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Second-order optimization control problem for McKean-Vlasov systems via L-derivatives

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Abstract

In this paper, we develop a second-order optimality condition for optimal regular-singular control in the integral form of McKean-Vlasov stochastic differential equations. The coefficients of the dynamic depend on the state process as well as on its probability law. The control process has two components, the first being regular and absolutely continuous and the second is an increasing process (componentwise), continuous on the left with limits on the right with bounded variation. The regular control variable is allowed to enter into both drift and diffusion coefficients. The control domain is assumed to be convex. Our main result is proved by applying the L-derivatives with respect to probability law.

Keywords: Second-order necessary conditions, Optimal stochastic control, L-derivatives with respect to measure, McKean-vlasov systems, Regular-singular control, Stochastic differential equation
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1 Introduction

The McKean-Vlasov equation was proposed by Kac [23] as a stochastic system for the Vlasov-Kinetic equation of plasma and the study of which was initiated by McKean system [26]. A general class of stochastic differential equations (SDEs) is governed by McKean-Vlasov equations in which the coefficients are not only functions of the state but also of the probability measure induced by the state itself. A special cases of McKean-Vlasov equation is the mean-field equation in which the coefficients depend not only on the state but also on its mean, Buckdahn [4]. Since then, the McKean-Vlasov theory has found important applications and has become a powerful tool in many fields, such as economics, mathematical finance, optimal control and mean-field games; see Huang, Caines, and Malhame [20] and Lasry and Lions [24]. Stochastic differential systems of the McKean-Vlasov type are Itô's SDEs, where the coefficients of the state equation depend on the state of the solution process as well as of its probability law. McKean-Vlasov type maximum principle for SDEs under partial information has been established in Wang et al. [28]. Optimal control of mean-field jump-diffusion systems with delay has been studied by Meng and Shen [27]. The necessary and sufficient conditions for mean-field SDEs governed by Teugels martingales associated to Lévy process have been studied in [13, 14]. First-order local maximum principle for optimal singular control for mean-field SDEs has been investigated

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by Hafayed [15]. First-order necessary conditions for mean-field FBSDEs of mean-field type have been studied by Hafayed et al. [16]. The McKean-Vlasov maximum principle for SDEs has been established in Buckdahn et al. [4]. Mean-field game has been studied by Lions [25]. The first-order maximum principle for mean-field delay SDE have been investigated in Shen et al. [29]. The general first-order maximum principle for optimal stochastic control has been established in Peng [21]. A Peng's type maximum principle for SDEs of mean-field type was proved by Buckdahn et al., [3] by using second-order derivatives with respect to measures. McKean-Vlasov forward-backward stochastic differential equations have been investigated in Carmona and Delarue [7]. Linear quadratic optimal control problem for conditional McKean-Vlasov equation with random coefficients has been investigated by Pham [22]. Infinite horizon optimal control problems for McKean-Vlasov delay system with semi-Markov modulated jump-diffusion processes have been investigated by Deepa and Muthukumar [8]. First-order necessary conditions for optimal singular control problem for general McKean-Vlasov SDEs have been investigated by Hafayed et al. [17].

Singular stochastic control problems have received considerable attention in the literature. First-order maximum principle for irregular stochastic control problem has been derived by Cadenillas and Haussmann [5]. First-order necessary conditions for general optimal singular stochastic control problems have been derived by Dufour and Miller [10]. Under partial-information, first-order singular control problem for McKean-Vlasov stochastic differential equations driven by Teugels martingales measures has been obtained by Hafayed et al. [14]. First-order necessary and sufficient conditions for near-optimal mean-field stochastic singular control have been established in Hafayed and Abbas [13]. The first-order convex maximum principle for singular optimal control for mean-field SDEs has been derived in Hafayed [15]. Irregular stochastic control problem with linear diffusion and optimal stopping have been studied in Alvarez [1].

Second-order maximum principle for stochastic optimal controls was established by Zhang and Zhang [31] where both drift and diffusion terms may contain the control variable $u(\cdot)$, and the control domain should be convex. The method was further developed in Zhang and Zhang [32] to derive a general pointwise second-order maximum principle, where the control domain is not assumed to be convex. First and second-order necessary conditions for stochastic optimal controls have been studied by [11] and [2]. A second-order maximum principle for singular optimal control for SDEs with uncontrolled diffusion coefficient has been obtained by Tang [30]. Second-order maximum principle for optimal control with recursive utilities has been obtained by Dong and Meng [9]. A second-order necessary conditions for singular optimal controls with recursive utilities of stochastic delay systems have been proved by Huo and Meng [18]. Singular optimal control problems with recursive utilities of mean-field type have been studied in Hao and Meng [19]. Pointwise second-order necessary conditions for stochastic optimal control with jump diffusions have been studied by Ghoul et al. [12].

Motived by the recent works above, in this paper we establish a pointwise second-order necessary conditions of optimal regular-singular control for McKean-Vlasov systems. The first and second order derivatives with respect to measure (in the sense of P-L. Lions) on *Wasserstein space* and the associate Itô formula with some appropriate estimates are applied to derive our result. The McKean-Vlasov dynamics (3.1) occur naturally in the probabilistic analysis of financial optimization problems. Our control model play an important role in different fields of finance and economics, such as conditional mean variance portfolio selection problem with discrete movement in incomplete market. Also, optimal consumption and portfolio problem under some proportional transaction costs. In this paper, we have based ourselves on the notion of first and second-order derivative with respect to the probability measure which was introduced by Lions [25], see also, [3, 6]. Our pointwise second-order optimal control problem is strongly motivated by the recent study of the mean-field games and the related mean-field stochastic control problem, provides also an interesting models in many applications such as mathematical finance. This result is a generalizes the results of Zhang et al. [31] to McKean-Vlasov pontwise second-order maximum principle for optimal regular-singular control. In our class of second-order stochastic control problem, there are two types of singularity:

1. Singularity in the control variable; where the control variable has two components $(u(\cdot), \xi(\cdot))$, the first $u(\cdot)$ being regular (absolutely continuous) and the second $\xi(\cdot)$ is singular. This singularity come since $\mathrm{d}\xi(t)$ may be irregular with respect to Lebesgue measure $\mathrm{d}t$. More precisely $\xi(\cdot)$ an increasing process (componentwise), continuous on the left with limits on the right with bounded variation (see *Definition 3.1*).

2. Following the ideas considered in [9, 18, 19, 31, 32], and in order to derive a second-order necessary conditions, one needs to assume that the first order condition degenerates in some sense. So we define a new type of singularity; in the classical sense for the regular control part and in maximum principle sense for the singular part of the control, (see *Definition 3.2*).

The rest of the paper is organized as follows. The formulation of the first and second-order derivatives with respect to probability measure, and basic notations are given in Section 2. The formulation of the irregular-singular optimal control problem is given in Section 3. In Sections 4 and 5, we prove our McKean-Vlasov type pointwise second-order maximum principle. The final section concludes the paper and outlines some of the possible future developments.

2 L-derivative in the sense of P-L. Lions

We recall briefly an important notion in McKean-Vlasov control problems: the L-derivatives with respect to probability law in *Wasserstein space* which was introduced by P.Lions [25]. The main idea is to identify a distribution $\mu \in Q_2(\mathbb{R}^n)$ with a random variable $x \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ so that $\mu = P_x$. We assume that probability space $(\Omega, \mathcal{F}, P)$ is *rich-enough* in the sense that for evry $\mu \in Q_2(\mathbb{R}^n)$, there is a random variable $x \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ such that $\mu = P_x$. We suppose that there is a sub-$\sigma-$field $\mathcal{F}_0 \subset \mathcal{F}$ such that $\mathcal{F}_0$ is *rich-enough i.e,*

$$Q_2(\mathbb{R}^n) := \left\{ P_x : x \in \mathbb{L}^2(\mathcal{F}_0, \mathbb{R}^n) \right\}. \tag{2.1}$$

By $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ we denote the filtration generated by $W(\cdot)$, completed and augmented by $\mathcal{F}_0$. Next, for any function $g : Q_2(\mathbb{R}^n) \to \mathbb{R}$ we define a function $\widetilde{g} : \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n) \to \mathbb{R}$ such that

$$\widetilde{g}(x) = g(P_x), \ x \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n). \tag{2.2}$$

Clearly, the function $\widetilde{g}$, called the *lift* of g, depends only on the law of $x \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ and is independent of the choice of the representative x, (see [3])

Definition 2.1. Let $g : Q_2(\mathbb{R}^n) \to \mathbb{R}$. The function g is differentiable at a distribution $\mu_0 \in Q_2(\mathbb{R}^n)$ if there exists $x_0 \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$, with $\mu_0 = P_{x_0}$ such that its lift $\widetilde{g}$ is *Fréchet-differentiable* at x_0. More precisely, there exists a continuous linear functional $\mathcal{D}\widetilde{g}(x_0) : \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n) \to \mathbb{R}$ such that

$$\widetilde{g}(x_0 + \zeta) - \widetilde{g}(x_0) = \langle \mathcal{D}\widetilde{g}(x_0) \cdot \zeta \rangle + o(\|\zeta\|_2) = \mathcal{D}_\zeta g(\mu_0) + o(\|\zeta\|_2), \tag{2.3}$$

where $\langle . \cdot . \rangle$ is the dual product on $\mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$. We called $\mathcal{D}_\zeta g(\mu_0)$ the *Fréchet-derivative* of g at μ_0 in the direction ξ. In this case we have

$$\mathcal{D}_\zeta g(\mu_0) = \langle \mathcal{D}\widetilde{g}(x_0) \cdot \zeta \rangle = \frac{\mathrm{d}}{\mathrm{d}t} \widetilde{g}(x_0 + t\zeta) \bigg|_{t=0}, \text{ with } \mu_0 = P_{x_0}. \tag{2.4}$$

By applying *Riesz representation theorem*, there is a unique random variable $\Theta_0 \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ such that $\langle \mathcal{D}\widetilde{g}(x_0) \cdot \zeta \rangle = (\Theta_0 \cdot \zeta)_2 = E[(\Theta_0 \cdot \zeta)_2]$ where $\zeta \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$. It was shown, (see [3]) that there exists a Boral function $\Phi[\mu_0](\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, depending only on the law $\mu_0 = P_{x_0}$ but not on the particular choice of the representative x_0 such that

$$\Theta_0 = \Phi[\mu_0](x_0). \tag{2.5}$$

Thus we can write (2.3) as

$$g(P_x) - g(P_{x_0}) = (\Phi[\mu_0](x_0) \cdot x - x_0)_2 + o(\|x - x_0\|_2), \ \forall x \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n).$$

We denote

$$\partial_\mu g(P_{x_0}, x) = \Phi[\mu_0](x), \ x \in \mathbb{R}^n.$$

Moreover, we have the following identities

$$\mathcal{D}\widetilde{g}(x_0) = \Theta_0 = \Phi[\mu_0](x_0) = \partial_\mu g(P_{x_0}, x_0), \tag{2.6}$$

and

$$\mathrm{D}_\xi g(P_{x_0}) = \langle \partial_\mu g(P_{x_0}, x_0) \cdot \zeta \rangle, \tag{2.7}$$

where $\zeta = x - x_0$. For each probability law $\mu \in Q_2(\mathbb{R}^n)$, $\partial_\mu g(P_x, \cdot) = \Phi[P_x](\cdot)$ is only defined in a $P_x(\mathrm{d}x) - a.e$ sense where $\mu = P_x$.

Among the different notions of differentiability of a function g defined over $Q_2(\mathbb{R}^n)$, we apply for our control problem that introduced by *Lions* [25]. We refer the reader to Buckdahn et al., [3].

Definition 2.2. We say that the function $g \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^n))$ if for all $x \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ there exists a P_x-modification of $\partial_\mu g\,(P_x, \cdot)$ (denoted by $\partial_\mu g$) such that $\partial_\mu g : Q_2\,(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$ is bounded and Lipchitz continuous. That is for some $C > 0$, it holds that

(1) $|\partial_\mu g(\mu, x)| \leq C$, for all $\mu \in Q_2(\mathbb{R}^n)$, $\forall x \in \mathbb{R}^n$.

(2) $|\partial_\mu g(\mu, x) - \partial_\mu g(\mu', x')| \leq C\,[\mathbb{T}\,(\mu, \mu') + |x - x'|]\,,\ \forall \mu, \mu' \in Q_2(\mathbb{R}^n),\ \forall x, x' \in \mathbb{R}^n.$

Noting that if $g \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^n))$ the version of $\partial_\mu g\,(P_x, \cdot)\,,\ x \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ indicate in *Definition 2.2* is unique (see [3, Remark 2.2], and [6]). We shall denote by $\partial_\mu g\,(t, x, \mu_0)$ the derivative with respect to μ computed at μ_0 whenever all the other variables (t, x) are held fixed.

We present a second order derivatives with respected to measure of probability.

Let $g \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^n))$ and consider the mapping $(\partial_\mu g\,(\cdot,\cdot)_1\,, \partial_\mu g\,(\cdot,\cdot)_2\,, ..., \partial_\mu g\,(\cdot,\cdot)_n)^\top : Q_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$.

Definition 2.3 (The space $\mathbb{C}_b^{2,1}(Q_2(\mathbb{R}^n))$). We say that the function $g \in \mathbb{C}_b^{2,1}(Q_2(\mathbb{R}^n))$ if $g \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^n))$ such that $\partial_\mu g(\cdot, x) : Q_2(\mathbb{R}^n) \to \mathbb{R}^n$

(1) $\partial_\mu g(\cdot, y) \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^n))$, $\forall y \in \mathbb{R}^n$ and $i \in \{1, 2, ..., n\}$.

(2) $\partial_\mu g(\mu, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable, for evry $\mu \in Q_2(\mathbb{R}^n)$.

