Int. J. Nonlinear Anal. Appl. 14 (2023) 10, 85–93
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.30429.4399

Existence of three weak solutions for an anisotropic quasi-linear elliptic problem

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(Communicated by Abdolrahman Razani)

Abstract

We consider in this paper a Neumann $\vec{p}(x)-$elliptic problems of the type

$$\begin{cases} -\Delta_{\vec{p}(x)} u + \lambda(x)|u|^{p_0(x)-2}u = \alpha f(x,u) + \beta g(x,u) & \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\gamma_i = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We prove the existence of three weak solutions in the framework of anisotropic Sobolev spaces with variable exponent $W^{1,\vec{p}(\cdot)}(\Omega)$ under some hypotheses. The approach is based on a recent three critical points theorem for differentiable functionals.

Keywords: Neumann elliptic problem, weak solutions, Variational principle, Anisotropic variable exponent Sobolev spaces
2020 MSC: Primary 35J57, 35D30; Secondary 34B15, 35A15

1 Introduction

In this paper, we study the existence of three weak solutions of the following non-linear anisotropic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\right) + \lambda(x)|u|^{p_0(x)-2}u = \alpha f(x,u) + \beta g(x,u) & \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\gamma_i = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, and let $\vec{\gamma}$ be the outward unit normal vector on $\partial\Omega$ and let γ_i, $i \in \{1 \dots N\}$, $\alpha, \beta > 0$ are real numbers, and $p_i(x) \in \mathcal{C}_+(\overline{\Omega})$ for $i = 1, \dots, N$. The functions $f(x,t)$ and $g(x,t)$ in the right-hand side of equation (1.1) satisfies some suitable conditions which will be specified later in this article.

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Received: April 2023 *Accepted:* May 2023

It's clear that this $\vec{p}(\cdot)$-Laplace operator

$$\Delta_{\vec{p}(\cdot)}u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right), \tag{1.2}$$

is a generalization of the $p(\cdot)$-Laplace operator

$$\Delta_{p(\cdot)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right). \tag{1.3}$$

The $p(\cdot)$-Laplacian is a meaningful generalization of the p-Laplacian operator

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \tag{1.4}$$

obtained in the case when p is a positive constant.

In the classical Sobolev spaces, G. Bonanno and P. Candito [7] have proved the existence of three solutions for the problem (1.1), for more results see [2, 6, 30].

In the Sobolev variable exponent setting, Pan et al. [20] solved the problem (1.1), see also [12, 21, 23] for related topics.

The study of nonlinear partial differential equations in this type of spaces is strongly motivated by numerous phenomena of physics, namely the problems related to non Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like shear rate, magnetic or electric field [4, 9, 11, 32].

It is not a surprise that, when passing from a variable exponent to an anisotropic variable exponent, new difficulties occur. To overpass these difficulties, we combine the classical techniques with the recent techniques that appeared when treating anisotropic problems with variable exponents. Many such problems that are related to our study were presented in [1, 10, 18]. Nonetheless, the hypotheses we use in this paper are totally different from those ones and so are our results.

The necessity for anisotropic spaces with variable exponents arises due to the varied behaviors exhibited by some materials in different directions. In [1], Ahmed, Hjiaj, and Touzani examined the Neumann $\vec{p}(\cdot)$-elliptic problem, as demonstrated in equation (1.1).They established the existence of multiple weak solutions under augmented conditions in the anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$. Razani and Soltani explored the multiplicity of solutions for a similar Neuman problem in [33] under the assumptions that $p_i(x) \geq 2$ for all $x \in \Omega$ and $i \in [\![1, N]\!]$, as well as an additional hypothesis regarding the second terms f and g. Additional related findings can be found in [26, 27, 28, 33, 29].

It is worth noting that exploring the existence and multiplicity of solutions for the equation (1.1) in different boundary conditions or in Orlicz and Musielak Sobolev spaces is a fascinating problem. For those interested, more information about these spaces can be found in [14, 17, 19], as well as in the references cited therein.

The following theorem plays an important role in this paper.

