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Nevanlinna's counting functions for difference operator and related results

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Abstract

The study of the Nevanlinna theory for difference operators was introduced independently by Halburd & Korhonen and Chiang & Feng in the years 2006 and 2008 respectively. Halburd and Korhonen proved the unique theorem for meromorphic functions associated with c-separated pairs. In this paper, we have generalised the result by introducing c-separated pairs of multiplicity p and their counting functions. We have deduced some analogues of certain uniqueness results of classical Nevanlinna theory due to Chen, Chen & Tsai; Gopalakrishna & Bhoosnurmath; and Lahiri & Pal. Thereafter, we have also discussed certain implications of the deduced results.

Keywords: Meromorphic functions, c-separated pairs, Nevanlinna's counting functions, difference operator, periodic functions
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1 Introduction

The value distribution of complex functions was revolutionised by the Finnish mathematician Rolf Nevanlinna, in a series of publications made by him during 1922-25 [21, 22]. The uniqueness theory of meromorphic functions developed as an offshoot of the value distribution theory when Nevanlinna himself put forward the five value theorem in 1929 [23], as an application of the two fundamental theorems [14]. Thereby, the uniqueness theory went on to become one of the cardinal domains of investigation in this field. In 1976, Gopalakrishna and Bhoosnurmath [10] generalised Nevanlinna's five value theorem further by using truncated sharing of points. In 2007, Chen, Chen and Tsai [5] generalised the results by introducing the partial sharing of small functions which were further relaxed to some extent by Lahiri and Pal [17] in 2015.

The first decade of the twenty-first century witnessed a major shift in paradigm when Halburd and Korhonen [12] and Chiang and Feng [9] independently proved the difference analogue of the logarithmic derivative lemma. The Nevanlinna theory for difference operators manifested largely due to the works of Halburd and Korhonen [13]; they introduced the concept of c-separated pairs and proved the equivalent of Nevanlinna's five value theorem for the difference operators. Thereafter, the value distribution of the difference operators and difference polynomials of meromorphic functions have been thoroughly studied [1, 7, 6, 2]. Uniqueness results in the difference realm have also

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been consistently worked upon. In this context, it is worth mentioning that among many others, Heittokangas et al. [16, 15], Liu [20], Charak et al. [3] investigated the value sharing conditions with the shift operators under which a meromorphic function is periodic. Further, research is being done to substantiate the value sharing conditions that yield $\Delta^n f \equiv f$ [18, 4, 25, 26].

However, the c-separated pairs have not been much scrutinized while studying the uniqueness results of meromorphic functions in the difference realm. In this article, we wish to use the notion of sharing of c-separated pairs, introduced by Halburd and Korhonen, to deduce certain uniqueness results.

2 Definitions and Results

For standard notations of the classical Nevanlinna theory, we refer the reader to [14, 27, 28]. Also, for any meromorphic function f, by $S(r, f)$ we mean any positive quantity such that

$$S(r, f) = o(T(r, f)),$$

outside a set of finite linear measure. And by a small function $a(z)$ of f we mean either $a = \infty$ or

$$T(r, a) = S(r, f).$$

We denote the set of all small functions of a meromorphic function f by $\mathscr{S}(f)$.

Moving in conformance with our principal motivation, we initiate our venture with defining the c-separated pairs and the allied counting functions as given by Halburd and Korhonen.

Definition 2.1. [13] Let $c \in \mathbb{C}$. For a non-constant meromorphic function f, and $a \in \mathbb{C} \cup \{\infty\}$, $n_c(r, a)$ is defined as the counting function of the number of points z_0, where $f(z_0) = a$ and $f(z_0 + c) = a$, in $|z| \leq r$, counted according to the number of equal terms in the beginning of the Laurent series expansions of $f(z)$ and $f(z + c)$ in a neighbourhood of z_0. The points z_0 and $z_0 + c$ are called the c-separated a-pairs of f. Also,

$$\begin{aligned} N_c(r, a) &= \int_0^r \frac{n_c(t, a) - n_c(0, a)}{t} dt + n_c(0, a) \log r, \\ N_c(r, \infty) &= \int_0^r \frac{n_c(t, \infty) - n_c(0, \infty)}{t} dt + n_c(0, \infty) \log r, \end{aligned}$$

where $n_c(r, \infty)$ is the number of c-separated pole pairs of f, which are exactly the c-separated 0-pairs of $1/f$. $N_c(r, \infty)$ is generally denoted by $N_c(r, f)$ and $N_c(r, a)$ is also denoted by $N_c\left(r, \dfrac{1}{f - a}\right)$.

If $c = 0$, then the above counting functions of f reduce to the classical Nevanlinna counting functions. So, henceforth, we shall consider $c \neq 0$.

Definition 2.2. [13] For the integrated counting functions $N(r, a)$ and $N_c(r, a)$, we have

$$\tilde{N}_c(r, a) = N(r, a) - N_c(r, a)$$

which is the number of a-points of f ignoring all the c-separated pairs of f. $\tilde{N}_c(r, a)$ is an analogue of $\overline{N}(r, a)$, denoting the integrated counting function of the distinct zeros of $f(z) = a$ in $|z| \leq r$.