(3) The mapps $\partial_x \partial_\mu g(\cdot,\cdot) : Q_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ and $\partial_\mu^2 g(P_{x_0}, y, Z) : Q_2(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ are bounded and Lipshitz continuous, where $\partial_\mu^2 g(P_{x_0}, y, Z) = \partial_\mu\,[\partial_\mu g(\cdot, y)]\,(P_{x_0}, Z)$. Similar, we define $\partial_u \partial_\mu g(\cdot,\cdot) : Q_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ by $\partial_u \partial_\mu g(P_{x_0}, y, Z) = \partial_u\,[\partial_\mu g(\cdot, y)]\,(P_{x_0}, Z)$.

Now, we give a second-order Taylor expansion that plays an essential role to establish our maximum principle. Let $g \in \mathbb{C}_b^{2,1}(Q_2(\mathbb{R}^n))$, for $i \in \{1, 2, ..., n\}$.

$$\begin{aligned}\mathcal{D}\widetilde{g_i}(x_0) - \mathcal{D}\widetilde{g_j}(x_0 - \xi) &= \int_0^1 \left\langle \mathcal{D}\widetilde{[\partial_\mu g]}_i\,(x_0 + \theta\xi, Z) \cdot \xi \right\rangle d\theta \bigg|_{Z = x_0} \\ &\quad + (\partial_x\,[\partial_\mu g]_i\,(P_{x_0}, x_0)\,, \xi) + o\,(\|\xi\|_2)\,.(d1) \end{aligned} \tag{2.8}$$

then, we obtain $\mathcal{D}\widetilde{[\partial_\mu g]}_i\,(x_0, y) = \left[\partial_\mu^2 g\right]_i (P_{x_0}, y, Z)\Big|_{Z = x_0}$.

Second-order derivatives of f at a measure μ_0. Let $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$ be a copy of the probability space $(\Omega, \mathcal{F}, P)$. For any pair of random variable $(Z, \xi) \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^d) \times \mathbb{L}^2(\mathcal{F}, \mathbb{R}^d)$, we let $(\widehat{Z}, \widehat{\xi})$ be an independent copy of (Z, ξ) defined on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$. We consider the product probability space $(\Omega \times \widehat{\Omega}, \mathcal{F} \otimes \widehat{\mathcal{F}}, P \otimes \widehat{P})$ and setting $(\widehat{Z}, \widehat{\xi})(w, \widehat{w}) = (Z(\widehat{w}), \xi(\widehat{w}))$ for any $(w, \widehat{w}) \in \Omega \times \widehat{\Omega}$.

Let $(\widehat{u^*}(t), \widehat{x^*}(t))$ is an independent copy of $(u^*(t), x^*(t))$, so that $P_{x^*(t)} = \widehat{P}_{\widehat{x^*}(t)}$. We denote by $\widehat{E}$ the expectation under probability measure $\widehat{P}$, where $\widehat{E}\,(X) = \int_{\widehat{\Omega}} X(\widehat{w}) \mathrm{d}\widehat{P}(\widehat{w})$.

Now, for any $\mu_0 \in Q_2(\mathbb{R}^n)$, in the direction ξ, we define the second-order derivatives of a function g at μ_0 with $\mu_0 = P_{x_0}$

$$\mathcal{D}_\xi^2 g\,(\mu_0) = E\left[\widehat{E}\left[tr\left(\partial_\mu^2 g(P_{x_0}, x_0, \widehat{x_0})\widehat{\xi} \otimes \xi\right)\right]\right] + E\left[tr\left(\partial_y \partial_\mu g(P_{x_0}, x_0)\xi \otimes \xi\right)\right], \tag{2.9}$$

where

$$\widehat{E}\left[tr\left(\partial_\mu^2 g(P_{x_0}, x_0, \widehat{x_0})\widehat{\xi} \otimes \xi\right)\right] = \int_{\widehat{\Omega}} tr\left[\partial_\mu^2 g(P_{x_0}, x_0\,(w)\,, \widehat{x_0}(\widehat{w}))\widehat{\xi} \otimes \xi(w, \widehat{w})\right] \mathrm{d}\widehat{P}(\widehat{w}). \tag{2.10}$$

and

$$E\left[\widehat{E}\left[tr\left[\partial_\mu^2 g(P_{x_0}, x_0, \widehat{x_0})\widehat{\xi} \otimes \xi\right]\right]\right] = \int_\Omega \int_{\widehat{\Omega}} tr\left[\partial_\mu^2 g(P_{x_0}, x_0\,(w)\,, \widehat{x_0}(\widehat{w}))\widehat{\xi} \otimes \xi(w, \widehat{w})\right] \mathrm{d}(P \otimes \widehat{P})(w, \widehat{w}). \tag{2.11}$$

For convenience, we will use the following notations throughout the paper, for $\psi = f, \sigma, \ell, h$:

$$\begin{array}{rcl} \delta\psi(t) & = & \psi(t, x^*(t), P_{x^*(t)}, u^*(t)) - \psi(t, x^\varepsilon(t), P_{x^\varepsilon(t)}, u^\varepsilon(t)); \\ \psi_x(t) & = & \dfrac{\partial\psi}{\partial x}(t, x^*(t), P_{x^*(t)}, u^*(t)); \\ \psi_u(t) & = & \dfrac{\partial\psi}{\partial u}(t, x^*(t), P_{x^*(t)}, u^*(t)); \\ \widehat{\psi}_\mu(t) & = & \partial_\mu\psi(t, x^*(t), P_{x^*(t)}, u^*(t); \widehat{x^*}(t)), \\ \widehat{\psi}^*_\mu(t) & = & \partial_\mu\psi(t, \widehat{x^*}(t), P_{x^*(t)}, \widehat{u^*}(t); x^*(t)), \end{array} \tag{2.12}$$

and similarly, we denote the second derivative processes:

$$\begin{array}{rcl} \psi_{xx}(t) & = & \dfrac{\partial^2\psi}{\partial x^2}(t, x^*(t), P_{x^*(t)}, u^*(t)), \\ \psi_{uu}(t) & = & \dfrac{\partial^2\psi}{\partial u^2}(t, x^*(t), P_{x^*(t)}, u^*(t)), \\ \widehat{\psi}_{\mu\mu}(t) & = & \partial^2_\mu\psi(t, x^*(t), P_{x^*(t)}, u^*(t); x^*(t), \widehat{x^*}(t)), \\ \psi_{x\mu}(t) & = & \partial_x\partial_\mu\psi(t, x^*(t), P_{x^*(t)}, u^*(t); x^*(t)), \\ \widehat{\psi}^*_{x\mu}(t) & = & \partial_x\partial_\mu\psi(t, \widehat{x^*}(t), P_{x^*(t)}, \widehat{u^*}(t); \widehat{x^*}(t)). \end{array} \tag{2.13}$$

3 Formulation of the regular-singular control problem

Let us formulate the optimal mean-field type control problem. Let T be a fixed strictly positive real number and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a fixed filtered probability space satisfying the usual conditions in which $one-$dimensional Brownian motion $W(t) = \{W(t) : 0 \leq t \leq T\}$ and $W(0) = 0$ is defined.

We study optimal solutions of general stochastic control problem driven by stochastic differential equation of McKean-Vlasov type:

$$\begin{cases} \mathrm{d}x^{u,\xi}(t) = f\left(t, x^{u,\xi}(t), P_{x^{u,\xi}}(t), u(t)\right)\mathrm{d}t + \sigma\left(t, x^{u,\xi}(t), P_{x^{u,\xi}}(t), u(t)\right)\mathrm{d}W(t) \\ \qquad\qquad + G(t)\mathrm{d}\xi(t), \ t \in [0,T] \\ x^{u,\xi}(0) = x_0. \end{cases} \tag{3.1}$$

The criteria to be minimized over the class of admissible controls has the form

$$J\left(u(\cdot), \xi(\cdot)\right) = E\left[h(x^{u,\xi}(T)) + \int_0^T \ell(t, x^{u,\xi}(t), P_{x^{u,\xi}}(t), u(t))\mathrm{d}t + \int_{[0,T]} M(t)\mathrm{d}\xi(t)\right]. \tag{3.2}$$

Here the regular-singular control variable is a pair $(u(\cdot), \xi(\cdot))$ of measurable $\mathbb{A}_1 \times \mathbb{A}_2$-valued, $\mathbb{F}-$adapted processes, where $\mathbb{A}_1$ is a closed convex subset of $\mathbb{R}^m$ and $\mathbb{A}_2 := [0,\infty)^m$

Definition 3.1. An admissible regular-singular control is a pair $(u(\cdot), \xi(\cdot))$ of measurable $\mathbb{A}_1\times\mathbb{A}_2-$valued, $\mathbb{F}-$adapted processes, such that $\xi(\cdot)$ is of bounded variation, non-decreasing continuous on the left with right limits and $\xi(0_-) = 0$. Moreover, $E\left[\sup_{t\in[0,T]} |u(t)|^2\right] < \infty$ and $E\left|\xi(T)\right|^2 < \infty$.

We should note that since $\mathrm{d}\xi(t)$ may be singular with respect to Lebesgue measure $\mathrm{d}t$, we call $\xi(\cdot)$ the irregular or singular part of the control and the process $u(\cdot)$ its regular or absolutely continuous part. This construction allows us to define integrals of the form $\int_{[0,T]} G(t)\mathrm{d}\xi(t)$ and $\int_{[0,T]} M(t)\mathrm{d}\xi(t)$. Denote by $\mathcal{A}_1 \times \mathcal{A}_2$ the set of $\mathcal{B}([0,T])\otimes\mathcal{F}$-measurable and $\mathbb{F}$-adapted stochastic processes valued in $\mathbb{A}_1 \times \mathbb{A}_2$. Any $(u(\cdot), \xi(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$ is called an admissible control. The stochastic optimal control problem considered in this paper is to find a pair of adapted processes $(u^*(\cdot), \xi^*(\cdot)) \in \mathcal{A}_1\times\mathcal{A}_2$ such that

$$J\left(u^*(\cdot), \xi^*(\cdot)\right) = \min_{(u(\cdot),\xi(\cdot))\in\mathcal{A}_1\times\mathcal{A}_2} J\left(u(\cdot), \xi(\cdot)\right). \tag{3.3}$$

Any admissible control $(u^*(\cdot), \xi^*(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$ satisfying (3.3) is called an optimal control. The corresponding state $x^*(\cdot)$ is called an optimal state, and $(x^*(\cdot), u^*(\cdot), \xi^*(\cdot))$ is called an optimal solution of the control problem (3.1)-(3.3). The maps

$$\begin{array}{rcl} f & : & [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times \mathbb{A}_1 \rightarrow \mathbb{R}^n \\ \sigma & : & [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times \mathbb{A}_1 \rightarrow \mathbb{M}^{n\times d}(\mathbb{R}) \\ \ell & : & [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times \mathbb{A}_1 \rightarrow \mathbb{R} \\ h & : & \mathbb{R}^n \times Q_2(\mathbb{R}^n) \rightarrow \mathbb{R} \\ G & : & [0,T] \times \Omega \rightarrow \mathcal{M}_{n\times m}(\mathbb{R}) \\ M & : & [0,T] \times \Omega \rightarrow [0,\infty)^m \end{array}$$

are given deterministic functions, where $Q_2(\mathbb{R}^n)$ is Wasserstein space of probability measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ with finite second-moment, i.e; $\int_{\mathbb{R}^n} |x|^2 \mu(\mathrm{d}x) < \infty$, endowed with the following $2-$*Wasserstein metric*: for $\mu_1, \mu_2 \in Q_2(\mathbb{R}^n)$,

$$\mathbb{T}_2(\mu_1, \mu_2) = \inf_{\rho(\cdot,\cdot)\in Q_2(\mathbb{R}^{2d})} \left\{ \left[\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^2 \rho(\mathrm{d}x, \mathrm{d}y) \right]^{\frac{1}{2}} \right\}, \tag{3.4}$$

where $\rho \in Q_2(\mathbb{R}^{2n})$, $\rho(\cdot, \mathbb{R}^n) = \mu_1$, and $\rho(\mathbb{R}^n, \cdot) = \mu_2$. This distance $\mathbb{T}_2(\cdot,\cdot)$ is just the *Monge-Kankorovich distance* when $p = 2$.

In order not to over complicate the already notational heavy presentation of this paper, in what follows we shall assume all processes are *one*-dimensional $(i.e., n = d = m = 1)$.

Assumptions. The following assumptions will be in force throughout this paper, where x denotes the state variable, and u the control variable.

- **Hypothesis (H1)** For fixed $\mu \in Q_2(\mathbb{R})$, for any $(x,u) \in \mathbb{R}^d \times \mathbb{A}_1$, the coefficients f, σ, ℓ are measurable in all variables and continuously differentiable up to order-2 with respect to x, u; and al their partial derivatives are uniformly bounded. The function h is continuously differentiable up to order-2 with respect to x and u.

  Moreover the second-order derivatives $\psi_{xx}, \psi_{uu}, \psi_{xu}$, for $\psi = f, \sigma, \ell$ are bounded and Lipshitz in (x,u). The derivative h_{xx} is bounded and Lipshitz in x.

$$\begin{array}{rcl} |\ell(t,x,\mu,u)| & \leq & C(1+|x|^2+|u|^2) \\ |h(x,u)| & \leq & C(1+|x|^2) \\ |\ell_x(t,x,\mu,u)| + |\ell_u(t,x,\mu,u)| & \leq & C(1+|x|+|u|) \\ |h_x(x,u)| & \leq & C(1+|x|). \end{array}$$

  where $C > 0$ is a generic positive constant, which may vary from line to line.