Theorem 1.1. ([31]). Let E be a separable and reflexive real Banach space; $\Psi : E \longrightarrow \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on E^*, $\Phi : E \longrightarrow \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that

(a)

$$\lim_{\|u\|_E \longrightarrow +\infty}\Big(\Psi(u) + \alpha\Phi(u)\Big) = +\infty \text{ for all } \alpha > 0;$$

and there are $r \in \mathbb{R}$ and $u_0, u_1 \in E$ such that

(b)

$$\Psi(u_0) < r < \Psi(u_1);$$

(c)

$$\inf_{u\in\Psi^{-1}(]-\infty,r])}\Phi(u) > \frac{\Big(\Psi(u_1) - r\Big)\Phi(u_0) + \Big(r - \Psi(u_0)\Big)\Phi(u_1)}{\Psi(u_1) - \Psi(u_0)}.$$

Then there exist an open interval $\Lambda \subset]0, +\infty[$ and a positive real number ρ such that for each $\alpha \in \Lambda$ and every continuously Gâteaux differentiable functional $J : E \longrightarrow \mathbb{R}$ with compact derivative, there exists $\sigma > 0$ such that for each $\beta \in [0, \sigma]$, the equation

$$\Psi'(u) + \alpha\Phi'(u) + \beta J'(u) = 0,$$

has at least three solutions in E whose norms are less than ρ.

This paper is organized as follows: In Section 2, we present some necessary preliminary knowledge on the anisotropic Sobolev spaces with variable exponents. We introduce in the Section 3, some assumptions for which our problem has a solutions. In the final section we state and prove the existence of three weak solutions for our Neumann elliptic problem. Our main results, it is a result nouveau. So, even if for the constant exponent case.

2 Preliminary

In this section we summarize notation, definitions and properties of our framework. For more details we refer to [13, 16, 26, 27, 28, 29, 33]. Let Ω be a bounded domain in $\mathbb{R}^N$, we define:

$$\mathcal{C}_+(\overline{\Omega}) = \Big\{\text{measurable function} \quad p(\cdot) : \overline{\Omega} \longrightarrow \mathbb{R} \quad \text{such that} \quad 1 < p^- \leq p^+ < \infty\Big\},$$

where

$$p^- = \operatorname{ess\,inf}\big\{p(x) \,/\, x \in \overline{\Omega}\big\} \quad \text{and} \quad p^+ = \operatorname{ess\,sup}\big\{p(x) \,/\, x \in \overline{\Omega}\big\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \longrightarrow \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)}\, dx,$$

is finite, then

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf\big\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1\big\},$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $\big(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}\big)$ is a separable Banach space. Moreover, the space $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\Big|\int_\Omega uv\; dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{(p^-)'}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}, \tag{2.1}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$. An important role in manipulating the generalized Lebesgue spaces is played by the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result.

Proposition 2.1. ([13]). If $u \in L^{p(\cdot)}(\Omega)$, then the following properties hold true:

(i) $\|u\|_{p(\cdot)} < 1 (= 1, > 1) \Longleftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1, > 1)$,

(ii) $\|u\|_{p(\cdot)} > 1 \Longrightarrow \|u\|^{p^-}_{p(\cdot)} < \rho_{p(\cdot)}(u) < \|u\|^{p^+}_{p(\cdot)}$,

(iii) $\|u\|_{p(\cdot)} < 1 \Longrightarrow \|u\|^{p^+}_{p(\cdot)} < \rho_{p(x)}(u) < \|u\|^{p^-}_{p(\cdot)}$.

We define the Sobolev space with variable exponent by:

$$W^{1,p(\cdot)}(\Omega) = \Big\{u \in L^{p(\cdot)}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{p(\cdot)}(\Omega)\Big\},$$

equipped with the following norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $\big(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)}\big)$ is a separable and reflexive Banach space. We refer to [13] for the elementary properties of these spaces.

Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of our main problem.

Let $p_0(x), p_1(x), \dots, p_N(x)$ be $N+1$ variable exponents in $\mathcal{C}_+(\overline{\Omega})$. We denote

$$\vec{p}(x) = \Big\{p_0(x), p_1(x), \dots, p_N(x)\Big\},\ D^0u = u \quad \text{and} \quad D^iu = \frac{\partial u}{\partial x_i} \quad \text{for} \quad i = 1, \dots, N.$$

We define

$$\underline{p} = \min\Big\{p_0^-, p_1^-, \dots, p_N^-\Big\} \qquad \text{then} \qquad \underline{p} > 1, \tag{2.2}$$

and

$$\overline{p} = \max\Big\{p_0^+, p_1^+, \dots, p_N^+\Big\}. \tag{2.3}$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follows

$$W^{1,\vec{p}(\cdot)}(\Omega) = \Big\{u \in L^{p_0(\cdot)}(\Omega) \quad \text{and} \quad D^iu \in L^{p_i(\cdot)}(\Omega), \quad i = 1, 2, \dots, N\Big\},$$

endowed with the norm

$$\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = \|u\|_{1,\vec{p}(\cdot)} = \|u\|_{L^{p_0(\cdot)}(\Omega)} + \sum_{i=1}^{N} \|D^iu\|_{L^{p_i(\cdot)}(\Omega)}. \tag{2.4}$$

(Cf. [5, 24, 25] for the constant exponent case). For the basic properties of $W^{1,\vec{p}(\cdot)}(\Omega)$, see [8, 22].

