth_n \wp = (1-\delta_n)\wp + \delta_n \eth((1-\tau_n)\wp + \tau_n \eth\wp), \quad \wp \in \Delta \tag{3.4}$$

Then, for all $\wp, \nu \in \Delta$,

$$\begin{aligned} \|\eth_n^\gamma \wp - \eth_n^\gamma \nu\|^q \leq\ & (1+\gamma_n)\|\wp - \nu\|^q - \delta_n[\vartheta^{q-1}q(1-\tau_n) - c_q\delta_n^{q-1}] \\ & \times \|\wp - \eth^\gamma(s_n(\wp)) - (\nu - \eth^\gamma(s_n(\nu)))\|^q, \end{aligned} \tag{3.5}$$

where $\eth_n^\gamma = (I-\gamma)I + \gamma\eth^\gamma, \gamma_n = 2q\delta_n\tau_n\vartheta^{q-1}d_q(1+\vartheta)^q + q\delta_n\vartheta_\star(1+\vartheta)^{q+1}\tau_n^{q-1}, s_n(\wp) = (1-\tau_n)\wp + \tau_n\eth^\gamma\wp$ and $s_n(\nu) = (1-\tau_n)\nu + \tau_n\eth^\gamma\nu$.

**Proof .** Since $\eth$ is $(\sigma,\vartheta)$-enriched strictly psuedocontractive, it follows from (3.1) and (3.2) that

$$\langle \eth^\gamma \wp - \eth^\gamma \nu, j_q(\wp - \nu)\rangle \leq \|\wp - \nu\|^q - \vartheta^{q-1}\|\wp - \eth^\gamma \wp - (\nu - \eth^\gamma \nu)\|^q,$$

where $\eth_\gamma = \eth^\gamma = (1-\gamma)I + \gamma\eth$. It is not difficult to see that the average operator $\eth^\gamma$ is a strict pseudocontraction. Now, define $\eth_n^\gamma : \Delta \longrightarrow \Delta$ by

$$\eth_n^\gamma \wp = (1-\delta_n)\wp + \delta_n\eth^\gamma((1-\tau_n)\wp + \tau_n\eth^\gamma\wp), \quad \wp \in \Delta. \tag{3.6}$$

Then, by setting $\eth_n^\gamma = T_n, \eth^\gamma = T, \wp = x,$ *and* $\nu = y$, it follows from the proof of Lemma 1 in [13] that

$$\begin{aligned} \|\eth_n^\gamma \wp - \eth_n^\gamma \nu\|^q \leq\ & (1+\gamma_n)\|\wp - \nu\|^q - \delta_n[\vartheta^{q-1}q(1-\tau_n) - c_q\delta_n^{q-1}] \\ & \times \|\wp - \eth^\gamma(s_n(\wp)) - (\nu - \eth^\gamma(s_n(\nu)))\|^q, \end{aligned} \tag{3.7}$$

where $\gamma_n = 2q\delta_n\tau_n\vartheta^{q-1}d_q(1+\vartheta)^q + q\delta_n\vartheta_\star(1+\vartheta)^{q+1}\tau_n^{q-1}, s_n(\wp) = (1-\tau_n)\wp + \tau_n\eth^\gamma\wp$ and $s_n(\nu) = (1-\tau_n)\nu + \tau_n\eth^\gamma\nu$. □

**Remark 3.4.** Let $\rho = \max\left\{1, \vartheta\left(\frac{q}{c_q}\right)^{\frac{1}{q-1}}\right\}$, choose any $\delta \in (0,\rho]$ and and put $\delta_n = \delta, \tau_n = 0, \forall n \geq 1$ in equation (3.6). Then, we get $\eth_\delta^\gamma : \Delta \longrightarrow \Delta$ defined for all $\wp \in \Delta$, by $\eth_\delta^\gamma = (I-\delta)\wp + \delta\eth^\gamma\wp$. In addition,

$$\begin{aligned} \|\eth_\delta^\gamma \wp - \eth_\delta^\gamma \nu\|^q \leq\ & \|\wp - \nu\|^q - \delta[q\vartheta^{q-1} - c_q\delta^{q-1}] \\ & \times \|\wp - \eth^\gamma\wp - (\nu - \eth^\gamma\nu)\|^q, \quad \forall \wp, \nu \in \Delta. \end{aligned} \tag{3.8}$$