- **Hypothesis (H2)** (1) For fixed $x \in \mathbb{R}$, for all $u(t) \in \mathbb{A}_1 : f, \sigma, \ell \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^d); \mathbb{R})$, and $h \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}); \mathbb{R})$.

  (2) All the derivatives with respect to measure f_μ, σ_μ, ℓ_μ, h_μ are bounded and Lipschitz continuous, with Lipschitz constants independent of u.

- **Hypothesis (H3)** (1) The coefficients f, σ, ℓ, h satisfy assumption (H2).

  (2) For all $u(t) \in \mathbb{A}_1$, $f, \sigma, \ell \in \mathbb{C}_b^{2,1}(Q_2(\mathbb{R}); \mathbb{R})$, and $h \in \mathbb{C}_b^{2,1}(Q_2(\mathbb{R}); \mathbb{R})$.

  (3) All the second-order derivatives of $\psi_{\mu\mu}$, $\psi_{x\mu}$ $\psi_{u\mu}$ for $\psi = f, \sigma, \ell$ are bounded and Lipschitz continuous in (x, μ, u) with Lipschitz constants independent of u.

  (4) The second-order derivative $h_{\mu\mu}$, $h_{x\mu}$ is bounded and Lipschitz in x and μ.

- **Hypothesis (H4)** The maps: $G(\cdot) : [0,T] \times \Omega \rightarrow \mathcal{M}_{n\times m}(\mathbb{R})$, and $M(\cdot) : [0,T] \times \Omega \rightarrow [0,\infty)^m$ are bounded and continuous.

From assumption (H3), Item 3, since the second-order derivatives are Lipschitz continuous, we have

$$\left\{ \begin{array}{l} \forall \mu, \mu' \in Q_2(\mathbb{R}^n), \forall x, x' \in \mathbb{R}^n, \forall u, u' \in \mathbb{A}_1 : \\ |(\psi_{\mu\mu}, \psi_{x\mu}, \psi_{u\mu})(t,x,\mu,u) - (\psi_{\mu\mu}, \psi_{x\mu}, \psi_{u\mu})(t,x',\mu',u')| \\ \leq C[\mathbb{T}_2(\mu,\mu') + |x-x'| + |u-u'|]. \end{array} \right. \tag{3.5}$$

Similarly for Item 4, we deduce $\forall \mu, \mu' \in Q_2(\mathbb{R}^n)$, and $\forall x, x' \in \mathbb{R}^n$:

$$|(h_{\mu\mu}, h_{x\mu})(x,\mu) - (h_{\mu\mu}, h_{x\mu})(x',\mu')| \leq C\left[\mathbb{T}_2(\mu,\mu') + |x-x'|\right]. \tag{3.6}$$

Under the assumptions $(\mathbf{H1})-(\mathbf{H4})$, for each $u(\cdot) \in \mathcal{A}_1$, Eq-(3.1) has a unique strong solution $x^{u,\xi}(\cdot)$ such that $E\left[\sup_{s\in[0,T]} |x^{u,\xi}(s)|^2\right] < +\infty$, and the functional $J(\cdot,\cdot)$ is well defined. Let $(u^*(\cdot), \xi^*(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$ is an optimal regular-singular control for the problem (3.1)-(3.2). The corresponding state process is $x^*(\cdot) = x^{u^*,\xi^\bullet}(\cdot)$.
We define for $t \in [0,T]$:

$$\begin{aligned} \mathcal{L}_{xx}(t,\varphi,z) &= \frac{1}{2}\partial_{xx}\varphi(t,x^*(t),P_{x^*(t)},u^*(t))z^2, \\ \mathcal{L}_{y\mu}(t,\widehat{\varphi},z) &= \frac{1}{2}\partial_y\partial_\mu\varphi(t,x^*(t),P_{x^*(t)},u^*(t);\widehat{x^*})z^2. \end{aligned} \tag{3.7}$$

The Hamiltonian. We define the Hamiltonian function associated to our control problem. For any $(t,x,\mu,u,p,q) \in [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$$H(t,x,\mu,u,p_1,q) = f(t,x,\mu,u), p_1 + \sigma(t,x,\mu,u)q_1 - \ell(t,x,\mu,u). \tag{3.8}$$

where $(p_1(\cdot), q_1(\cdot))$ be a pair of adapted processes, solution of the first-order adjoint equation (3.13). We denote

$$H(t) = H(t,x^*,P_{x^*},u^*,p_1,q_1). \tag{3.9}$$

We define

$$\begin{aligned} \delta H(t) &= \delta f(t)p_1(t) + \delta\sigma(t)\cdot q_1(t) - \delta\ell(t); \\ H_x(t) &= f_x(t)p_1(t) + \sigma_x(t)q_1(t) - \ell_x(t); \\ H_u(t) &= f_u(t)p_1(t) + \sigma_u(t)q_1(t) - \ell_u(t); \\ H_\mu(t) &= f_\mu(t)p_1(t) + \sigma_\mu(t)q_1(t) - \ell_\mu(t); \\ H_{xx}(t) &= f_{xx}(t)p_1(t) + \sigma_{xx}(t) \otimes q_1(t) - \ell_{xx}(t). \\ H_{uu}(t) &= f_{uu}(t)p_1(t) + \sigma_{uu}(t) \otimes q_1(t) - \ell_{uu}(t). \\ H_{x\mu}(t) &= f_{x\mu}(t)p_1(t) + \sigma_{x\mu}(t) \otimes q_1(t) - \ell_{x\mu}(t). \\ H_{\mu\mu}(t) &= f_{\mu\mu}(t)p_1(t) + \sigma_{\mu\mu}(t) \otimes q_1(t) - \ell_{\mu\mu}(t). \end{aligned} \tag{3.10}$$

To establish our integral-type second-order necessary condition for stochastic optimal control, we introduce the following notion.

Definition 3.2 (Singularity in the classical sense). We call an admissible control $(\overline{u}(\cdot), \overline{\xi}(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$ a singular pair in the classical sense if $(\overline{u}(\cdot), \overline{\xi}(\cdot))$ satisfies

$$\begin{cases} H_u(t,\overline{x}(t),\overline{u}(t),\overline{p}_1(t),\overline{q}_1(t)) = 0, \quad a.s.\ a.e.t \in [0,T], \\ H_{uu}(t,\overline{x}(t),\overline{u}(t),\overline{p}_1(t),\overline{q}_1(t)) + \overline{p}_2(t)\sigma_u(t,\overline{x}(t),\overline{u}(t))^2] = 0, \\ \qquad\qquad a.s.\ a.e.\ t \in [0,T], \end{cases} \tag{3.11}$$

and

$$E\int_{[0,T]} (M(t) - \overline{p}_1(t)G(t))\,\mathrm{d}\xi(t) = E\int_{[0,T]} (M(t) - \overline{p}_1(t)G(t))\,\mathrm{d}\overline{\xi}(t), \tag{3.12}$$

for any $(u(\cdot), \xi(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$.

Other type of singularity have been studied by some authors. Singularity in classical sense has been considered in [19, Definition 2.4] and [31, Definition 3.3], singularity in Pontryagin-type maximum principle sense has been investigated in [32, Definition 3.2] and partially singular control in classical sense in [11, Definition 4.1]. We introduce the adjoint equations involved in the stochastic maximum principle for our control problem.

First-order adjoint equation. We consider the first-order adjoint equation, which has the following McKean-Vlasov linear BSDE:

$$\begin{cases} -\mathrm{d}p_1(t) = \Big[f_x(t)p_1(t) + \widehat{E}\Big[\widehat{f}^*_\mu(t)\widehat{p_1}(t)\Big] + \sigma_x(t)q_1(t) + \widehat{E}\left[\widehat{\sigma}^*_\mu(t)\widehat{q_1}(t)\right] \\ \qquad\qquad - \ell_x(t) - \widehat{E}\Big[\widehat{\ell}^*_\mu(t)(t)\Big]\Big]\,\mathrm{d}t - q_1(t)\mathrm{d}W(t), \\ p_1(T) = -h_x(T) - \widehat{E}[\widehat{h}^*_\mu(T)]. \end{cases} \tag{3.13}$$

Here, from (2.13), $t \in [0,T]$, for $\varphi = f, \sigma, \ell$

$$\begin{aligned} \widehat{E}\left[\partial_\mu \widehat{\varphi^*}(t)\right] &= \widehat{E}\left[\partial_\mu \varphi(t, \widehat{x}(t), P_{x^*(t)}, \widehat{u}^*(t); z)\right]\Big|_{z=x^*(t)} \\ &= \int_{\widehat{\Omega}} \partial_\mu \varphi(t, \widehat{x}(t,\widehat{w}), P_{x^*(t,w)}, \widehat{u}^*(t,\widehat{w}); x^*(t,w))\mathrm{d}\widehat{P}(\widehat{w}), \end{aligned} \tag{3.14}$$

and the same argument allows to show that

$$\begin{aligned} \widehat{E}\left[\partial_\mu \widehat{h}^*(T)\right] &= \widehat{E}\left[\partial_\mu h(\widehat{x}(T), P_{x^*(T)}; z)\right]\Big|_{z=x^*(t)} \\ &= \int_{\widehat{\Omega}} \partial_\mu h(\widehat{x}(T,\widehat{w}), P_{x(T,w)}; x^*(T,w))\mathrm{d}\widehat{P}(\widehat{w}). \end{aligned} \tag{3.15}$$

Second-order adjoint equation. Consider the following linear BSDE:

$$\begin{cases} \mathrm{d}p_2(t) = -\Big\{2(f_x(t) + \widehat{E}[\widehat{f}^*_\mu(t)])p_2(t) + [\sigma_x(t) + \widehat{E}(\widehat{\sigma}_\mu(t))]^2 p_2(t) \\ \qquad\qquad + 2(\sigma_x(t) + \widehat{E}[\widehat{\sigma}_\mu(t)])q_2(t) + (H_{xx}(t) + \widehat{E}[\widehat{H}_{\mu y}(t)])\Big\}\,\mathrm{d}t + q_2(t)\mathrm{d}W(t), \\ p_2(T) = -(h_{xx}(T) + \widehat{E}[\widehat{h}^*_{\mu y}(T)]). \end{cases} \tag{3.16}$$

Similar to (3.14) and (3.15), we have

$$\begin{aligned} \widehat{E}[\widehat{H}^*_{\mu y}(t)]) &= \widehat{E}\left[\partial_\mu \partial_y H(t, \widehat{x}(t), P_{x^*(t)}, \widehat{u}^*(t), \widehat{p_1}(t), \widehat{q_1}(t); y)\right]\Big|_{y=x^*(t)} \\ &= \int_{\widehat{\Omega}} \partial_\mu \partial_y H(t, \widehat{x}(t,\widehat{w}), P_{x^*(t)}, \widehat{u}^*(t,\widehat{w}), \widehat{p_1}(t), \widehat{q_1}(t); x^*(t))\mathrm{d}\widehat{P}(\widehat{w}). \end{aligned}$$

Since the derivatives $f_x, f_\mu, \sigma_x, \sigma_\mu, \ell_x, \ell_\mu, h_x, h_\mu$ are bounded, (from assumptions (H1) and (H2)), the BSDE (3.13) admits a unique $\mathcal{F}_t$-adapted strong solution $(p_1(\cdot), q_1(\cdot))$ such that

$$E\left[\sup_{s\in[0,T]} |p_1(s)|^2 + \int_0^T |q_1(s)|^2\,\mathrm{d}s\right] < \infty. \tag{3.17}$$

Also, from the boundness of the first and second-order derivatives of the coefficients f, σ, ℓ, and h with respect to (x, μ), Eq-(3.16) has a unique $\mathcal{F}_t-$adapted strong solution $(p_2(\cdot), q_2(\cdot))$ which satisfies the following estimate

$$E\left[\sup_{s\in[0,T]} |p_2(s)|^2 + \int_0^T |q_2(s)|^2\,\mathrm{d}s\right] < \infty. \tag{3.18}$$

If the coefficients f, σ, ℓ, and h do not explicitly depend on law of the solution, the McKean-Vlasov BSDE-(3.13) and (3.16) reduce to a standard BSDE (see Zhang and Zhang [31]. Peng [21, Equation 19, page 974]), or Buckdahn et al., ([3]).

4 Second-order necessary conditions for McKean-Vlasov optimal regular-singular control

The aim of the stochastic maximum principle is to establish a set of necessary conditions for optimality satisfied by an optimal mixed control. In our paper, the goal is to derive a set of second-order necessary conditions for the optimal control, where the system evolves according to controlled McKean-Vlasov SDEs. To derive our main result, the approach that we use is based on the convex perturbation of the optimal regular-singular control. This perturbation is described by the following method:

Let $(x^*(\cdot), u^*(\cdot), \xi^*(\cdot))$ be an optimal solution and $(u(\cdot), \xi(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$ be any given admissible control. Let $\varepsilon \in (0,1)$, and write

$$u^\varepsilon(\cdot) = u^*(\cdot) + \varepsilon v(\cdot) \quad \text{where } v(\cdot) = u(\cdot) - u^*(\cdot), \tag{4.1}$$

and

$$\xi^\varepsilon(t) = \xi^*(t) + \varepsilon\zeta(t) \text{ where } \zeta(t) = \xi(t) - \xi^*(t). \tag{4.2}$$

where ε a sufficiently small $\varepsilon > 0$. Denote by $x^\varepsilon = x^{u^\varepsilon, \xi^\varepsilon}$ the state of (3.1) with respect to $(u^\varepsilon(\cdot), \xi^\varepsilon(\cdot))$.

We introduce the following new variational equations for our control problem.