Gopalakrishna and Bhoosnurmath [10] defined $N_p(r, a, f)$ as the counting function of the number of zeros of $f - a$ with multiplicity $\leq p$ in $|z| \leq r$. We have similaryly tried to add the notion of multiplicity for the c-separated pairs as well.

Definition 2.3. Let $c \in \mathbb{C} \setminus \{0\}$ and f be a meromorphic function of finite order. Let p be a positive integer and $a \in \mathbb{C}$. Then the point z_0 is said to be of c-multiplicity p in association with the a-pairs of f, if $f(z_0) = a$ and $f(z_0 + c) = a$ holds simultaneously; and the Taylor series expansions of $f(z)$ and $f(z + c)$ have first p terms equal; that is, z_0 is counted p times in $n_c(r, a)$. We also define $n_c^{p)}(r, a)$ (resp. $n_c^{(p}(r, a)$) as the counting function of all those points z_0 in $|z| \leq r$ having c-multiplicity at most p (resp. at least p), in association with the c-separated a-pairs of f. Finally, $n_c^p(r, a)$ counts the number of a-points of f with c-multiplicity p exact.

For example, if $f(z_0)=a$ and $f(z_0+c)=a$ with the first p terms of their Taylor series in a neighbourhood of z_0 as follows

$$f(z)=a_0+a_1(z-z_0)+a_2(z-z_0)^2+a(z-z_0)^3+\ldots$$

and

$$f(z+c)=a_0+a_1(z-z_0)+a_2(z-z_0)^2+b(z-z_0)^3+\ldots,$$

where $a\neq b$, then z_0 is said to be of c-multiplicity 3 and $\Delta_c f(z)$ has a zero of order 3 at z_0. The integrated counting function corresponding to the above terms are defined in the following.

Definition 2.4. Let $c\in\mathbb{C}\setminus\{0\}$ and f be a meromorphic function. Then

$$\begin{aligned} N_c^{p)}(r,a) &= \int_0^r \frac{n_c^{p)}(t,a)-n_c^{p)}(0,a)}{t}dt+n_c^{p)}(0,a)\log r \\ N_c^{(p}(r,a) &= \int_0^r \frac{n_c^{(p}(t,a)-n_c^{(p}(0,a)}{t}dt+n_c^{(p}(0,a)\log r, \end{aligned}$$

and we define

$$\begin{aligned} \tilde{N}_c^{p)}(r,a) &= N(r,a)-N_c^{p)}(r,a) \\ \tilde{N}_c^{(p}(r,a) &= N(r,a)-N_c^{(p}(r,a) \end{aligned}$$

which counts the number of a-points of f, ignoring the c-separated pairs of c-multiplicity at most p and at least p respectively in association with the c-separated a-pairs of f. We evidently have

$$\tilde{N}_c^{p)}(r,a)+\tilde{N}_c^{(p+1}(r,a)=N(r,a)+\tilde{N}_c(r,a).$$

for some positive integer p.

If we consider the c-separated pairs and its counting functions on a particular subset $A\subset\mathbb{C}$, then they are respectively denoted by $n_c^A(r,a)$, $N_c^A(r,a)$ and $\tilde{N}_c^A(r,a)$. It is needless to say that $n_c^A(r,a)$ cannot exceed $n_c(r,a)$.

For the sake of convenience, the shared points are collectively denoted by a set in the classical theory. The definition is as given below.

Definition 2.5. (see [10],[27]) Let f be a meromorphic function and a be an extended complex number. Then $E^{p)}(a,f)$ is the subset of $\mathbb{C}$ consisting of all the zeros of $f-a$ of order $\leq p$. In particular,

$$E^{\infty)}(a,f)=\{z\in\mathbb{C}|\ f(z)=a\}$$

and the above set is denoted simply by $E(a,f)$. The set of distinct zeros of $f-a$ is denoted by $\overline{E}(a,f)$.

Analogous definition in the difference realm is given below for the c-separated pairs.

Definition 2.6. Let $c\in\mathbb{C}$ and f be a meromorphic function. We define $\tilde{E}_c(a,f)$ to be the set of all a-points of f, ignoring the c-separated pairs. Also, let $\tilde{E}_c^{p)}(a,f)$ be the set of all a-points of f ignoring all the c-separated pairs of c-multiplicity at most p. Then

$$\tilde{E}_c^{\infty)}(a,f)=\tilde{E}_c(a,f).$$

All the above definitions can also be given similarly, if a were a small function of f.

Nevanlinna proved his celebrated five-point uniqueness theorem in the year 1929 [23]. It states that two meromorphic functions sharing five complex values IM are identical. In the year 1976, Gopalakrishna and Bhoosnurmath generalised the classical Nevanlinna's five value theorem and proved the following.