Since $[q\vartheta^{q-1} - c_q\delta_n^{q-1}] \geq 0$, by virtue of the position of $\delta$, it follows from inequality (3.8) that

$$\|\eth_\delta^\gamma \wp - \eth_\delta^\gamma \nu\| \leq \|\wp - \nu\|, \quad \forall \wp, \nu \in \Delta. \tag{3.9}$$

Hence, $\eth_\delta^\gamma$ is nonexpansive and $F(\eth_\delta^\gamma) = F(\eth^\gamma)$.

**Lemma 3.5.** Let $\Psi$ and $\Delta$ be as described in Lemma 3.3. Let $\eth : \Delta :\longrightarrow \Delta$ be a $(\sigma,\vartheta)$-enriched strictly psuedocontractive mapping with $F(\eth) \neq \emptyset$. Let $\{\delta_n\}_{n=1}^\infty$ and $\{\tau_n\}_{n=1}^\infty$ be real sequences in $[0,1]$ satisfying the conditions:

(*i*) $0 \leq \delta_n, \tau_n \leq 1, \quad n \geq 1$;

(*ii*) $0 \leq e \leq \delta_n^{q-1} \leq f < \left(q\dfrac{\vartheta}{c_q}\right)(1-\tau_n)$, for all $n \geq 1$ and for some $e, f \in (0,1)$;

(*iii*) $\sum_{n=1}^{\infty} \delta_n^{\ell} < \infty,$ *where* $\ell = \min\{1, (q-1)\}$.

Let $\{\wp_n\}_{n=1}^{\infty}$ be a sequence developed from any point $\wp_1 \in \Delta$ by

$$\begin{cases} \nu_n = (1-\tau_n)\wp_n + \tau_n \partial \wp_n, \quad n \geq 1 \\ \\ \wp_{n+1} = (\delta_n)\wp_n + \delta_n \partial \nu_n, \quad n \geq 1. \end{cases} \tag{3.10}$$

Then:

(*a*) $\lim_{n\to\infty} \|\wp_n - \wp^\star\|$ exists for every $\wp^\star \in F(\partial)$;

(*b*) $\lim_{n\to\infty} \|\nu_n - \partial \nu_n\| = 0$.

**Proof .** Since $\partial$ is $(\sigma, \vartheta)$-enriched strictly psuedocontractive, it follows from inequality (3.1) and inequality (3.2) that the average operator $\partial^\gamma = \partial_\gamma = (1-\gamma)I + \gamma\partial$ is strictly pseudocontractive..

In view of this, equation (3.10) can be rewritten as follows:

$$\begin{cases} \nu_n = (1-\tau_n)\wp_n + \tau_n \partial^\gamma \wp_n, \quad n \geq 1 \\ \\ \wp_{n+1} = (\delta_n)\wp_n + \delta_n \partial^\gamma \nu_n, \quad n \geq 1, \end{cases} \tag{3.11}$$

and as a consequence, we prove (*b*) by showing that $\lim_{n\to\infty} \|\nu_n - \partial^\gamma \nu_n\| = 0$. Note that if $\wp^\star \in F(\partial^\gamma)$, then $\partial^\gamma \wp^\star = \wp^\star = (1-\gamma)\wp^\star + \gamma\partial\wp^\star$. Thus, $\wp^\star \in F(\partial^\gamma) = F(\partial)$.