First-order variational equation: let $t \in [0,T]$

$$\begin{cases} \mathrm{d}y_1(t) = \Big[f_x(t)y_1(t) + \widehat{E}[\widehat{f}_\mu(t)\widehat{y}_1(t)] + f_u(t)v(t)\Big]\,\mathrm{d}t \\ \qquad\qquad + \Big[\sigma_x(t)y_1(t) + \widehat{E}[\widehat{\sigma}_\mu(t)\widehat{y}_1(t)] + \sigma_u(t)v(t)\Big]\,\mathrm{d}W(t) \\ \qquad\qquad + G(t)\mathrm{d}\zeta(t),\ \ t \in [0,T] \\ y_1(0) = 0. \end{cases} \tag{4.3}$$

Here the process $y_1(\cdot)$ is called the *first-order variational process* which is depend explicitly on irregular control. Since the coefficients $f_x, f_\mu, f_u, \sigma_x, \sigma_\mu, \sigma_u$ in (4.3) are bounded, it follows that there exists a unique solution $y_1(\cdot)$ such that $k \geq 2$

$$E\left[\sup_{t\in[0,T]} |y_1(t)|^k\right] < C_k. \tag{4.4}$$

We note that unless specified, for each $k \in \mathbb{R}_+$, we denote by $C_k > 0$ a generic positive constant depending only on k, which may vary from line to line.

Second-order variational equation:

$$\begin{cases} \mathrm{d}y_2(t) = \Big[f_x(t)y_2(t) + \widehat{E}[\widehat{f}_\mu(t)\widehat{y}_2(t)] + f_{xx}(t)y_1^2(t) + \widehat{E}[\widehat{f}_{x\mu}(t)\widehat{y}_1(t)]y_1(t) \\ \qquad\qquad + 2f_{xu}(t)y_1(t)v(t) + 2\widehat{E}[\widehat{f}_{u\mu}(t)\widehat{y}_1(t)]v(t) + f_{uu}(t)v^2(t)\Big]\,\mathrm{d}t \\ \qquad\qquad + \Big[\sigma_x(t)y_2(t) + \widehat{E}[\widehat{\sigma}_\mu(t)\widehat{y}_2(t)] + \sigma_{xx}(t)y_1^2(t) + \widehat{E}[\widehat{\sigma}_{x\mu}(t)\widehat{y}_1(t)]y_1(t) \\ \qquad\qquad + 2\sigma_{xu}(t)y_1(t)v(t) + 2\widehat{E}[\widehat{\sigma}_{u\mu}(t)\widehat{y}_1(t)]v(t) + \sigma_{uu}(t)v^2(t)\Big]\,\mathrm{d}W(t), \\ y_2(0) = 0. \end{cases} \tag{4.5}$$

Here the stochastic process $y_2(\cdot)$ is called the *second-order variational process.* Moreover, under assumptions (H1) and (H2), equation (4.5) admits a unique $\mathbb{F}$-adapted strong solution such that: for any $k \geq 1$ we have

$$E(\sup_{t\in[0,T]} |y_2(t)|^k) \leq C_k. \tag{4.6}$$

We derive some fundamental estimates that will play the crucial roles to establish our result.

Proposition 4.1. Let $x^\varepsilon(\cdot)$ and $x^*(\cdot)$ be the states of (4.7) associated to $u^\varepsilon(\cdot)$ and $u^*(\cdot)$ respectively. Let $y_1(\cdot)$ be the solutions of (4.3). Then the following estimates hold:

$$E\left[\sup_{t\in[0,T]} |x^\varepsilon(t) - x^*(t)|^{2k}\right] \leq C_k \varepsilon^{2k}, \tag{4.7}$$

$$\lim_{\varepsilon\to 0} E\left[\sup_{0\leq t\leq T} \left|\frac{x^\varepsilon(t) - x^*(t)}{\varepsilon} - y_1(t)\right|^2\right] = 0. \tag{4.8}$$

Proof . The proof of estimate (4.7) follows immediately from [3, Proposition 4.2, estimate (4.8)]. Let us turn to estimate (4.8). We put

$$\gamma^{\varepsilon}(t)=\frac{x^{\varepsilon}(t)-x^{*}(t)}{\varepsilon}-y_{1}(t),\ t\in[0,T]\,. \tag{4.9}$$

Since $D_{\xi}f(P_{Z_0})=\left\langle D\widetilde{f}(Z_0)\cdot\xi\right\rangle=\frac{d}{dt}\widetilde{f}\left(Z_0+t\xi\right)\Big|_{t=0}$, we have the following simple form of the *Taylor expansion*

$$f(P_{Z_0+\eta})-f(P_{Z_0})=D_{\xi}f(P_{Z_0})+\mathcal{R}(\eta),$$

where $\mathcal{R}(\eta)$ is of order $o\left(\|\xi\|_2\right)$ with $o\left(\|\eta\|_2\right)\to 0$ for $\eta\in\mathbb{L}^2\left(\mathcal{F},\mathbb{R}^d\right)$.

$$\begin{aligned}
\gamma^{\varepsilon}(t) \;=\;& \frac{1}{\varepsilon}\int_0^t\left[f\left(s,x^{\varepsilon}(s),P_{x^{\varepsilon}(s)},u^{\varepsilon}(s)\right)-f\left(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)\right)\right]\mathrm{d}s\\
&+\frac{1}{\varepsilon}\int_0^t\left[\sigma\left(s,x^{\varepsilon}(s),P_{x^{\varepsilon}(s)},u^{\varepsilon}(s)\right)-\sigma\left(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)\right)\right]\mathrm{d}W(s)\\
&-\int_0^t\Big\{f_x\left(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)\right)y_1(s)+\widehat{E}\left[f_{\mu}(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s);\widehat{x}^{*}(s))\widehat{y_1}(s)\right]\\
&+f_u(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s))v(s)\Big\}\mathrm{d}s\\
&-\int_0^t\Big\{\sigma_x\left(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)\right)y_1(s)+\widehat{E}\left[\sigma_{\mu}(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s);\widehat{x}^{*}(s))\widehat{y_1}(s)\right]\\
&+\sigma_u(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s))v(s)\Big\}\mathrm{d}W(s),+\frac{1}{\varepsilon}\int_{[0,t]}G(s)\mathrm{d}\left(\xi^{\varepsilon}-\xi^{*}\right)(s)-\int_{[0,t]}G(s)\mathrm{d}\zeta(s).
\end{aligned}$$

Since $\zeta(t)=\xi(t)-\xi^{*}(t)$, then by simple calculation, we shows that

$$\frac{1}{\varepsilon}\int_{[0,t]}G(s)\mathrm{d}\left(\xi^{\varepsilon}-\xi^{*}\right)(s)-\int_{[0,t]}G(s)\mathrm{d}\zeta(s)=\frac{1}{\varepsilon}\left[\int_{[0,t]}G(s)\mathrm{d}\left(\xi^{\varepsilon}-\xi^{*}\right)(s)-\varepsilon\int_{[0,t]}G(s)\mathrm{d}\zeta(s)\right]=0.$$

We decompose the integral $\frac{1}{\varepsilon}\int_0^t[f(s,x^{\varepsilon}(s),P_{x^{\varepsilon}(s)},u^{\varepsilon}(s))-f(s,x^{*}(s),P_{x^{*}(s)},x^{*}(s))]\mathrm{d}s$ into the following parts:

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t(f(s,x^{\varepsilon}(s),P_{x^{\varepsilon}(s)},u^{\varepsilon}(s))-f(s,x^{*}(s),P_{x^{*}(s)},x^{*}(s)))\mathrm{d}s\\
=\;&\frac{1}{\varepsilon}\int_0^t(f(s,x^{\varepsilon}(s),P_{x^{\varepsilon}(s)},u^{\varepsilon}(s))-f(s,x^{*}(s),P_{x^{\varepsilon}(s)},u^{\varepsilon}(s)))\mathrm{d}s\\
&+\frac{1}{\varepsilon}\int_0^t(f(s,x^{*}(s),P_{x^{\varepsilon}(s)},u^{\varepsilon}(s))-f(s,x^{*}(s),P_{x^{*}(s)},u^{\varepsilon}(s)))\mathrm{d}s\\
&+\frac{1}{\varepsilon}\int_0^t(f(s,x^{*}(s),P_{x^{*}(s)},u^{\varepsilon}(s))-f(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)))\mathrm{d}s.
\end{aligned}$$

We notice that by simple computation, we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t(f(s,x^{\varepsilon}(s),P_{x^{\varepsilon}(s)},u^{\varepsilon}(s))-f(s,x^{*}(s),P_{x^{\varepsilon}(s)},u^{\varepsilon}(s)))\mathrm{d}s\\
=\;&\int_0^t\int_0^1\left[f_x\left(s,x^{*}(s)+\lambda\varepsilon(\gamma^{\varepsilon}(s)+Y(s)),P_{x^{\varepsilon}(s)},u^{\varepsilon}(s)\right)(\gamma^{\varepsilon}(s)+y_1(s))\right]\mathrm{d}\lambda\mathrm{d}s\\
=\;&\int_0^t\int_0^1\widehat{E}\left[\partial_{\mu}f(s,x^{\varepsilon}(s),P_{x^{*}(s)+\lambda\varepsilon(\gamma(s)+Y(s))},u^{\varepsilon}(s);\widehat{x}^{*}(s))(\widehat{\gamma}(s)+\widehat{y_1}(s))\right]\mathrm{d}\lambda\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t(f(s,x^{*}(s),P_{x^{*}(s)},u^{\varepsilon}(s))-f(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)))\mathrm{d}s\\
=\;&\int_0^t\int_0^1\left[f_u\left(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)+\lambda\varepsilon(v(s)-u^{*}(s)\right)v(s)\right]\mathrm{d}\lambda\mathrm{d}s.
\end{aligned}$$

Similarly, for the coefficient σ. We have

$$\begin{aligned} E\left[\sup_{s\in[0,t]}|\gamma^{\varepsilon}(s)|^{2}\right] &\leq C(t)\left[E\int_{0}^{t}\int_{0}^{1}\left|f_{x}\left(s,x^{*}(s)+\lambda\varepsilon(\gamma(s)+Y(s)),P_{x^{*}(s)},u^{\varepsilon}(s)\right)\gamma^{\varepsilon}(s)\right|^{2}d\lambda ds\right. \\ &+E\int_{0}^{t}\int_{0}^{1}\widehat{E}\left|f_{\mu}(s,x^{\varepsilon}(s),P_{x^{*}(s)+\lambda\varepsilon(\widehat{\gamma}(s)+\widehat{Y}(s))},u^{\varepsilon}(s);\widehat{x}^{*}(s))\widehat{\gamma}^{\varepsilon}(s)\right|^{2}d\lambda ds \\ &+E\int_{0}^{t}\int_{0}^{1}\left|\sigma_{x}\left(s,x^{*}(s)+\lambda\varepsilon(\gamma(s)+Y(s)),P_{x^{\varepsilon}(s)},u^{\varepsilon}(s)\right)\gamma^{\varepsilon}(s)\right|^{2}d\lambda ds \\ &+E\int_{0}^{t}\int_{0}^{1}\widehat{E}\left|\sigma_{\mu}(s,x^{\varepsilon}(s),P_{x^{*}(s)+\lambda\varepsilon(\widehat{\gamma}(s)+\widehat{Y}(s))},u^{\varepsilon}(s);\widehat{x}^{*}(s))\widehat{\gamma}^{\varepsilon}(s)\right|^{2}d\lambda ds \\ &\left.+E\left[\sup_{s\in[0,t]}|\beta^{\varepsilon}(s)|^{2}\right]\right], \end{aligned}$$

where

$$\begin{aligned} \beta^{\varepsilon}(t) = &\int_{0}^{t}\int_{0}^{1}[f_{x}\left(s,x^{*}(s)+\lambda\varepsilon(\gamma^{\varepsilon}(s)+y_{1}(s)),P_{x^{*}(s)},u^{\varepsilon}(s)\right)-f_{x}\left(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)\right)]y_{1}(s)d\lambda ds \\ &+\int_{0}^{t}\int_{0}^{1}\widehat{E}\left[f_{\mu}(s,x^{\varepsilon}(s),P_{x^{*}(s)+\lambda\varepsilon(\widehat{\gamma}^{\varepsilon}(s)+\widehat{y_{1}}(s))},u^{\varepsilon}(s);\widehat{x}^{*}(s))-f_{\mu}(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s);\widehat{x}^{*}(s))\right]\widehat{y_{1}}(s)d\lambda ds \\ &+\int_{0}^{t}\int_{0}^{1}[f_{u}\left(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)+\lambda\varepsilon v(t)\right)-f_{u}(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)]v(t)d\lambda ds \\ &+\int_{0}^{t}\int_{0}^{1}[\sigma_{x}\left(s,x^{*}(s)+\lambda\varepsilon(\gamma^{\varepsilon}(s)+y_{1}(s)),P_{x^{*}(s)},u^{\varepsilon}(s)\right)-\sigma_{x}\left(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)\right)]y_{1}(s)\mathrm{d}\lambda\mathrm{d}W(s) \\ &+\int_{0}^{t}\int_{0}^{1}\widehat{E}\left[\sigma_{\mu}(s,x^{\varepsilon}(s),P_{x^{*}(s)+\lambda\varepsilon(\widehat{\gamma}^{\varepsilon}(s)+\widehat{y_{1}}(s))},u^{\varepsilon}(s);\widehat{x}^{*}(s))\right. \\ &-\sigma_{\mu}(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s);\widehat{x}^{*}(s))]\widehat{y_{1}}(s)\mathrm{d}\lambda\mathrm{d}W(s)+\int_{0}^{t}\int_{0}^{1}\sigma_{u}\left(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)+\lambda\varepsilon v(t)\right) \\ &-\sigma_{u}(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s))]v(t)\mathrm{d}\lambda\mathrm{d}W(s). \end{aligned}$$