Theorem 2.7. [10] Let f and g be two meromorphic functions. If there exist distinct extended complex numbers $a_1, \ldots, a_k$ such that $E^{p_j)}(a_j, f) = E^{p_j)}(a_j, g)$ for $j = 1(1)k$ for some $p_1, p_2, \ldots, p_k$ each of which is a positive integer or ∞ such that $p_1 \geq p_2 \geq \cdots \geq p_k$, and

$$\sum_{j=2}^{k} \frac{p_j}{p_j+1} - \frac{p_1}{p_1+1} > 2,$$

then $f \equiv g$.

In 2007, Chen Chen and Tsai generalised the uniqueness theorem of the classical Nevanlinna theory in the following manner.

Theorem 2.8. [5] Let f and g be two non-constant meromorphic functions. If $a_1, a_2, \ldots,\ a_k,\ k \geq 5$ are distinct functions in $\mathscr{S}(f) \cap \mathscr{S}(g)$ such that

$$\overline{E}(a_j, f) \subseteq \overline{E}(a_j, g),$$

and

$$\liminf_{r \to \infty} \frac{\sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{f-a_j}\right)}{\sum_{j=1}^{k} \overline{N}\left(r, \frac{1}{g-a_j}\right)} > \frac{1}{k-3},$$

then $f \equiv g$.

In the year 2015, Lahiri and Pal generalised theorem 2.7 and stated the following.

Theorem 2.9. [17] Let f and g be two non-constant meromorphic functions and a_j are distinct small functions of f and g for $j = 1(1)k$, $k \geq 5$. Suppose that $p_1 \geq p_2 \geq \cdots \geq p_k$ are positive integers or infinity and $\delta(\geq 0)$ is such that

$$\frac{1}{p_1} + \left(1 + \frac{1}{p_1}\right) \sum_{j=2}^{k} \frac{1}{1+p_j} + 1 + \delta < (k-2)\left(1 + \frac{1}{p_1}\right).$$

Let $A_j = E^{p_j)}(a_j, f) \setminus E^{p_j)}(a_j, g)$, for $j = 1(1)k$. If

$$\sum_{j=1}^{k} \overline{N}_{A_j}\left(r, \frac{1}{f-a_j}\right) \leq \delta T(r, f)$$

and

$$\liminf_{r \to \infty} \frac{\sum_{j=1}^{k} \overline{N}_{p_j)}\left(r, \frac{1}{f-a_j}\right)}{\sum_{j=1}^{k} \overline{N}_{p_j)}\left(r, \frac{1}{g-a_j}\right)} > \frac{p_1}{(1+p_1)(k-2) - p_1(1+\delta) - 1 - (1+p_1)\sum_{j=2}^{k} \frac{1}{1+p_j}},$$

then $f \equiv g$.

As stated earlier in the text, Halburd and Korhonen proved a difference analogue of the Nevanlinna five-value theorem, which is stated below.

Theorem 2.10. [13] Let $c \in \mathbb{C}$, and let f and g be meromorphic functions of finite order. If there are five distinct periodic functions $a_k \in \mathscr{S}(f)$ such that f and g share a_k, ignoring c-separated pairs, for $k = 1, 2, \ldots, 5$ then, either $f \equiv g$ or both f and g are periodic functions of period c.

We now proceed to state the main results of this paper.

Theorem 2.11. Let $c \in \mathbb{C}$ and let f and g be two finite-ordered meromorphic functions. If $a_1, a_2, \ldots, a_k$, $k \geq 5$ are distinct periodic functions of period c in $\mathscr{S}(f) \cap \mathscr{S}(g)$ such that

$$\tilde{E}_c(a_j, f) \subseteq \tilde{E}_c(a_j, g), \tag{2.1}$$

and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{f-a_j}\right)}{\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{g-a_j}\right)} > \frac{1}{k-3},$$

then, either $f \equiv g$ or both f and g are periodic with period c.

Theorem 2.12. Let $c \in \mathbb{C}$ and f and g be two finite-ordered meromorphic functions. If a_j, $j = 1(1)k$, with $k \geq 5$ are distinct periodic functions of period c in $\mathscr{S}(f) \cap \mathscr{S}(g)$. If

$$\tilde{E}_c^{p_j)}(a_j, f) = \tilde{E}_c^{p_j)}(a_j, g), \quad j = 1(1)k,$$

for some $p_1, \ldots p_k$ each of which is a positive integer or infinity with $p_1 \geq p_2 \geq \cdots \geq p_k$ and

$$\sum_{j=2}^{k} \frac{p_j}{p_j+1} - \frac{p_1}{p_1+1} > 2.$$

Then, either $f \equiv g$ or both f and g are periodic with period c.