Now, by setting $\wp = \wp_n$ *and* $\nu = \wp^\star$ in Lemma 3.3, we obtain

$$\begin{aligned} \|\wp_{n+1} - \wp^\star\|^q \quad \leq \quad & (1+\gamma_n)\|\wp_n - \wp^\star\|^q - \delta_n[(1-\tau_n)q\vartheta^{q-1} - c_q\delta_n^{q-1}] \\ & \times\|\wp_n - \partial^\gamma \nu_n\|^q, \quad \forall \wp, \nu \in \Delta. \end{aligned} \tag{3.12}$$

Since, from condition (*ii*),

$$(1-\tau_n)q\vartheta^{q-1} - c_q\delta_n^{q-1} \geq [(1-\tau_n)q\vartheta^{q-1} - c_q f] > 0, \quad \forall n \geq 1, \tag{3.13}$$

it follows from (3.12) that

$$\|\wp_{n+1} - \wp^\star\| \quad \leq \quad (1+\gamma_n)\|\wp_n - \wp^\star\|, \quad n \geq 1 \tag{3.14}$$

Again, since $\sum_{n=1}^{\infty} \gamma_n < \infty$ (by condition (iii)), we obtain from inequality (3.14) that $\{\|\wp_n - \wp^\star\|\}_{n=1}^{\infty}$ is bounded. Set $\|\wp_n - \wp^\star\| \leq Q, n \geq 1$, and inequality (3.14) becomes

$$\|\wp_{n+1} - \wp^\star\| \quad \leq \quad (1+\gamma_n)\|\wp_n - \wp^\star\| + Q^q\gamma_n, \quad , \quad n \geq 1 \tag{3.15}$$

for all $(\wp, \wp^\star) \in \Delta \times F(\partial)$. Inequality (3.15) and Lemma 2.6 imply $\lim_{n\to\infty} \|\wp_n - \wp^\star\|$ exists, completing the proof of (*a*).

From inequalities (3.12) and (3.13), we get

$$\begin{aligned} \|\wp_{n+1} - \wp^\star\|^q \quad \leq \quad & \|\wp_n - \wp^\star\|^q - \delta_n[(1-\tau_n)q\vartheta^{q-1} - c_q f]\|\wp_n - \partial^\gamma \nu_n\|^q \\ & + Q^q\gamma_n. \end{aligned} \tag{3.16}$$

In view of the fact that

$$\lim_{n\to\infty}[(1-\tau_n)q\vartheta^{q-1} - c_q\delta_n^{q-1}] = q\vartheta^{q-1} - c_q f > 0,$$

we can find a positive integer $D_0$ with the property that guarantees

$$(1-\tau_n)q\vartheta^{q-1} - c_q\delta_n^{q-1} \geq \frac{1}{2}(q\vartheta^{q-1} - c_q f) > 0, \quad \forall n \geq D_0.$$

With the above information, and following similar approach as in the proof of Lemma 1 in [13], we obtain $\sum_{n=1}^{\infty} \|\wp_n - \eth^\gamma \nu_n\| < \infty$

Observe that

$$\|\wp_n - \eth^\gamma \wp_n\| \leq \|\wp_n - wp^\star\| + \|\wp^\star - \eth^\gamma \wp_n\| \leq (1+\vartheta)\|\wp_n - wp^\star\|,$$

so that

$$\begin{aligned} 0 \leq \|\nu_n - \eth^\gamma \nu_n\| &\leq \|\nu_n - \wp_n\| + \|\wp_n - \eth^\gamma \nu_n\| \\ &\leq \tau_n\|\wp_n - \eth^\gamma \wp_n\| + \|\wp_n - \eth^\gamma \nu_n\| \\ &\leq (1+\vartheta)\tau_n Q + \|\wp_n - \eth^\gamma \nu_n\| \to 0 \quad (as\ n \to \infty). \end{aligned}$$

Consequently, $\gamma \lim_{n\to\infty} \|\nu_n - \eth\nu_n\| = \lim_{n\to\infty} \|\nu_n - \eth^\gamma \nu_n\| = 0$, completing the proof of Lemma 3.4. □

**Corollary 3.6.** Let $\Psi, \Delta, \eth, \{\delta_n\}_{n=1}^{\infty}, \{\tau_n\}_{n=1}^{\infty}$ and $\{\wp_n\}_{n=1}^{\infty}$ be as described in Lemma 3.5. If $\{\wp_n\}_{n=1}^{\infty}$ clusters strongly at some point $\eth$, then $\eth \in F(\eth)$ and $\{\wp_n\}_{n=1}^{\infty}$ converges strongly to $\eth$.