Now, since the derivatives of f and σ with respect to x,μ,u are Lipschitz continuous in (x,μ,u), we get

$$\lim_{\varepsilon\to 0}E\left[\sup_{s\in[0,T]}|\beta^{\varepsilon}(s)|^{2}\right]=0.$$

Since the derivatives of f and σ with respect to variables $x,\mu,$ and u are bounded, we have:

$$E\left[\sup_{s\in[0,t]}|\gamma^{\varepsilon}(s)|^{2}\right]\leq C(t)\left(E\int_{0}^{t}|\gamma^{\varepsilon}(s)|^{2}ds+E\left[\sup_{s\in[0,t]}|\beta^{\varepsilon}(s)|^{2}\right]\right),$$

Now, by applying *Gronwall's Lemma*, we have: for any $t\in[0,T]$

$$E\left[\sup_{s\in[0,t]}|\gamma^{\varepsilon}(s)|^{2}\right]\leq C(t)E\left[\sup_{s\in[0,t]}|\beta^{\varepsilon}(s)|^{2}\right]\exp\left\{\int_{0}^{t}C(s)ds\right\}.$$

Finally, by putting $t=T$ and letting ε go to zero, the proof of is complete. □

Proposition 4.2. Let $y_1(\cdot)$ and $y_2(\cdot)$ be the solutions of (4.3), (4.5), respectively. Let assumptions (H1)-(H4) hold. Then, for any $k\geq 1$, and $\varepsilon>0$, the we have

$$E\left[\sup_{t\in[0,T]}\left|x^{\varepsilon}(t)-x^{*}(t)-\varepsilon y_{1}(t)-\frac{\varepsilon^{2}}{2}y_{2}(t)\right|^{2k}\right]\leq C_{k}\varepsilon^{6k}. \tag{4.10}$$

Proof . From (3.1), (4.3) and (4.5), then by simple calculation, we obtain

$$
\begin{aligned}
\left|x^{\varepsilon}(t)-x^{*}(t)-\varepsilon y_{1}(t)-\frac{\varepsilon^{2}}{2}y_{2}(t)\right|^{2k} = & \left|\int_{0}^{t}\left\{f\left(s,x^{\varepsilon}(s),P_{x^{\varepsilon}(s)},u^{\varepsilon}(s)\right)-f\left(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)\right)\right.\right. \\
& -\varepsilon\left[f_{x}(s)y_{1}(s)+\widehat{E}[\widehat{f}_{\mu}(t)\widehat{y}_{1}(t)]+f_{u}(s)v(s)\right] \\
& -\frac{\varepsilon^{2}}{2}\left[f_{x}(s)y_{2}(s)+\widehat{E}[\widehat{f}_{\mu}(s)\widehat{y}_{2}(s)]+f_{xx}(s)y_{1}^{2}(s)+2f_{xu}(s)y_{1}(s)v(s)\right. \\
& \left.\left.+\ \widehat{E}[\widehat{f}_{x\mu}(t)\widehat{y}_{1}(t)]y_{1}(t)+2\widehat{E}[\widehat{f}_{u\mu}(s)\widehat{y}_{1}(s)]v(s)+f_{uu}(s)v^{2}(s)\right]\right\}\mathrm{d}s \\
& +\int_{0}^{t}\left\{\sigma\left(s,x^{\varepsilon}(s),P_{x^{\varepsilon}(s)},u^{\varepsilon}(s)\right)-\sigma\left(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)\right)\right. \\
& -\varepsilon\left[\sigma_{x}(s)y_{1}(s)+\widehat{E}[\widehat{\sigma}_{\mu}(t)\widehat{y}_{1}(t)]+\sigma_{u}(s)v(s)\right] \\
& -\frac{\varepsilon^{2}}{2}\left[\sigma_{x}(s)y_{2}(s)+\widehat{E}[\widehat{\sigma}_{\mu}(s)\widehat{y}_{2}(s)]+\sigma_{xx}(s)y_{1}^{2}(s)+2\sigma_{xu}(s)y_{1}(s)v(s)\right. \\
& \left.\left.\left.+\ \widehat{E}[\widehat{\sigma}_{x\mu}(t)\widehat{y}_{1}(t)]y_{1}(t)+2\widehat{E}[\widehat{\sigma}_{u\mu}(s)\widehat{y}_{1}(s)]v(s)+\sigma_{uu}(s)v^{2}(s)\right]\right\}\mathrm{d}W(s)\right|^{2k}.
\end{aligned} \tag{4.11}
$$

A straightforward calculation, we get

$$
\left|x^{\varepsilon}(t)-x^{*}(t)-\varepsilon y_{1}(t)-\frac{\varepsilon^{2}}{2}y_{2}(t)\right|^{2k}\leq A_{1}^{\varepsilon}(t)+A_{2}^{\varepsilon}(t), \tag{4.12}
$$

where

$$
\begin{aligned}
A_{1}^{\varepsilon}(t) = & \left|\int_{0}^{t}\left\{f\left(s,x^{\varepsilon}(s),P_{x^{\varepsilon}(s)},u^{\varepsilon}(s)\right)-f\left(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)\right)\ -\varepsilon\left[f_{x}(s)y_{1}(s)+f_{u}(s)v(s)\right]\right.\right. \\
& \left.-\frac{\varepsilon^{2}}{2}\left[f_{x}(s)y_{2}(s)+[f_{xx}(s)y_{1}(s)+f_{xu}(s)v(s)]y_{1}(s)+\ (f_{xu}(s)y_{1}(s))v(s)+f_{uu}(s)v^{2}(s)\right]\right\}\mathrm{d}s \\
& +\int_{0}^{t}\left\{\sigma\left(s,x^{\varepsilon}(s),P_{x^{\varepsilon}(s)},u^{\varepsilon}(s)\right)-\sigma\left(s,x^{*}(s),P_{x^{*}(s)},u^{*}(s)\right)\ -\varepsilon\left[\sigma_{x}(s)y_{1}(s)+\sigma_{u}(s)v(s)\right]\right. \\
& \left.\left.-\frac{\varepsilon^{2}}{2}\left[\sigma_{x}(s)y_{2}(s)+[\sigma_{xx}(s)y_{1}(s)+\sigma_{xu}(s)v(s)]y_{1}(s)\ +(\sigma_{xu}(s)y_{1}(s))v(s)+\sigma_{uu}(s)v^{2}(s)\right\}\mathrm{d}W(s)\right|^{2k},
\end{aligned} \tag{4.13}
$$

and

$$
\begin{aligned}
A_{2}^{\varepsilon}(t)= & \left|\int_{0}^{t}\left\{-\varepsilon\widehat{E}[\widehat{f}_{\mu}(t)\widehat{y}_{1}(t)]\ -\frac{\varepsilon^{2}}{2}\left[\widehat{E}[\widehat{f}_{\mu}(s)\widehat{y}_{2}(s)]+\widehat{E}[\widehat{f}_{x\mu}(s)\widehat{y}_{2}(s)]+\ \widehat{E}[\widehat{f}_{u\mu}(s)\widehat{y}_{1}(s)]v(s)\right]\right\}\mathrm{d}s\right. \\
& \left.+\int_{0}^{t}\left\{-\varepsilon\widehat{E}[\widehat{\sigma}_{\mu}(t)\widehat{y}_{1}(t)]-\frac{\varepsilon^{2}}{2}\left[\widehat{E}[\widehat{\sigma}_{\mu}(s)\widehat{y}_{2}(s)]+\widehat{E}[\widehat{\sigma}_{x\mu}(s)\widehat{y}_{2}(s)]+\ \widehat{E}[\widehat{\sigma}_{u\mu}(s)\widehat{y}_{1}(s)])v(s)\right]\right\}\mathrm{d}W(s)\right|^{2k}.
\end{aligned} \tag{4.14}
$$

Using similar arguments developed in Proposition 3.1 by Zhang and Zhang [31], we obtain

$$
E\left[\sup_{t\in[0,T]}|A_{1}^{\varepsilon}(t)|^{2k}\right]\leq C_{k}\varepsilon^{6k}. \tag{4.15}
$$

Now, by applying similar arguments proved in Proposition 4.3 in Buckdahn et al. [3], we get

$$
E\left[\sup_{t\in[0,T]}|A_{2}^{\varepsilon}(t)|^{2k}\right]\leq C_{k}\varepsilon^{6k}. \tag{4.16}
$$

Finally, by combining (4.12), (4.13), (4.14), (4.15) and (4.16), then the desired result (4.10) is fulfilled, which completes the proof. □

Lemma 4.3. Let $(p_1(\cdot), q_1(\cdot))$ and $(p_2(\cdot), q_2(\cdot))$ be the solution to the adjoint equation (3.13) and (3.16) respectively. Let $y_1(\cdot)$ and $y_2(\cdot)$ be the solutions to the first and second order variational equations (4.3) and (4.5), respectively associated to $u^*(\cdot)$. Then the following duality relations hold

$$\begin{aligned} E\left[h_x(x^*(T)\, y_1(T) + \widehat{E}[\widehat{h}_\mu(T)\widehat{y}_1(T)]\right] =& - E\int_0^T \left[p_1(t)\, f_u(t)\, v(t) + q_1(t)\sigma_u(t)\, v(t) + y_1(t)\left(\ell_x(t) + \widehat{E}[\widehat{\ell}_\mu(t)]\right)\right] \mathrm{d}t \\ & - E\int_{[0,T]} p_1(t)G(t)\mathrm{d}\zeta(t), \end{aligned} \tag{4.17}$$

$$\begin{aligned} E\left[h_x(T)y_2(T) + \widehat{E}[\widehat{h}_\mu(T)]y_2(T)\right] =& - E\int_0^T \Big\{p_1(t)\left[[f_{xx}(t)y_1(t) + \widehat{E}[\widehat{f}_{x\mu}(t)\widehat{y}_1(t)] + 2f_{xu}(t)v(t)]y_1(t)\right. \\ & \left.+\widehat{E}[\widehat{f}_{u\mu}(t)\widehat{y}_1(t)]v(t) + f_{uu}(t)v^2(t)\right] + q_1(t)\left[\sigma_{xx}(t)y_1^2(t) + \widehat{E}[\widehat{\sigma}_{x\mu}(t)\widehat{y}_1(t)]y_1(t)\right. \\ & \left.+ 2\sigma_{xu}(t)y_1(t)v(t) + \widehat{E}[\widehat{\sigma}_{u\mu}(t)\widehat{y}_1(t)]v(t) + \sigma_{uu}(t)v^2(t)\right] \\ & +\ell_x(t)y_2(t) + \widehat{E}(\widehat{\ell}_\mu(t))y_2(t)\Big\} \mathrm{d}t, \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} E\left[h_{xx}(x^*(T))\, y_1^2(T) + \widehat{E}[\widehat{h}_{\mu\mu}(T)\, \widehat{y}_1^2(T)]\right] =& - E\int_0^T \{[2y_1(t)\, [p_2(t)\, (f_u(t) + \sigma_x(t)\, \sigma_u(t)) + q_2(t)\, \sigma_u(t)] \\ & + p_2(t)\left[2\sigma_u(t)\, \widehat{E}[\widehat{\sigma}_\mu(t)\widehat{y}_1(t)] + \sigma_u^2(t)\, v(t)\right]\Big]\, v(t) \\ & -y_1^2(t)(H_{xx}(t) + \widehat{E}[\widehat{H}_{\mu y}(t)])\Big\}\, \mathrm{d}t - E\int_{[0,T]} 2y_1(t)p_2(t)G(t)\mathrm{d}\zeta(t). \end{aligned} \tag{4.19}$$

Proof . The proof of this lemma follows from Itô's formula to $p_1(T)\, y_1(T)$, $p_1(T)\, y_2(T)$ and $p_2(t)\, y_1^2(t)$.□

Proposition 4.4. Let assumption (H1)-(H4) hold. Then the following variational equality holds: $\forall (u(\cdot), \xi(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$,

$$\begin{aligned} J(u^\varepsilon(\cdot), \xi^\varepsilon(\cdot)) - J(u^*(\cdot), \xi^*(\cdot)) =& - E\int_0^T \left[\varepsilon H_u(t)v(t) + \frac{\varepsilon^2}{2}\left(H_{uu}(t) + p_2(t)\sigma_u^2(t)\right)v^2(t) + \varepsilon^2\mathcal{S}(t)y_1(t)\, v(t)\right]\mathrm{d}t \\ & + \varepsilon E\int_{[0,T]} (M(t) - p_1(t)G(t))\, \mathrm{d}\zeta(t) - \varepsilon^2 E\int_{[0,T]} y_1(t)p_2(t)G(t)\mathrm{d}\zeta(t) + o(\varepsilon^2), \end{aligned} \tag{4.20}$$

where $\varepsilon \to 0^+$, $v(t) = u(t) - u^*(t)$, $\zeta(\cdot) = \xi(\cdot) - \xi^*(\cdot)$, and $\mathcal{S}(t)$ has a McKean-Vlasov type

$$\begin{aligned} \mathcal{S}(t) &= \mathcal{S}(t, x, u, \mu, p_1, q_1, p_2, q_2) \\ &= H_{xu}(t) + \widehat{E}[\widehat{H}_{\mu u}(t)] + f_u(t, x, \mu, u)p_2(t) + \sigma_u(t, x, \mu, u)q_2(t) \\ &\quad +p_2(t)\sigma_u(t, x, \mu, u)(\sigma_x(t, x, \mu, u) + \widehat{E}[\widehat{\sigma}_\mu(t, x, \mu, u)]). \end{aligned} \tag{4.21}$$