Theorem 2.13. Let $c \in \mathbb{C}$ and let f and g be two finite-ordered meromorphic functions. If a_j, $j = 1(1)k$, with $k \geq 5$ are distinct periodic functions of period c in $\mathscr{S}(f) \cap \mathscr{S}(g)$. Suppose that $p_1 \geq p_2 \geq \cdots \geq p_k$ are positive integers or infinity and $\delta(\geq 0)$ be such that

$$\frac{1}{p_1} + \left(1 + \frac{1}{p_1}\right) \sum_{j=2}^{k} \frac{1}{1+p_j} + 1 + \delta < (k-2)\left(1 + \frac{1}{p_1}\right).$$

Let $A_j = \tilde{E}_c^{p_j)}(a_j, f) \setminus \tilde{E}_c^{p_j)}(a_j, g)$, for $j = 1(1)k$. If

$$\sum_{j=1}^{k} \tilde{N}_c^{A_j}\left(r, \frac{1}{f-a_j}\right) \leq \delta T(r,f)$$

and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \tilde{N}_c^{p_j)}\left(r, \frac{1}{f-a_j}\right)}{\sum_{j=1}^{k} \tilde{N}_c^{p_j)}\left(r, \frac{1}{g-a_j}\right)} > \frac{p_1}{(1+p_1)(k-2) - p_1(1+\delta) - 1 - (1+p_1)\sum_{j=2}^{k} \frac{1}{1+p_j}},$$

then either $f \equiv g$ or both f and g are periodic with period c.

It is worth mentioning that if f and g does not have any c-separated pairs then the above theorems reduce to theorems 2.8, 2.7 and 2.9 respectively.

3 Lemmas

In this section, we state certain lemmas that we will use in the sequel.

Lemma 3.1. [12] Let $c \in \mathbb{C}$, and let f be a meromorphic function of finite order such that $\Delta_c f \not\equiv 0$. Let $k \geq 2$, and let $a_1(z), a_2(z), \ldots, a_k(z)$ be distinct meromorphic periodic functions with period c such that $a_j \in \mathscr{S}(f)$ for $j = 1, \ldots, k$. Then

$$(k-2)T(r,f) \leq \sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{f-a_j}\right) + S(r,f)$$

where the exceptional set associated with $S(r,f)$ is of at most finite logarithmic measure.

We state a result analogous to the Picard's theorem for classical Nevanlinna theory.

Lemma 3.2. [12] Let $c \in \mathbb{C}$. If a finite-order meromorphic function f has three exceptional paired values with separation c, then f is a periodic function with period c.

Lemma 3.3. Let f be a meromorphic function and c be a non-zero complex number. Then, for positive integers p, q and an extended complex number a, if $n_c^p(r,a)$, $n_c^{p)}(r,a)$ and $n_c^{(p}(r,a)$ are the counting functions associated with the c-separated pairs of c-multiplicity p (or q), then the following hold.

1. $n_c^p(r,a) \leq n_c(r,a)$; $n_c^{p)}(r,a) \leq n_c(r,a)$; $n_c^{(p}(r,a) \leq n_c(r,a)$;
2. $n_c^1(r,a) + n_c^2(r,a) + \ldots + n_c^p(r,a) = n_c^{p)}(r,a)$;
3. $n_c^{(p}(r,a) = n_c^p(r,a) + n_c^{p+1}(r,a) + \ldots$;
4. $n_c(r,a) = n_c^1(r,a) + n_c^2(r,a) + \ldots = \displaystyle\sum_{i=1}^{\infty} n_c^i(r,a)$;
5. $n_c^{p)}(r,a) + n_c^{(p}(r,a) = n_c^p(r,a) + n_c(r,a)$;
6. $n_c^{p)}(r,a) + n_c^{(p+1}(r,a) = n_c(r,a)$;
7. For two positive integers p and q such that $p < q$, we have, $n_c^{p)}(r,a) \leq n_c^{q)}(r,a)$ and $n_c^{(q}(r,a) \leq n_c^{(p}(r,a)$;
8. $n_c^{\infty)}(r,a) = n_c(r,a)$ and $n_c^{(\infty}(r,a) = 0$; however, if f is a periodic function with period c, then for sufficiently large r, we may have $n_c^{(\infty}(r,a) = n_c(r,a) = n_c^{\infty}(r,a)$.

The proof is obvious from the definitions.

Lemma 3.4. Let $c \in \mathbb{C}$ and f be a meromorphic function. Then we have, for any positive integer p,

$$\begin{array}{rrcl} & p\tilde{N}_c(r,a) & \leq & p\tilde{N}_c^{p)}(r,a) + N_c(r,a) \\ \text{and} & (p+1)\tilde{N}_c(r,a) & \leq & p\tilde{N}_c^{p)}(r,a) + N(r,a), \\ & p\tilde{N}_c(r,a) & \leq & p\tilde{N}_c^{(p}(r,a) + N_c(r,a). \end{array}$$