**Proof .** Since $\eth$ is $(\sigma, \vartheta)$-enriched strictly psuedocontractive, using the same argument as in Lemma 3.5, we have that the average operator $\eth^\gamma = \eth_\gamma = (1-\gamma)I + \gamma\eth$ is strictly pseudocontractive. Hence, applying similar technique as in the proof of Corollary 1 in [13], we have

$$\|\eth - \eth^\gamma \eth\| = \|\eth - ((1-\gamma)\eth + \gamma\eth)\eth\| = \gamma\|\eth - \eth\eth\| = 0,$$

so that $\eth \in F(\eth)$. □

**Remark 3.7.** From Corollary 3.6, it is evident that if $\Delta$ is also closed in Lemma 3.5, then either $\{\wp_n\}_{n=1}^{\infty}$ strongly converges to a member of $F(\eth)$ or there is no subsequence $\{\wp_{n_k}\}_{k=1}^{\infty}$ of $\{\wp_n\}_{n=1}^{\infty}$ that converges strongly. If, in particular, $\Delta$ is compact, then $\{\wp_n\}_{n=1}^{\infty}$conveges strongly to a member of $F(\eth)$.

**Corollary 3.8.** Let $\Psi$ be as in Lemma 3.5 and $\emptyset \neq \Delta \subset \Psi$ be closed and convex. Let $\eth$ ia a $\eth : \Delta \longrightarrow \Delta$ be a $(\sigma, \vartheta)$-enriched strictly psuedocontractive and demicompact mapping such that $F(\eth) \neq \emptyset$. Let $\{\delta_n\}_{n=1}^{\infty}, \{\tau_n\}_{n=1}^{\infty}$ and $\{\wp_n\}_{n=1}^{\infty}$ be as described in Lemma 3.5. Then, $\{\wp_n\}_{n=1}^{\infty}$conveges strongly to a member of $F(\eth)$.

**Proof .** Since $\eth$ is a $(\sigma, \vartheta)$-enriched strictly psuedocontractive, then the average operator $\eth^\gamma = \eth_\gamma = (1-\gamma)I + \gamma\eth$ is strictly pseudocontractive (see inequalities (3.1) and (3.2)). From this fact, and following similar proof technique of Corollary 2 in [13], we get (as $n \to \infty$) that $\wp_n \to \eth \in F(\eth^\gamma)$. Since

$$0 = \|\eth - \eth^\gamma \eth\| = \gamma\|\eth - \eth\eth\|, \tag{3.17}$$

it follows that $\eth \in F(\eth) = F(\eth^\gamma)$ □

**Lemma 3.9.** Let $\Psi$ be a real $q$-uniformly smooth Banach space which is also uniformly convex and $\emptyset \neq \Delta \subset \Psi$. Let $\eth, \{\delta_n\}_{n=1}^{\infty}, \{\tau_n\}_{n=1}^{\infty}$ and $\{\wp_n\}_{n=1}^{\infty}$ be as described in Lemma 3.5. Then, for all $\eth_1, \eth_2 \in F(\eth)$, the limit

$$\lim_{n\to\infty} \|\Im\wp_n + (1-\Im)\eth_1 - \eth_2\|$$

exists for all $\Im \in [0,1]$.

**Proof .** Let $a_n(\Im) = \|\Im\wp_n + (1-\Im)\eth_1 - \eth_2\|$. Then, $\lim_{n\to\infty} a_n(0) = \|\eth_1 - \eth_2\|$, and from Lemma 3.5, $\lim_{n\to\infty} a_n(1) = \|\wp_n - \eth_2\|$ exists. It now suffices to show that Lemma 3.9 is true for $\Im \in (0,1)$. Let $\eth_n^\gamma$ be as described in Lemma 3.3. Then

$$\|\eth_n^\gamma \wp - \eth_n^\gamma \nu\| \leq (1+\gamma_n)\|\wp - \nu\| = \zeta_n\|\wp - \nu\|, \quad \forall \wp, \nu \in \Delta,$$

where $\zeta_n = 1 + \gamma_n$. Since $\sum_{n=1}^{\infty} \gamma_n < \infty$, then $\prod_{n=1}^{\infty} \zeta_n < \infty$. Set

$$S_{n,m} = \eth_{n+m-1}\eth_{n+m-2}\eth_{n+m-3}\cdots Game_n, m \geq 1.$$

Then,

$$\|S_{n.m}\wp - S_{n,m}\nu\| \leq \Big(\prod_{k=n}^{n+m-1} \zeta_k\Big)\|\wp - \nu\|, \quad \wp, \nu \in \Delta,$$