Proof . From (3.2), we have

$$J(u^\varepsilon(\cdot), \xi^\varepsilon(\cdot)) - J(u^*(\cdot), \xi^*(\cdot)) = E\int_0^T \delta\ell(t)\, \mathrm{d}t + E[h(x^\varepsilon(T), P_{x^\varepsilon(T)}) - h(x^*(T), P_{x^*(T)})] + E\int_{[0,T]} M(t)\mathrm{d}(\xi^\varepsilon - \xi^*)(t).$$

Applying *Taylor-Young's formula* for the function ℓ and since $\xi^\varepsilon(t) - \xi^*(t) = \varepsilon(\xi(t) - \xi^*(t)) = \varepsilon\zeta(t)$, by applying

Proposition 4.1, we can easily find

$$
\begin{aligned}
&J\left(u^{\varepsilon}(\cdot), \xi^{\varepsilon}(\cdot)\right)-J\left(u^{*}(\cdot), \xi^{*}(\cdot)\right) \\
&=E\left[\int_{0}^{T}\left\{\varepsilon \ell_{x}(t) y_{1}(t)+\varepsilon \widehat{E}\left[\widehat{\ell}_{\mu}(t) \widehat{y}_{1}(t)\right]+\frac{\varepsilon^{2}}{2} \ell_{x}(t) y_{2}(t)+\frac{\varepsilon^{2}}{2} \widehat{E}\left[\widehat{\ell}_{x}(t) \widehat{y}_{2}(t)\right]+\varepsilon \ell_{u}(t) v(t)\right.\right. \\
&+\frac{\varepsilon^{2}}{2}\left(\ell_{xx}(t) y_{1}(t)^{2}+\widehat{E}\left[\widehat{\ell}_{\mu\mu}(t) \widehat{y}_{1}(t)^{2}\right]+\ell_{uu}(t) v(t)^{2}\right. \\
&\left.\left.+\ 2 \ell_{xu}(t) y_{1}(t) v(t)+2 \widehat{E}(\widehat{\ell}_{\mu u}(t) \widehat{y}_{1}(t)) v(t)+2 \widehat{E}(\widehat{\ell}_{x\mu}(t) \widehat{y}_{1}(t)) y_{1}(t)\right)\right\} \mathrm{d}t \\
&+E\left[\varepsilon[h_{x}(x^{*}(T) y_{1}(T)+\widehat{E}[\widehat{h}_{\mu}(T) \widehat{y}_{1}(T)]]+\frac{\varepsilon^{2}}{2}[h_{x}(x^{*}(T)) y_{2}(T)+\widehat{E}[\widehat{h}_{\mu}(x^{*}(T)) \widehat{y}_{2}(T)]]\right. \\
&\left.+\frac{\varepsilon^{2}}{2}[h_{xx}(x^{*}(T)) y_{1}^{2}(T)+\widehat{E}[\widehat{h}_{\mu\mu}(T) \widehat{y}_{1}^{2}(T)]\right]+E \int_{[0,T]} \varepsilon M(t) \mathrm{d}\zeta(t)+o\left(\varepsilon^{2}\right), \quad\left(\varepsilon \longrightarrow 0^{+}\right).
\end{aligned} \tag{4.22}
$$

Further, from Lemma 5.1, (4.21), (3.8) and (3.10), we get

$$
\begin{aligned}
&J\left(u^{\varepsilon}(\cdot), \xi^{\varepsilon}(\cdot)\right)-J\left(u^{*}(\cdot), \xi^{*}(\cdot)\right) \\
&=-E \int_{0}^{T}\left[\varepsilon H_{u}(t) v(t)+\frac{\varepsilon^{2}}{2} H_{uu}(t) v^{2}(t)+\varepsilon^{2} \mathcal{S}(t) y_{1}(t) v(t)+\frac{\varepsilon^{2}}{2} p_{2}(t) \sigma_{u}^{2}(t) v^{2}(t)\right] \mathrm{d}t+\varepsilon E \int_{[0,T]}(M(t)-p_{1}(t) G(t)) \mathrm{d}\zeta(t) \\
&-\varepsilon^{2} E \int_{[0,T]} y_{1}(t) p_{2}(t) G(t) \mathrm{d}\zeta(t)+o\left(\varepsilon^{2}\right) \quad\left(\varepsilon \longrightarrow 0^{+}\right).
\end{aligned} \tag{4.23}
$$

This completes the proof. □

From Proposition 4.4, we can derive the following second-order necessary condition in integral form for our stochastic optimal control (3.1)-(3.2).

Theorem 4.5 (McKean-Vlasov maximum principle for regular-singular control in integral form). Let assumption (H1)-(H4) hold. If the regular control $u^{*}(\cdot)$ is a singular in the classical sense for the control problem (3.1)-(3.2). Then we obtain

$$
E \int_{0}^{T} \mathcal{S}(t) y_{1}(t)(u(t)-u^{*}(t)) \mathrm{d}t+E \int_{[0,T]} y_{1}(t) p_{2}(t) G(t) \mathrm{d}(\xi-\xi^{*})(t) \leq 0, \tag{4.24}
$$

$$
E \int_{[0,T]}(M(t)-p_{1}(t) G(t)) \chi_{\{(w,t) \in \Omega \times[0,T]:(M(t)-p_{1}(t) G(t)) \geq 0\}} \mathrm{d}\xi^{*}(t)=0, \tag{4.25}
$$

for any $(u(\cdot), \xi(\cdot)) \in \mathcal{A}_{1} \times \mathcal{A}_{2}$, where $\mathcal{S}(t)$ is defined by the formula (4.21) and $y_{1}(\cdot)$ solution of first-order adjoint equation (4.3).

Proof . By applying Proposition 4.4, for $v(t)=u(t)-u^{*}(t)$ and $\zeta(t)=\xi(t)-\xi^{*}(t)$ we have

$$
\begin{aligned}
0 \quad \leq \quad & \frac{1}{\varepsilon^{2}}\left[J\left(u^{\varepsilon}(\cdot), \xi^{\varepsilon}(\cdot)\right)-J\left(u^{*}(\cdot), \xi^{*}(\cdot)\right)\right] \\
= \quad & -E \int_{0}^{T}\left[\frac{1}{\varepsilon} H_{u}(t) v(t)+\frac{1}{2} H_{uu}(t) v^{2}(t)+\mathcal{S}(t) y_{1}(t) v(t)+\frac{1}{2} p_{2}(t) \sigma_{u}^{2}(t) v^{2}(t)\right] \mathrm{d}t \\
& +\frac{1}{\varepsilon} E \int_{[0,T]}(M(t)-p_{1}(t) G(t)) \mathrm{d}\zeta(t)-E \int_{[0,T]} y_{1}(t) p_{2}(t) G(t) \mathrm{d}\zeta(t)+o\left(\varepsilon^{2}\right) \quad\left(\varepsilon \longrightarrow 0^{+}\right).
\end{aligned}
$$

From Definition 3.1, we deduce $\frac{1}{\varepsilon} H_{u}(t) v(t)=0$, and

$$
\frac{1}{2} H_{uu}(t) v^{2}(t)+\frac{1}{2} p_{2}(t) \sigma_{u}^{2}(t) v^{2}(t)=\frac{1}{2}\left[H_{uu}(t)+p_{2}(t) \sigma_{u}^{2}(t)\right] v^{2}(t)=0,
$$

the desired result (4.24) follows immediately. Now let us turn to prove (4.25). From the singularity in (3.11) holds for any $\xi(\cdot) \in \mathcal{A}_{2}$.

$$
E \int_{[0,T]}(M(t)-p_{1}(t) G(t)) \mathrm{d}(\xi-\xi^{*})(t)=0.
$$

Let $\xi(\cdot) \in \mathcal{A}_2$ be defined by

$$\mathrm{d}\xi(t) = \begin{cases} 0 \text{ if } (M(t) - p_1(t)G(t)) \geq 0, \\ \mathrm{d}\xi^*(t) \text{ if } (M(t) - p_1(t)G(t)) < 0, \end{cases} \tag{4.26}$$

Let $\mathcal{N}$ a set be defined by $\mathcal{N} = \{(t,w) \in [0,T] \times \Omega : (M(t) - p_1(t)G(t)) \geq 0\}$. Then we have

$$\begin{aligned} \mathrm{d}\xi(t) &= \chi_{\mathcal{N}}\mathrm{d}\xi(t) + \chi_{\mathcal{N}^c}\mathrm{d}\xi(t) \\ &= \chi_{\{(t,w)\in[0,T]\times\Omega:(M(t)-p_1(t)G(t))<0\}}(t)\mathrm{d}\xi^*(t). \end{aligned} \tag{4.27}$$

By a simple computations, it is easy to see that $\xi(\cdot)$ is in $\mathcal{A}_2$. Moreover, we have

$$\begin{aligned} 0 &= E\int_{[0,T]} (M(t) - p_1(t)G(t))\,\mathrm{d}(\xi - \xi^*)(t) \\ &= E\int_{[0,T]} (M(t) - p_1(t)G(t))\,\chi_{\{(t,w)\in[0,T]\times\Omega:(M(t)-p_1(t)G(t))<0\}}\mathrm{d}(\xi^* - \xi^*)(t), \\ &\quad +E\int_{[0,T]} (M(t) - p_1(t)G(t))\,\chi_{\{(t,w)\in[0,T]\times\Omega:(M(t)-p_1(t)G(t))\geq 0\}}\mathrm{d}(-\xi^*)(t), \end{aligned}$$

then we conclude that

$$E\int_{[0,T]} (M(t) - p_1(t)G(t))\,\chi_{\mathcal{N}}(t)\mathrm{d}\xi^*(t) = 0. \tag{4.28}$$

This completes the proof. □

From Theorem 4.5, we have the following corollary

Corollary 4.6. For any $(u(\cdot), \xi(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$, we have

$$\begin{aligned} &E\int_0^T \mathcal{S}(t)y_1(t)u^*(t))\mathrm{d}t + E\int_{[0,T]} y_1(t)p_2(t)G(t)\mathrm{d}\xi^*(t) \\ = &\max_{(u(\cdot),\xi(\cdot))\in\mathcal{A}_1\times\mathcal{A}_2} \left[E\int_0^T \mathcal{S}(t)y_1(t)u(t)\,\mathrm{d}t + E\int_{[0,T]} y_1(t)p_2(t)G(t)\mathrm{d}\xi(t)\right]. \end{aligned}$$

5 Pointwise McKean-Vlasov second-order necessary conditions for optimal regular-singular control

In this section, by using the property of Itô's integrals and the martingale representation theorem, we aim to establish the second-order necessary condition for optimal controls, which is pointwise McKean-Vlasov maximum principle in terms of the martingale with respect to the time variable t. The following lemma play an important role to prove our main result.

Lemma 5.1. The first-order variational equation (4.3) admits a unique strong solution $y_1(\cdot)$, which is given by the following equation:

$$y_1(t) = \Phi(t)\left[\int_0^t \Psi(s)\left(f_u(s) - (\sigma_x(s) + \widehat{E}[\widehat{\sigma}_\mu(s)])\sigma_u(s)\right)v(s)\mathrm{d}s + \int_0^t \Psi(s)\sigma_u(s)v(s)\mathrm{d}W(s) + \int_{[0,t]} \Psi(s)G(s)\mathrm{d}\zeta(s)\right], \tag{5.1}$$

where the stochastic process $\Phi(\cdot)$ is a defined by the following linear stochastic differential equation:

$$\begin{cases} d\Phi(t) &= \left[f_x(t) + \widehat{E}(\widehat{f}_\mu(t))\right]\Phi(t)\mathrm{d}t + \left[\sigma_x(t) + \widehat{E}[\widehat{\sigma}_\mu(t)]\right]\Phi(t)\mathrm{d}W(t), \\ \Phi(0) &= 1, \end{cases} \tag{5.2}$$

and $\Psi(t)$ its inverse.