Proof of Lemma 3.4

We have,

$$pn_c^{p)}(r,a) \leq (p+1)n_c(r,a),$$

which gives

$$pN_c^{p)}(r,a) + N(r,a) \leq (p+1)N_c(r,a) + N(r,a),$$

since $N(r,a)$ is a positive quantity. Further simplifications give us

$$(p+1)\tilde{N}_c(r,a) \leq p\tilde{N}_c^{p)}(r,a) + N(r,a),$$

and since $\tilde{N}_c(r,a) = N(r,a) - N_c(r,a)$, so we get

$$p\tilde{N}_c(r,a) \leq p\tilde{N}_c^{p)}(r,a) + N_c(r,a).$$

We similarly have for $\tilde{N}_c^{(p}(r,a)$,

$$(p+1)\tilde{N}_c(r,a) \leq p\tilde{N}_c^{(p}(r,a) + N(r,a),$$

and

$$p\tilde{N}_c(r,a) \leq p\tilde{N}_c^{(p}(r,a) + N_c(r,a).$$

□

4 Proof of the Main Theorems

Proof of theorem 2.11

Suppose first that f is periodic with period c. Then all the a-points of f are paired and by the equation (2.1), g has at least five exceptional paired values. Thus, g has to be periodic by lemma 3.2. So, assume that neither f nor g are periodic with period c and that $f \not\equiv g$. We have, by lemma and the properties of $\tilde{N}_c(r, f)$,

$$(k-2)T(r,f) \leq \sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{f-a_j}\right) + S(r,f), \tag{4.1}$$

outside a set of finite logarithmic measure. Similarly, we have for g,

$$(k-2)T(r,g) \leq \sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{g-a_j}\right) + S(r,g), \tag{4.2}$$

outside a set of finite logarithmic measure. Since $f \not\equiv g$, by equations (4.1) and (4.2), we have,

$$\begin{aligned}
\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{f-a_j}\right) &\leq N\left(r, \frac{1}{f-g}\right) \\
&\leq T\left(r, \frac{1}{f-g}\right) \\
&= T(r, f-g) + O(1) \\
&\leq T(r,f) + T(r,g) + O(1) \\
&\leq \frac{1}{k-2}\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{f-a_j}\right) + \frac{1}{k-2}\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{g-a_j}\right) \\
&+ S(r,f) + S(r,g) + O(1) \\
&\leq \left\{\frac{1}{k-2} + o(1)\right\}\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{f-a_j}\right) \\
&+ \left\{\frac{1}{k-2} + o(1)\right\}\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{g-a_j}\right)
\end{aligned}$$

from which it follows that

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{f-a_j}\right)}{\sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{g-a_j}\right)} \leq \frac{1}{k-3},$$

which is a contradiction to our assumption. Hence f and g must be identical. □

Proof of theorem 2.12

First, let f be a periodic function with period c. Thus, all the a-points of f are paired. Since $\tilde{E}_c^{p_j)}(a_j, f) = \tilde{E}_c^{p_j)}(a_j, g)$ for $j = 1(1)k$, $k \geq 5$, so g has at least five exceptionally paired values. Hence, by lemma 3.2, g is also a periodic function of period c.

Next, suppose that neither f nor g are periodic of period c. Also, let $f \not\equiv g$. So by lemma 3.1,

$$(k-2)T(r,f) \leq \sum_{j=1}^{k} \tilde{N}_c\left(r, \frac{1}{f-a_j}\right) + S(r,f),$$

which gives

$$\begin{aligned}(k-2)T(r,f) &\le \sum_{j=1}^{k}\frac{1}{p_j+1}\left\{p_j\tilde{N}_c^{p_j)}\left(r,\frac{1}{f-a_j}\right)+N\left(r,\frac{1}{f-a_j}\right)\right\}+S(r,f)\\ &\le \sum_{j=1}^{k}\frac{p_j}{p_j+1}\tilde{N}_c^{p_j)}\left(r,\frac{1}{f-a_j}\right)+\sum_{j=1}^{k}\frac{1}{p_j+1}T(r,f)+S(r,f).\end{aligned}$$

This gives

$$\left\{\sum_{j=1}^{k}\frac{p_j}{p_j+1}-2\right\}T(r,f)\le\sum_{j=1}^{k}\frac{p_j}{p_j+1}\tilde{N}_c^{p_j)}\left(r,\frac{1}{f-a_j}\right)+S(r,f). \tag{4.3}$$

Similarly, for the function g, we get

$$\left\{\sum_{j=1}^{k}\frac{p_j}{p_j+1}-2\right\}T(r,g)\le\sum_{j=1}^{k}\frac{p_j}{p_j+1}\tilde{N}_c^{p_j)}\left(r,\frac{1}{g-a_j}\right)+S(r,g). \tag{4.4}$$

Adding equations (4.3) and (4.4), we get

$$\left\{\sum_{j=1}^{k}\frac{p_j}{p_j+1}-2\right\}\{T(r,f)+T(r,g)\}\le\sum_{j=1}^{k}\frac{p_j}{p_j+1}\{\tilde{N}_c^{p_j)}\left(r,\frac{1}{f-a_j}\right)+\tilde{N}_c^{p_j)}\left(r,\frac{1}{g-a_j}\right)\}+S(r,f)+S(r,g). \tag{4.5}$$