$S_{n,m}\wp_n = \wp_{n+1}$. Now, put

$$b_{n,m} = \|S_{n,m}(\Im\wp_n + (1-\Im)\eth_1) - \Im S_{n,m}(\Im\wp_n - (1-\Im)S_{n,m}\eth_1\|$$

and

$$D = \Big(\prod_{k=n}^{\infty} \zeta_k\Big)^2\|\wp - \eth_1\|.$$

Let $\delta$ represent the modulus of convexity of $\Psi$. Then, the rest of the proof follows similarly as in the proof of Lemma 3 in [13] with $t = \Im, x_n = \wp_n, p = \eth$ *and* $k_j = \zeta_k$. Therefore, $\limsup_{n\to\infty} a_n(\eth) \leq \liminf a_n(\eth)$, completing the proof of Lemma 3.9. □

**Lemma 3.10.** Let $\Psi$ be $q$-uniformly smooth Banach space which is also uniformly convex. Let $\emptyset \neq \Delta \subset \Psi$ be convex and $\eth : \Delta \longrightarrow \Delta$ be an enriched strictly pseudocontractive mapping. Let $\{\delta_n\}_{n=1}^{\infty}, \{\tau_n\}_{n=1}^{\infty}$ and $\{\wp_n\}_{n=1}^{\infty}$ be as described in Lemma 3.5. Then, $\forall \eth_1, \eth_2 \in F(\eth), \lim_{n\to\infty}\langle\wp_n, j(\eth_1 - \eth_2)\rangle$ exists. In addition, if $\omega_\omega(\wp_n)$ represents the set of weak subsequential limits of $\{\wp_n\}_{n\geq 1}$, then $\langle\bar{a} - \bar{b}, j(\eth_1 - \eth_2)\rangle = 0, \quad \forall \eth_1, \eth_2 \in F(\eth)$ and $\forall \bar{a}, \bar{b} \in \omega_\omega(\wp_n)$.

**Proof .** Since $\Psi$ is both uniformly smooth and uniformly convex, it has a Frechet differentiable norm. By setting $\wp = \eth_1 - \eth_2, h = \Im(\wp_n - \eth_1)$ in (2.4) and acknowledging the fact that $\eth_\gamma = \eth^\gamma = (1-\gamma) + \gamma\eth$ is strictly pseudocontractive whenever $\eth : \Delta \longrightarrow \Delta$ is a $(\sigma, \vartheta)$-enriched strictly psuedocontractive (see inequalities (3.1) and (3.2)) with $F(\eth^\gamma) = F(\eth)$, we obtain, by following similar method of prove as in proof of Lemma 4 in [13] with $\wp_n = x_n$ *and* $t = \Im$, that

$$\limsup_{n\to\infty}\langle\wp_n, j(\eth_1 - \eth_2\rangle \leq) \liminf_{n\to}\langle\wp_n, i(\eth_1 - \eth_2\rangle + \frac{b(\Im(Q))}{\Im}.$$

Since $\lim_{\Im\to\infty} \dfrac{b(\Im(Q))}{\Im} = 0$, $\lim_{n\to\infty}\langle\wp_n, j(\eth_1 - \eth_2\rangle$ exists. Again, since

$$\lim_{n\to\infty}\langle\wp_n, j(\eth_1 - \eth_2\rangle = \langle\bar{a}, j(\eth_1 - \eth_2\rangle, \quad \forall \bar{a} \in \omega_\omega(\wp_n),$$

we have

$$\langle\bar{a} - \bar{b}, j(\eth_1 - \eth_2\rangle = 0,$$

for all $\eth_1, \eth_2 \in F(\eth)$ and for all $\bar{a}, \bar{b} \in \omega_\omega(\wp_n)$, completing the proof of Lemma 3.10. □

**Theorem 3.11.** Let $\Psi$ be as in Lemma 3.10 and $\emptyset \neq \Delta \subset \Psi$ be closed and convex. Let $\eth : \Delta \longrightarrow \Delta$ be an enriched strictly pseudocontractive mapping. Then, $(I - \eth)$ is demiclosed at zero.