Proof . Equation (5.2) is linear with bounded coefficients, then it admits a unique strong solution. Moreover, this solution is inversible and its inverse $\Psi(t)=\Phi^{-1}(t)$ given by the following McKean-Vlasov equation:

$$d\Psi(t)=\left[\left(\sigma_x(t)+\widehat{E}[\widehat{\sigma}_\mu(t)]\right)^2\Psi(t)-f_x(t)\Psi(t)-\widehat{E}(\widehat{f}_\mu(t))\Psi(t)\right]\mathrm{d}t-\left(\sigma_x(t)+\widehat{E}[\widehat{\sigma}_\mu(t)]\right)\Psi(t)\mathrm{d}W(t),\ \Psi(0)=1. \quad (5.3)$$

By Itô's formula to $\Psi(t)y_1(t)$, we have

$$\begin{aligned} d\left[\Psi(t)y_1(t)\right]&=y_1(t)\,\mathrm{d}\Psi(t)+\Psi(t)\mathrm{d}y_1(t)-\left[(\sigma_x(t)+\widehat{E}[\widehat{\sigma}_\mu(t)])\Psi(t)\right]\left[\sigma_x(t)y_1(t)+\widehat{E}[\widehat{\sigma}_\mu(t)\widehat{y}_1(t)]+\sigma_u(t)v(t)\right]\mathrm{d}t\\ &=\mathbb{I}_1(t)+\mathbb{I}_2(t)+\mathbb{I}_3(t), \end{aligned} \quad (5.4)$$

where

$$\begin{aligned} \mathbb{I}_1(t)&=y_1(t)\,\mathrm{d}\Psi(t) \quad (5.5)\\ &=\left[y_1(t)\left(\sigma_x(t)+\widehat{E}[\widehat{\sigma}_\mu(t)]\right)^2\Psi(t)-y_1(t)f_x(t)\Psi(t)-y_1(t)\widehat{E}(\widehat{f}_\mu(t))\Psi(t)\right]\mathrm{d}t-y_1(t)\left(\sigma_x(t)+\widehat{E}[\widehat{\sigma}_\mu(t)]\right)\Psi(t)\mathrm{d}W(t). \end{aligned}$$

By simple computations, we can get

$$\begin{aligned} \mathbb{I}_2(t)&=\Psi(t)\mathrm{d}y_1(t)\\ &=\left[\Psi(t)f_x(t)y_1(t)+\Psi(t)\widehat{E}[\widehat{f}_\mu(t)\widehat{y}_1(t)]+\Psi(t)f_u(t)v(t)\right]\mathrm{d}t \quad (5.6)\\ &+\left[\Psi(t)\sigma_x(t)y_1(t)+\Psi(t)\widehat{E}[\widehat{\sigma}_\mu(t)\widehat{y}_1(t)]+\Psi(t)\sigma_u(t)v(t)\right]\mathrm{d}W(t)+\Psi(t)G(t)\mathrm{d}\zeta(t), \end{aligned}$$

and

$$\mathbb{I}_3(t)=-\left[(\sigma_x(t)+\widehat{E}[\widehat{\sigma}_\mu(t)])\Psi(t)\right]\left[\sigma_x(t)y_1(t)+\widehat{E}[\widehat{\sigma}_\mu(t)\widehat{y}_1(t)]+\sigma_u(t)v(t)\right]\mathrm{d}t \quad (5.7)$$

By substituting (5.4), (5.5), and (5.6) into (5.4), we obtain

$$\begin{aligned} \Psi(t)y_1(t)-\Psi(0)y_1(0)&=\left[\int_0^t\Psi(s)\left[f_u(s)-\left(\sigma_x(s)+\widehat{E}[\widehat{\sigma}_\mu(s)]\right)\sigma_u(s)\right]v(s)\mathrm{d}s\right.\\ &+\int_0^t\Psi(s)\sigma_u(s)v(s)\mathrm{d}W(s)+\int_{[0,t]}\Psi(s)G(s)\mathrm{d}\zeta(s). \end{aligned} \quad (5.8)$$

Finally, since $y_1(0)=0$ and $\Psi^{-1}(t)=\Phi(t)$, from (5.8) the desired result (5.2) is fulfilled. This completes the proof. □

To prove the main theorem we need the following technical Lemma.

Lemma 5.2. Let assumptions (H1)-(H4) hold. Then we have
(1) $\mathcal{S}(\cdot)\in\mathbb{L}^2_{\mathbb{F}}([0,T];\mathbb{R})$.

(2) For any $v\in\mathbb{A}_1$, there exists a unique process $Q_v(\cdot,t)\in\mathbb{L}^2_{\mathbb{F}}([0,T];\mathbb{R})$, with $E\left(\left[\int_0^T|Q_v(s,t)|^2\,ds\right]^2\right)<\infty$ such that

$$\begin{aligned} \mathcal{S}(t)(v-\bar{u}(t))=E\left[\mathcal{S}(t)(v-\overline{u}(t))\right]+\int_0^t Q_v(s,t)\,\mathrm{d}W(s) \quad (5.9)\\ a.e.\ t\in[0,T],\ P-a.s. \end{aligned}$$

Proof . Since the derivatives, f_{xu}, $f_{\mu u}$, σ_{xu}, $\sigma_{\mu u}$, ℓ_{xu}, $\ell_{\mu u}$, f_u, σ_u, σ_x, and σ_μ, are bounded (see assumptions (H2) and (H4)), we obtain $E\left[\left(\int_0^T|\mathcal{S}(t)|^2\,dt\right)^2\right]<\infty$, the desired result in (1) follows immediately. By applying *Martingale Representation Theorem*, the second item follows. □

The following theorem constitutes the main contribution of this paper.

Theorem 5.3. Let assumptions (H1)-(H4) hold. If $\overline{u}(\cdot)$ is a singular optimal control in the classical sense for the stochastic control (3.1)-(3.2), then for any $(v,\zeta)\in\mathbb{A}_1\times\mathbb{A}_2$, it holds that

$$\begin{aligned}&E\left[\mathcal{S}(\tau)f_u(\tau)(v-\overline{u}(\tau))^2\right]+\partial_\tau^+\left(\mathcal{S}(\tau)\sigma_u(\tau)(v-\overline{u}(\tau))^2\right)+\left[\left(\mathcal{S}\left(\tau\right)+P(\tau)f_u(\tau)\right)G(\tau)(v-u^*(\tau))\zeta(\tau)\right]\\&+\left[p_2(\tau)G^2(\tau)\zeta^2\left(\tau\right)\right]\right]\leq 0,\qquad a.e.\ \tau\in[0,T],\end{aligned}\tag{5.10}$$

where $\mathcal{S}(\tau)$ has a McKean-Vlasov form, given by (4.21) at τ

$$\mathcal{S}(\tau)\;=\;H_{xu}(\tau)+\widehat{E}[\widehat{H}_{\mu u}(\tau)]+p_2(\tau)\left[f_u(\tau)+\sigma_u(\tau)\left(\sigma_x(\tau)+\widehat{E}[\widehat{\sigma}_\mu(\tau)]\right)\right]+\sigma_u(\tau)q_2(\tau).$$

and

$$\partial_\tau^+\left(\mathcal{S}(\tau)\sigma_u(\tau)(v-\overline{u}(\tau))^2\right)=2\lim_{\epsilon\to 0^+}\sup\frac{1}{\epsilon^2}E\int_\tau^{\tau+\epsilon}\int_\tau^t\left[Q_v(s,t)\Phi\left(\tau\right)\Psi\left(s\right)\sigma_u(s)(v-\overline{u}(s))\right]\mathrm{d}s\mathrm{d}t,\tag{5.11}$$

Here $Q_v(\cdot,t)$ is given by (5.9), and $\Psi\left(\cdot\right)$ is determined by (5.3).

Proof . Now, in order to derive a pointwise second order necessary condition from the integral form in (4.24), we need to choose the following *spike variation* for the optimal control $(u^*\left(\cdot\right),\xi^*(\cdot))$ by the form:

$$(u(t),\xi(t))=\begin{cases}(v,\xi^*(t)+\epsilon\zeta(t)),\ t\in E_\epsilon\\(u^*(t),\xi^*(t)+\epsilon\zeta(t)),\ t\in[0,T]\mid E_\epsilon.\end{cases}\tag{5.12}$$

For any $(v,\zeta)\in\mathbb{A}_1\times\mathbb{A}_2$, $\tau\in[0,T)$, and $\epsilon\in(0,T-\tau)$, let $E_\epsilon=[\tau,\tau+\epsilon)$, and define $u(\cdot)$ as that in (5.12). Then $v(\cdot)=u(\cdot)-u^*(\cdot)=\left(v-u^*(\cdot)\right)\chi_{E_\epsilon}(\cdot)$. From (5.12), we have $v(\cdot)=u(\cdot)-\overline{u}(\cdot)=(v-\overline{u}(\cdot))\chi_{\mathcal{G}_\epsilon}(\cdot)$ and the corresponding solution $y_1(\cdot)$ to (5.1) is given by the following McKean-Vlasov equation:

$$\begin{aligned}y_1(t)&=\Phi\left(t\right)\int_0^t\Psi\left(s\right)\left(f_u(s)-(\sigma_x(s)+\widehat{E}[\widehat{\sigma}_\mu(s)])\sigma_u(s)\right)(v-\overline{u}(s))\chi_{\mathcal{G}_\epsilon}(s)\mathrm{d}s\\&+\Phi\left(t\right)\int_0^t\Psi\left(s\right)\sigma_u(s)(v-\overline{u}(s))\chi_{\mathcal{G}_\epsilon}(s)\mathrm{d}W(s)+\Phi\left(t\right)\int_{[0,t]}\Psi(s)G(s)\mathrm{d}\zeta(s).\end{aligned}\tag{5.13}$$

Substituting $v(\cdot)=(v-\overline{u}(\cdot))\chi_{\mathcal{G}_\epsilon}(\cdot)$ and (5.13) into (4.24), we have

$$\begin{aligned}0&\geq\frac{1}{\epsilon^2}E\int_\tau^{\tau+\epsilon}\left[\mathcal{S}\left(t\right)y_1(t)\left(v-\overline{u}(t)\right)\right]\mathrm{d}t\\&=\frac{1}{\epsilon^2}E\int_\tau^{\tau+\epsilon}\left[\mathcal{S}\left(t\right)\Phi\left(t\right)\int_\tau^t\Psi\left(s\right)\left(f_u(s)-(\sigma_x(s)+\widehat{E}[\widehat{\sigma}_\mu(s)])\sigma_u(s)\right)\times(v-\overline{u}(s))\mathrm{d}s\left(v-\overline{u}(t)\right)\right]\mathrm{d}t\\&+\frac{1}{\epsilon^2}E\int_\tau^{\tau+\epsilon}\left[\mathcal{S}\left(t\right)\Phi\left(t\right)\int_\tau^t\Psi\left(s\right)\sigma_u(s)(v-\overline{u}(s))\mathrm{d}W(s)\left(v-\overline{u}(t)\right)\right]\mathrm{d}t\\&+\frac{1}{\epsilon^2}E\int_{[\tau,\tau+\epsilon]}y_1(t)p_2(t)G(t)\mathrm{d}(\xi-\xi^*)\left(t\right)\\&=\mathbb{J}_1^\epsilon\left(\tau\right)+\mathbb{J}_2^\epsilon\left(\tau\right)+\mathbb{J}_3^\epsilon\left(\tau\right).\end{aligned}\tag{5.14}$$

where

$$\mathbb{J}_1^\epsilon\left(\tau\right)-\frac{1}{\epsilon^2}E\int_\tau^{\tau+\epsilon}\left[\mathcal{S}\left(t\right)\Phi\left(t\right)\int_\tau^t\Psi\left(s\right)\left(f_u(s)-(\sigma_x(s)+\widehat{E}[\widehat{\sigma}_\mu(s)])\sigma_u(s)\right)\times(v-\overline{u}(s))\mathrm{d}s\left(v-\overline{u}(t)\right)\right]\mathrm{d}t,\tag{5.15}$$

$$\mathbb{J}_2^\epsilon\left(\tau\right)=\frac{1}{\epsilon^2}E\int_\tau^{\tau+\epsilon}\left[\mathcal{S}\left(t\right)\Phi\left(t\right)\int_\tau^t\Psi\left(s\right)\sigma_u(s)(v-\overline{u}(s))\mathrm{d}W(s)\left(v-\overline{u}(t)\right)\right]\mathrm{d}t.\tag{5.16}$$

and $\mathbb{J}_3^\epsilon\left(\tau\right)$ is given by

$$\mathbb{J}_3^\epsilon\left(\tau\right)=\frac{1}{\epsilon^2}E\int_{[\tau,\tau+\epsilon]}y_1(t)p_2(t)G(t)\mathrm{d}(\xi-\xi^*)\left(t\right).\tag{5.17}$$

Applying similar arguments developed in [31], we get

$$\lim_{n\to+\infty}\mathbb{J}_1^\epsilon(\tau)=\frac{1}{2}E\left[\mathcal{S}(\tau)\left(f_u(\tau)-(\sigma_x(\tau)+\widehat{E}[\widehat{\sigma}_\mu(\tau)])\sigma_u(\tau)\right)(v-\overline{u}(\tau))^2\right]. \tag{5.18}$$

Let us turn to estimate the second term $\mathbb{J}_2^\epsilon(\tau)$. From (5.2), we can obtain

$$\begin{aligned}
\mathbb{J}_2^\epsilon(\tau) &= \frac{1}{\epsilon^2}E\int_\tau^{\tau+\epsilon}\left[\mathcal{S}(t)\,\Phi(\tau)\int_\tau^t\Psi(s)\,\sigma_u(s)(v-\overline{u}(s))\mathrm{d}W(s)\,(v-\overline{u}(t))\right]\mathrm{d}t\\
&\quad+\frac{1}{\epsilon^2}E\int_\tau^{\tau+\epsilon}\left\{\mathcal{S}(t)\int_\tau^t(f_x(s)+\widehat{E}(\widehat{f}_\mu(s)))\Phi(s)\mathrm{d}s\times\int_\tau^t\Psi(s)\,\sigma_u(s)(v-\overline{u}(s))\mathrm{d}W(s)\,(v-\overline{u}(t))\right\}\mathrm{d}t\\
&\quad+\frac{1}{\epsilon^2}E\int_\tau^{\tau+\epsilon}\left\{\mathcal{S}(t)\int_\tau^t(\sigma_x(s)+\widehat{E}[\widehat{\sigma}_\mu(s)])\Phi(s)\mathrm{d}W(s)\times\int_\tau^t\Psi(s)\,\sigma_u(s)(v-\overline{u}(s))\mathrm{d}W(s)\,(v-\overline{u}(t))\right\}\mathrm{d}t\\
&= J_{2,1}^\epsilon(\tau)+J_{2,2}^\epsilon(\tau)+J_{2,3}^\epsilon(\tau).
\end{aligned} \tag{5.19}$$

Now, we proceed to derive estimates for $J_{2,1}^\epsilon(\tau)$, $J_{2,2}^\epsilon(\tau)$, and $J_{2,3}^\epsilon(\tau)$. Arguing as in [31, Eq-(4.8)], with the helps of Lemma 5.2, we get

$$\begin{aligned}
\lim_{\epsilon\to0^+}\sup J_{2,1}^\epsilon(\tau) &= \lim_{\epsilon\to0^+}\sup\frac{1}{\epsilon^2}E\int_\tau^{\tau+\epsilon}\left[\mathcal{S}(t)\,\Phi(\tau)\int_\tau^t\Psi(s)\,\sigma_u(s)(v-\overline{u}(s))\mathrm{d}W(s)\times(v-\overline{u}(t))\right]\mathrm{d}t\\
&= \frac{1}{2}\partial_\tau^+\left(\mathcal{S}(\tau)\,(v-\overline{u}(\tau))^2\sigma_u(\tau)\right),\ \forall\tau\in[0,T].
\end{aligned} \tag{5.20}$$