Since $\tilde{E}_c^{p_j)}(a_j,f)=\tilde{E}_c^{p_j)}(a_j,g)$ for $j=1(1)k,\ k\ge 5$, we have, for all j,

$$\tilde{N}_c^{p_j)}\left(r,\frac{1}{f-a_j}\right)=\tilde{N}_c^{p_j)}\left(r,\frac{1}{g-a_j}\right)=\tilde{N}_0(r,a_j),\quad \text{say.}$$

Also, since $p_1\ge\cdots\ge p_k$, and the sequence $\{j/(j+1)\}$ is increasing, we get from equation (4.5),

$$\left\{\sum_{j=1}^{k}\frac{p_j}{p_j+1}-2\right\}(T(r,f)+T(r,g))\le\frac{2p_1}{p_1+1}\sum_{j=1}^{k}\tilde{N}_0(r,a_j)+S(r,f)+S(r,g). \tag{4.6}$$

Now, since $f\not\equiv g$, we get

$$\begin{aligned}\sum_{j=1}^{k}\tilde{N}_0(r,a_j) &= \sum_{j=1}^{k}\tilde{N}_c^{p_j)}\left(r,\frac{1}{g-a_j}\right)\\ &\le N\left(r,\frac{1}{f-g}\right)\\ &\le T\left(r,\frac{1}{f-g}\right)\\ &= T(r,f-g)+O(1)\\ &\le T(r,f)+T(r,g)+O(1).\end{aligned}$$

Thus, using these, we get, from equation (4.6),

$$\left\{\sum_{j=1}^{k}\frac{p_j}{p_j+1}-2\right\}\{T(r,f)+T(r,g)\}\le\frac{2p_1}{p_1+1}\{(T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

which gives

$$\left\{\sum_{j=2}^{k}\frac{p_j}{p_j+1}-\frac{2kp_1}{p_1+1}-2\right\}\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g). \tag{4.7}$$

Now, since

$$\sum_{j=2}^{k}\frac{p_j}{p_j+1}-\frac{p_1}{p_1+1}-2>0,$$

we get, from equation (4.7),

$$T(r,f)+T(r,g)=o\{T(r,f)+T(r,g)\}$$

for sufficiently large r, outside a set of finite logarithmic measure, which is absurd. Thus, f must be equivalent to g. $\square$

Proof of theorem 2.13

First let, f is periodic with period c. Thus, all the a-points of f are paired. And since

$$\sum_{j=1}^{k}\tilde{N}_c^{A_j}\left(r,\frac{1}{f-a_j}\right)\le \delta T(r,f),$$

so, the a_j-points of f are shared by g with the same multiplicity p_j. Thus, g has at least five exceptional paired values and thus, by lemma 3.2, we say that g is also a periodic function of period c.

Next, suppose that neither f nor g is periodic with period c. If possible, let $f\not\equiv g$. By lemmas 3.1 and 3.4, we get

$$\begin{aligned}(k-2)T(r,f) &\le \sum_{j=1}^{k}\tilde{N}_c\left(r,\frac{1}{f-a_j}\right)+S(r,f)\\ &= \sum_{j=1}^{k}\frac{1}{1+p_j}\left\{p_j\tilde{N}_c^{p_j)}\left(r,\frac{1}{f-a_j}\right)+N\left(r,\frac{1}{f-a_j}\right)\right\}+S(r,f)\\ &\le \sum_{j=1}^{k}\frac{p_j}{1+p_j}\tilde{N}_c^{p_j)}\left(r,\frac{1}{f-a_j}\right)+T(r,f)\sum_{j=1}^{k}\frac{1}{1+p_j}+S(r,f),\end{aligned}$$

which gives

$$\left\{k-2-\sum_{j=1}^{k}\frac{1}{p_j+1}+o(1)\right\}T(r,f)\le \sum_{j=1}^{k}\frac{p_j}{p_j+1}\tilde{N}_c^{p_j)}\left(r,\frac{1}{f-a_j}\right). \tag{4.8}$$

Similarly, for g, we get

$$\left\{k-2-\sum_{j=1}^{k}\frac{1}{p_j+1}+o(1)\right\}T(r,g)\le \sum_{j=1}^{k}\frac{p_j}{p_j+1}\tilde{N}_c^{p_j)}\left(r,\frac{1}{g-a_j}\right). \tag{4.9}$$

Now, let us take the set $B_j=\tilde{E}_c^{p_j)}(a_j,f)\setminus A_j$ for all $j=1(1)k$. Since $f\not\equiv g$, we get

$$\begin{aligned}\sum_{j=1}^{k}\tilde{N}_c^{p_j)}\left(r,\frac{1}{f-a_j}\right) &= \sum_{j=1}^{k}\tilde{N}_c^{A_j}\left(r,\frac{1}{f-a_j}\right)+\sum_{j=1}^{k}\tilde{N}_c^{B_j}\left(r,\frac{1}{f-a_j}\right)\\ &\le \delta T(r,f)+N\left(r,\frac{1}{f-g}\right)\\ &\le \delta T(r,f)+T\left(r,\frac{1}{f-g}\right)\\ &= \delta T(r,f)+T(r,f-g)+O(1)\\ &\le (\delta+1)T(r,f)+T(r,g)+O(1).\end{aligned}$$