**Proof .** Let $\{\wp_n\}_{n\geq 1}$ be a sequence in $\Delta$ such that $\wp_n \rightharpoonup \eth$ and $(I - \eth)\wp_n \to 0$. Let $\delta$ *and* $\eth_\delta^\gamma$ be as described in Remark 3.4. Then,

$$(I - \eth_\delta^\gamma)\wp_n = \delta(I - \eth)\wp_n \to 0 \quad as\ n \to \infty.$$

Since $\eth_\delta^\gamma$ is nonexpansive, it follows from Theorem 2.4 that $(I - \eth_\delta^\gamma)$ is demiclosed at 0, so that $(I - \eth_\delta^\gamma)\eth = (I - ((I-\gamma)I + \gamma\eth))\eth = \gamma(I - \eth)\eth = 0$. Hence, $(I - \eth)\eth = 0$, completing the proof of Theorem 3.11. □

**Theorem 3.12.** Let $\Psi, \Delta$ *and* $\eth$ be as described in Theorem 3.11. Let $\{\delta_n\}_{n=1}^{\infty}, \{\tau_n\}_{n=1}^{\infty}$ and $\{\wp_n\}_{n=1}^{\infty}$ be as described in Lemma 3.5. Then, $\{\wp_n\}_{n=1}^{\infty}$ converges weakly to a member of $F(\eth)$.

**Proof .** In view of the boundedness of $\{\wp_n\}_{n=1}^{\infty}$, we can find a weakly convergent subsequent of $\{\wp_n\}_{n=1}^{\infty}$. Suppose $\wp_{n_k} \rightharpoonup \eth$. Then, $\eth \in \Delta$ since $\Delta$ is weakly closed. Recall that if $\eth : \Delta \longrightarrow \Delta$ is an enriched strictly pseudocontractive mapping, then $\eth_\gamma = \eth^\gamma = (1-\gamma) + \gamma\eth$ is strictly pseudocontractive (see inequalities (3.1) and (3.2)). Hence, by following similar method proof as in the prove of Theorem 2 in [13], we have that $\omega_\omega(\wp_n)$ is a singleton, so that $\{\wp_n\}_{n=1}^{\infty}$ converges weakly to a point $\eth \in F(\eth^\gamma) = F(\eth)$. □

**Remark 3.13.** If we set $\sigma = 0$, the lemmas, corollaries and theorems established in this paper reduce to the corresponding results in [13].

**Remark 3.14.** If $\tau_n = 0$, for all $n \geq 1$ in our Lemmas, Corollaries and Theorems, we get the corresponding results for the Mann iteration scheme, whih in some sense generalise the results in [1].

**Remark 3.15.** Since Hilbert spaces are 2-uniformly smooth Banach spaces and satisfy (4) with $c_q = 1$, it follows that:

1. Theorem 3 of [1] follows from Corollary 3.8 by setting $q = 2, c_q = 1, \tau_n = 0$ *and* $\delta \in (0, 1-k)$ for all $n \geq 1$.
2. Theorem 5 of [1] follows from Lemma 3.10 and Theorem 3.12 by setting $q = 2, c_q = 1, \tau_n = 0$ *and* $\delta \in (0, 1-k)$ for all $n \geq 1$.

## 4 Conclusion

In this paper, the idea of enriched strictly pseudocontractive mapping is introduced in the setup of real $q$-uniformly smooth Banach spaces. Furthermore, we demonstrated that weak and strong convergence theorems (and demiclosedness principle) for this class of mappings could be obtained in such spaces using Mann and Ishkawa iterative methods. Our results gave an affirmative answer to Question 1.1.

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