Let us turn to second term $J_{2,2}^\epsilon(\tau)$ in the right-hand side of (5.19). Since f_x, f_μ are bounded, then by applying similar arguments developed in [31, Eq-(4.9)], we have

$$\begin{aligned}
\lim_{\epsilon\to0^+}\sup J_{2,2}^\epsilon(\tau) &= \lim_{\epsilon\to0^+}\sup\frac{1}{\epsilon^2}E\int_\tau^{\tau+\epsilon}\left\{\mathcal{S}(t)\int_\tau^t(f_x(s)+\widehat{E}(\widehat{f}_\mu(s)))\Phi(s)\mathrm{d}s\right.\\
&\quad\left.\times\int_\tau^t\Psi(s)\,\sigma_u(s)(v-\overline{u}(s))\mathrm{d}W(s)\,(v-\overline{u}(t))\right\}\mathrm{d}t\\
&= 0,\quad a.e.\ \tau\in[0,T].
\end{aligned} \tag{5.21}$$

Let us turn to third term $J_{2,3}^\epsilon(\tau)$ in the right-hand side of (5.19). Since

$$E\left|\int_\tau^t|\sigma_x(s)\Phi(s)|^2\,ds\right|^2\qquad\text{and}\qquad E\left|\int_\tau^t\left|\widehat{E}[\widehat{\sigma}_\mu(s)])\Phi(s)\right|^2ds\right|^2$$

are bounded, then by applying similar arguments developed in [31, Eq-(4.10)], we have

$$\begin{aligned}
\lim_{\epsilon\to0^+}\sup J_{2,3}^\epsilon(\tau) &= \lim_{\epsilon\to0^+}\sup\frac{1}{\epsilon^2}E\int_\tau^{\tau+\epsilon}\left\{\mathcal{S}(t)\int_\tau^t(\sigma_x(s)+\widehat{E}[\widehat{\sigma}_\mu(s)])\Phi(s)\mathrm{d}W(s)\right.\\
&\quad\left.\times\int_\tau^t\Psi(s)\,\sigma_u(s)(v-\overline{u}(s))\mathrm{d}W(s)\,(v-\overline{u}(t))\right\}\mathrm{d}t\\
&= \frac{1}{2}E\left[\mathcal{S}(\tau)\,(\sigma_x(\tau)+\widehat{E}[\widehat{\sigma}_\mu(\tau)])\sigma_u(\tau)(v-\overline{u}(\tau))^2\right].\ a.e.\ \tau\in[0,T].
\end{aligned} \tag{5.22}$$

Substituting (5.20), (5.21), (5.22) in (5.19), we have

$$\begin{aligned}
\lim_{\epsilon\to0^+}\sup\mathbb{J}_2^\epsilon(\tau) &= \frac{1}{2}E\left[\mathcal{S}(\tau)\,(\sigma_x(\tau)+\widehat{E}[\widehat{\sigma}_\mu(\tau)])\sigma_u(\tau)(v-\overline{u}(\tau))^2\right]+\frac{1}{2}\partial_\tau^+\left(\mathcal{S}(\tau)\,(v-\overline{u}(\tau))^2\sigma_u(\tau)\right),\\
a.e.\ \tau &\in [0,T].
\end{aligned} \tag{5.23}$$

Estimate of (5.17). Now, let us turn to estimate $\mathbb{J}_3^\epsilon(\tau)$. From (5.14), we have

$$\begin{aligned}
\mathbb{J}_3^\epsilon(\tau) &= \frac{1}{\epsilon^2}E\int_{[\tau,\tau+\epsilon]}y_1(t)p_2(t)G(t)\mathrm{d}(\xi-\xi^*)(t)\\
&= J_3^{\epsilon1}(\tau)+J_3^{\epsilon2}(\tau)+J_3^{\epsilon3}(\tau).
\end{aligned} \tag{5.24}$$

where

$$J_3^{\epsilon 1}(\tau) = \frac{1}{\epsilon^2} E \int_{[\tau,\tau+\epsilon]} \Phi(t) \left[\int_\tau^t \Psi(s)\sigma_u(s)(v-u^*(t))\mathrm{d}W(s) \right] p_2(t)G(t)\mathrm{d}\zeta(t) \tag{5.25}$$

$$J_3^{\epsilon 2}(\tau) = \frac{1}{\epsilon^2} E \int_{[\tau,\tau+\epsilon]} \Phi(t) \left[\int_\tau^t \Psi(s) \left[f_u(s) - (\sigma_x(s) + \widehat{E}[\widehat{\sigma}_\mu(s)])\sigma_u(s) \right] (v-u^*(t))\mathrm{d}s \right] \times p_2(t)G(t)\mathrm{d}\zeta(t) \tag{5.26}$$

$$J_3^{\epsilon 3}(\tau) = \frac{1}{\epsilon^2} E \int_{[\tau,\tau+\epsilon]} \Phi(t) \left[\int_\tau^t \Psi(s)G(s)\mathrm{d}\zeta(s) \right] p_2(t)G(t)\mathrm{d}\zeta(t). \tag{5.27}$$

Estimate of (5.25). From [31, Eq-(3.21)], we have

$$\Phi(t) = \Phi(\tau) + \int_\tau^t \Phi(s)f_x(s)\mathrm{d}s + \int_\tau^t \Phi(s)f_x(s)\mathrm{d}W(s). \tag{5.28}$$

Substituting (5.28) into (5.25), we obtain

$$\begin{aligned} J_3^{\epsilon 1}(\tau) &= \frac{1}{\epsilon^2} E \int_{[\tau,\tau+\epsilon]} \Phi(t) \left[\int_\tau^t \Psi(s)\sigma_u(s)(v-u^*(s))\mathrm{d}W(s) \right] p_2(t)G(t)\mathrm{d}\zeta(t) \\ &= J_3^{\epsilon 1,1}(\tau) + J_3^{\epsilon 1,2}(\tau) + J_3^{\epsilon 1,3}(\tau), \end{aligned} \tag{5.29}$$

where

$$\begin{aligned} J_3^{\epsilon 1,1}(\tau) &= \frac{1}{\epsilon^2} E \int_{[\tau,\tau+\epsilon]} \Phi(\tau) \left[\int_\tau^t \Psi(s)\sigma_u(s)(v-u^*(s))\mathrm{d}W(s) \right] p_2(t)G(t)\mathrm{d}\zeta(t) \\ J_3^{\epsilon 1,2}(\tau) &= \frac{1}{\epsilon^2} E \int_{[\tau,\tau+\epsilon]} \left[\int_\tau^t \Phi(s)f_x(s)\mathrm{d}s \right] p_2(t)G(t)\mathrm{d}\zeta(t) \times \left[\int_\tau^t \Psi(s)\sigma_u(s)(v-u^*(s))\mathrm{d}W(s) \right] \\ J_3^{\epsilon 1,3}(\tau) &= \frac{1}{\epsilon^2} E \int_{[\tau,\tau+\epsilon]} \left[\int_\tau^t \Phi(s)\sigma_x(s)\mathrm{d}W(s) \right] p_2(t)G(t)\mathrm{d}\zeta(t) \times \left[\int_\tau^t \Psi(s)\sigma_u(s)(v-u^*(s))\mathrm{d}W(s) \right]. \end{aligned}$$

From [31, Eq-(3.23)], we have

$$\begin{aligned} \limsup_{\epsilon\to 0^+} J_3^{\epsilon 1,1}(\tau) &= \limsup_{\epsilon\to 0^+} \frac{1}{\epsilon^2} E \int_{[\tau,\tau+\epsilon]} \left[\int_\tau^t \Phi(\tau)\Psi(s)\sigma_u(s)(v-u^*(s))\mathrm{d}W(s) \right] \times p_2(t)G(t)\mathrm{d}\zeta(t) \\ &= 0. \end{aligned} \tag{5.30}$$

Similar as in [31, p 2288], with the helps of Cauchy-Schwartz inequality, we can prove that

$$\begin{aligned} \limsup_{\epsilon\to 0^+} J_3^{\epsilon 1,2}(\tau) &= \limsup_{\epsilon\to 0^+} \frac{1}{\epsilon^2} E \int_{[\tau,\tau+\epsilon]} \left[\int_\tau^t \Phi(s)f_x(s)\mathrm{d}s \right] \times \left[\int_\tau^t \Psi(s)\sigma_u(s)(v-u^*(s))\mathrm{d}W(s) \right] p_2(t)G(t)\mathrm{d}\zeta(t) \\ &= 0. \end{aligned} \tag{5.31}$$

By [31, Lemma 4.1 and Eq-(4.10)], with the help of *Dominate convergence theorem,* we have

$$\begin{aligned} \lim_{\epsilon\to 0^+} J_3^{\epsilon 1,3}(\tau) &= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon^2} E \int_{[\tau,\tau+\epsilon]} \left[\int_\tau^t \Phi(s)\sigma_x(s)\mathrm{d}W(s) \right] \times \left[\int_\tau^t \Psi(s)\sigma_u(s)(v-u^*(s))\mathrm{d}W(s) \right] p_2(t)G(t)\mathrm{d}\zeta(t) \\ &= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon^2} \int_{[\tau,\tau+\epsilon]} E \left[\int_\tau^t \sigma_x(s)\sigma_u(s)(v-u^*(s))\mathrm{d}s \right] p_2(t)G(t)\mathrm{d}\zeta(t) \\ &= \frac{1}{2} E \left[p_2(\tau) G(\tau)\sigma_x(\tau)\sigma_u(\tau)(v-u^*(\tau))\zeta(\tau) \right]. \end{aligned} \tag{5.32}$$

Substituting (5.30), (5.31), (5.32) into (5.29), we obtain

$$\lim_{\epsilon\to 0^+} J_3^{\epsilon 1}(\tau) = \frac{1}{2} E \left[p_2(\tau) G(\tau)\sigma_u(\tau)\sigma_x(\tau)(v-u^*(\tau))\zeta(\tau) \right]. \tag{5.33}$$

Estimate of (5.26). We proceed to estimate the second term $J_3^{\epsilon 2}(\tau)$. By Lemma 4.1 in [31], we have

$$\begin{aligned}
\lim_{\epsilon\to 0^+} J_3^{\epsilon 2}(\tau) &= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon^2} E\int_{[\tau,\tau+\epsilon]} \Phi(t)p_2(t)G(t)\times\left[\int_\tau^t \Psi(s)\left[f_u(s)-\sigma_x(s)\sigma_u(s)\right](v-u^*(t))\mathrm{d}s\right]\mathrm{d}\zeta(t) \\
&= \frac{1}{2}E\left[p_2(\tau)G(\tau)\left[f_u(\tau)-\sigma_x(\tau)\sigma_u(\tau)\right](v-u^*(\tau))\zeta(\tau)\right]. \quad (5.34)
\end{aligned}$$

Estimate of (5.27). Applying [31, Lemma 4.1, Eq (3.21)], we obtain

$$\begin{aligned}
\lim_{\epsilon\to 0^+} J_3^{\epsilon 3}(\tau) &= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon^2} E\int_{[\tau,\tau+\epsilon]} \Phi(t)p_2(t)G(t)\left[\int_\tau^t \Psi(s)G(s)\mathrm{d}\zeta(s)\right]\mathrm{d}\zeta(t) \quad (5.35)\\
&= \frac{1}{2}E\left[p_2(\tau)G^2(\tau)\zeta^2(\tau)\right],
\end{aligned}$$

substituting (5.33), (5.34), (5.35) into (5.24), we have

$$\lim_{\epsilon\to 0^+} \mathbb{J}_3^{\epsilon}(\tau) = \frac{1}{2}E\left[p_2(\tau)G(\tau)f_u(\tau)(v-u^*(\tau))\zeta(\tau)\right]+\frac{1}{2}E\left[p_2(\tau)G^2(\tau)\zeta^2(\tau)\right], \quad (5.36)$$

by substituting (5.23), (5.18) in (5.14), we can easily find

$$\begin{aligned}
0 &\geq \frac{1}{2}E\left[\mathcal{S}(\tau)\Big(f_u(\tau)-(\sigma_x(\tau)+\widehat{E}[\widehat{\sigma}_\mu(\tau)])\sigma_u(\tau)\Big)(v-u^*(\tau))^2\right] \\
&\quad +\frac{1}{2}E\left[\mathcal{S}(\tau)(\sigma_x(\tau)+\widehat{E}[\widehat{\sigma}_\mu(\tau)])\sigma_u(\tau)(v-u^*(\tau))^2\right] \\
&\quad +\frac{1}{2}E\left[p_2(\tau)G^2(\tau)\zeta^2(\tau)\right]+\frac{1}{2}\partial_\tau^+\left(\mathcal{S}(\tau)\sigma_u(\tau)(v-u^*(\tau))^2\right). \\
a.e.\ \tau &\in [0,T],
\end{aligned}$$

This completes the proof. □

6 Concluding remarks and future developments

In this paper, second-order necessary conditions for optimal singular stochastic control for systems governed by general McKean-Vlasov stochastic differential equation have been established. If the coefficients f,σ,ℓ,h depend only on the state variable and the control variable with $G=M=0$, our results coincides with pointwise second-order maximum principle developed by Zhang and Zhang [31, Theorem 3.5]. Apparently, there are many problems left unsolved such as: the case when the control domain is not assumed to be convex (general action space). One possible problem is to consider the second-order maximum principle for optimal singular control for fully coupled forward backward stochastic differential equation (FBSDE) We plane to study these interesting problems in forthcoming papers.

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