Thus, equations (4.8) and (4.9) give,

$$\begin{aligned}
&\left\{k-2-\sum_{j=1}^{k}\frac{1}{p_j+1}+o(1)\right\}\sum_{j=1}^{k}\tilde{N}_c^{p_j)}\left(r,\frac{1}{f-a_j}\right)\\
\leq\ &\left\{k-2-\sum_{j=1}^{k}\frac{1}{p_j+1}+o(1)\right\}\{(1+\delta)T(r,f)+T(r,g)\}\\
\leq\ &(1+\delta)\sum_{j=1}^{k}\frac{p_j}{p_j+1}\tilde{N}_c^{p_j)}\left(r,\frac{1}{f-a_j}\right)+\{1+o(1)\}\sum_{j=1}^{k}\frac{p_j}{p_j+1}\tilde{N}_c^{p_j)}\left(r,\frac{1}{g-a_j}\right)
\end{aligned}$$

Since $\{j/(j+1)\}$ is an increasing sequence, and $p_1 \geq p_2 \geq \cdots \geq p_k$, so from the above inequality, we get,

$$\left\{k-2-\sum_{j=1}^{k}\frac{1}{p_j+1}-\frac{(1+\delta)p_1}{1+p_1}+o(1)\right\}\sum_{j=1}^{k}\tilde{N}_c^{p_j)}\left(r,\frac{1}{f-a_j}\right)\leq\{1+o(1)\}\frac{p_1}{p_1+1}\sum_{j=1}^{k}\frac{p_j}{p_j+1}\tilde{N}_c^{p_j)}\left(r,\frac{1}{g-a_j}\right),$$

for a sequence of values of r tending to ∞. Thus, we get from the above inequality,

$$\liminf_{r\to\infty}\frac{\sum\limits_{j=1}^{k}\tilde{N}_c^{p_j)}\left(r,\frac{1}{f-a_j}\right)}{\sum\limits_{j=1}^{k}\tilde{N}_c^{p_j)}\left(r,\frac{1}{g-a_j}\right)}\leq\frac{p_1}{(1+p_1)(k-2)-p_1(1+\delta)-1-(1+p_1)\sum\limits_{j=2}^{k}\frac{1}{1+p_j}},$$

which is a contradiction to our assumption. Thus, f and g have to be identical. □

5 Consequences of the results

Consequences of theorem 2.11

The following corollary immediately follows from 2.11 putting $k = 5$.

Corollary 5.1. Let $c \in \mathbb{C}$ and let f and g be two finite-ordered meromorphic functions. If a_j, $j = 1(1)5$ are five distinct periodic functions of period c in $\mathscr{S}(f) \cap \mathscr{S}(g)$ such that

$$\tilde{E}_c(a_j; f) \subseteq \tilde{E}_c(a_j; g), \tag{5.1}$$

and

$$\liminf_{r\to\infty}\frac{\sum\limits_{j=1}^{5}\tilde{N}_c\left(r,\frac{1}{f-a_j}\right)}{\sum\limits_{j=1}^{5}\tilde{N}_c\left(r,\frac{1}{g-a_j}\right)}>\frac{1}{2},$$

then, either $f \equiv g$ or both f and g are periodic with period c.

Consequences of theorem 2.12

We will discuss certain consequences of theorem 2.12 in the following:

1. Let f and g are finite-ordered meromorphic functions both of which are not periodic. Let there are seven periodic functions $a_1, \ldots, a_7$ of period $c \in \mathbb{C}$ in $\mathscr{S}(f) \cap \mathscr{S}(g)$ such that $\tilde{E}_c^{p_j)}(a_j, f) = \tilde{E}_c^{p_j)}(a_j, g)$ for $j = 1(1)7$, where each p_j is either a positive integer or infinity with $p_1 \geq \cdots \geq p_7$ and $p_2 \geq 2$ if $p_1 = 0$. Then $p_1/(1+p_1) \leq 1$ with equality holding only when $p_1 = \infty$ and $p_j/(1+p_j) \geq 1/2$ for $i = 2(1)7$ with $p_2/(1+p_2) \geq 2/3$ if $p_1 = \infty$. Then

$$\sum_{j=2}^{7}\frac{p_j}{1+p_j}-\frac{p_1}{1+p_1}>2.$$

and thus by theorem 2.12, f and g are either identical.

In particular, if $p_1 = p_2 = \cdots = p_7 = \infty$, it follows that $\tilde{E}_c(a_j, f) = \tilde{E}_c(a_j, g)$ for $j = 1(1)7$. Thus, from theorem 2.12, we can say that $f \equiv g$. Further, if we use five distinct periodic small functions, then the above theorem takes the form of theorem 2.10.

2. Let f and g are finite-ordered meromorphic functions both of which are not periodic. Let there are six periodic functions $a_1, \ldots, a_6$ of period $c \in \mathbb{C}$ in $\mathscr{S}(f) \cap \mathscr{S}(g)$ such that $\tilde{E}_c^{p_j)}(a_j, f) = \tilde{E}_c^{p_j)}(a_j, g)$ for $j = 1(1)6$, where each p_j is either a positive integer or infinity with $p_1 \geq \cdots \geq p_6$ and $p_3 \geq 2$ and
$$\frac{p_1}{1+p_1} < \frac{p_2}{1+p_2} + \frac{1}{6}.$$
Then, clearly,
$$\sum_{j=2}^{6} \frac{p_j}{1+p_j} - \frac{p_1}{1+p_1} > 2.$$
Hence, by theorem 2.12, we have, $f \equiv g$.

3. Let f and g are finite-ordered meromorphic functions both of which are not periodic. Let there are five periodic functions $a_1, \ldots, a_5$ of period $c \in \mathbb{C}$ in $\mathscr{S}(f) \cap \mathscr{S}(g)$ such that $\tilde{E}_c^{p_j)}(a_j, f) = \tilde{E}_c^{p_j)}(a_j, g)$ for $j = 1(1)5$, where each p_j is either a positive integer or infinity with $p_1 \geq \cdots \geq p_5$ and $p_4 \geq 4$ and
$$\frac{p_1}{1+p_1} < \frac{p_2}{1+p_2} + \frac{1}{10}.$$
Then, clearly,
$$\sum_{j=2}^{5} \frac{p_j}{1+p_j} - \frac{p_1}{1+p_1} > 2.$$
Hence, by theorem 2.12, we again have, $f \equiv g$.

5.1 Consequences of theorem 2.13

We will discuss a corollary of theorem 2.13 which is stated as follows.

Corollary 5.2. Let $c \in \mathbb{C}$ and let f and g are non-periodic functions. Also, let there are k periodic functions $a_1, \ldots, a_k$ of period $c \in \mathbb{C}$ in $\mathscr{S}(f) \cap \mathscr{S}(g)$ such that $\tilde{E}_c^{p_j)}(a_j, f) \subset \tilde{E}_c^{p_j)}(a_j, g)$, for $j = 1(1)k$, $k \geq 5$, where p_j are either positive integers or infinity satisfying $p_1 \geq \cdots \geq p_k$. Let
$$B = \liminf_{r \to \infty} \frac{\sum\limits_{j=1}^{k} \tilde{N}_c^{p_j)}\left(r, \frac{1}{f-a_j}\right)}{\sum\limits_{j=1}^{k} \tilde{N}_c^{p_j)}\left(r, \frac{1}{g-a_j}\right)}.$$

Then
$$\sum_{j=2}^{k} \frac{p_j}{1+p_j} \leq \frac{p_1}{B(1+p_1)} + 2.$$

Proof . To prove the above corollary, we take $\delta = 0$. Also, $A_j = \emptyset$, for all j. Thus, by theorem, we get
$$B \leq \frac{p_1}{(1+p_1)(k-2) - p_1(1+\delta) - 1 - (1+p_1)\sum\limits_{j=2}^{k} \frac{1}{1+p_j}},$$
and thus, by simplifying, we get the desired result. The above corollary is also true if we take $\tilde{E}_c^{p_j)}(a_j, f) = \tilde{E}_c^{p_j)}(a_j, g)$, for $j = 1(1)k$. In that case, $B = 1$. $\square$

6 Conclusion and discussions

In this article, we have theoretically deduced the conditions under which two finite ordered meromorphic functions sharing c-separated pairs, are identical. In theorem 2.11, we saw that imposing a certain condition on the counting function $\tilde{N}\left(r, \frac{1}{f-a}\right)$ on the meromorphic functions partially sharing certain periodic small functions serves our purpose. Theorems 2.12 and 2.13 however take into account the c-separated pairs of c-multiplicity p_i, a finite decreasing sequence of positive integers. Since the difference analogue of the log-derivative lemma is also true for meromorphic functions of hyper-order < 1 [11], so it would be interesting to investigate for similar results of uniqueness for such functions as well. Having traversed one extreme end of the scale, it is worth mentioning the other extreme too. Chern [8] introduced the idea of logarithmic order to classify meromorphic functions of zero order. We conjecture that the conditions in our results can be relaxed further in case of zero ordered meromorphic functions. Till now, we have only discussed the scope of improvement for meromorphic functions depending upon their growth property. But, one may also attempt at improving, by considering the order of the c-separated pairs shared. In 2022, Pal and Choudhury [24] introduced the notion of c-separated pairs of order k. It would be an invigorating attempt to check the degree of sustainability of the results when functions share c-separated pairs of order k.

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