Proposition 2.2. ([15]). The space $\big(W^{1,\vec{p}(\cdot)}(\Omega), \|\cdot\|_{1,\vec{p}(\cdot)}\big)$ is a separable and reflexive Banach space, if $p_i^- > 1$ for $i = 1, \dots, N$.

From now on, we always assume that

$$\underline{p} > N. \tag{2.5}$$

Remark 2.3. Since $W^{1,\vec{p}(\cdot)}(\Omega)$ is continuously embedded in $W^{1,\underline{p}}(\Omega)$, and $W^{1,\underline{p}}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ (the space of continuous functions), thus $W^{1,\vec{p}(\cdot)}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$.

Set

$$C_0 = \sup_{u \in W^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}} \frac{\|u\|_{L^\infty(\Omega)}}{\|u\|_{1,\vec{p}(\cdot)}}. \tag{2.6}$$

Then C_0 is a positive constant.

3 Basic assumptions

Throughout this paper, we assume the following assumptions.

(H1) $\lambda(\cdot) \in L^\infty(\Omega)$, with $\lambda^- = \operatorname{ess\,inf} \lambda(x) > 0$.

(H2) $\alpha, \beta > 0$ are real numbers.

We assume that $f(x,u)$ and $g(x,u)$ satisfy the following general conditions:

(H3) $f,\ g : \Omega \times \mathbb{R} \longmapsto \mathbb{R}$ are Carathéodory functions and satisfies

$$f(x,t) \le c_1 + c_2|t|^{h(x)-1}, \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

$$g(x,t) \le c_1' + c_2'|t|^{k(x)-1}, \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $h(x), k(x) \in \mathcal{C}(\overline{\Omega})$ and $1 < h(x) \le h^+ = \max_{x\in\overline{\Omega}} h(x) < \underline{p}$, $1 < k(x) \le k^+ = \max_{x\in\overline{\Omega}} k(x) < \underline{p}$ and c_1, c_2, c_1', c_2' are positive constants.

(H4) There exists a constant $t_0 > 0$ such that

$$f(x,t) < 0 \quad \text{when } |t| \in [0, t_0],$$

$$f(x,t) > M > 0 \quad \text{when } |t| \in [t_0, +\infty[,$$

where M is a positive constant.

We set

$$F(x,t) = \int_0^t f(x,s)ds \qquad \text{and} \qquad G(x,t) = \int_0^t g(x,s)ds. \tag{3.1}$$

Before stating the result to be proved, we introduce the functionals Ψ, $\Phi : W^{1,\vec{p}(\cdot)}(\Omega) \longrightarrow \mathbb{R}$ by

$$\Psi(u) = \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx + \int_\Omega \frac{\lambda(x)}{p_0(x)} |u|^{p_0(x)}\, dx, \tag{3.2}$$

$$\Phi(u) = -\int_\Omega F(x,u)\, dx, \tag{3.3}$$

and

$$J(u) = -\int_\Omega G(x,u)\, dx. \tag{3.4}$$

Let us start by giving the definition of weak solution for the problem (1.1).

Definition 3.1. A measurable function $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ is called a weak solution of the Neumann elliptic problem (1.1) if

$$\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, dx + \int_\Omega \lambda(x)|u|^{p_0(x)-2}uv\, dx - \alpha\int_\Omega f(x,u)v\, dx - \beta\int_\Omega g(x,u)v\, dx, \tag{3.5}$$

for all $v \in W^{1,\vec{p}(\cdot)}(\Omega)$.

We recall the following results concerning the functionals Φ, Ψ and J.

Lemma 3.2. ([10]). The functionals Ψ, Φ and J are well-defined on $W^{1,\vec{p}(\cdot)}(\Omega)$. In addition, Ψ, Φ and J are of class $C^1(W^{1,\vec{p}(\cdot)}(\Omega), \mathbb{R})$ and

$$\langle \Psi'(u), v\rangle = \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, dx + \int_\Omega \lambda(x)|u|^{p_0(x)-2}uv\, dx,$$

$$\langle \Phi'(u), v\rangle = -\int_\Omega f(x,u)v\, dx,$$

and

$$\langle J'(u), v\rangle = -\int_\Omega g(x,u)v\, dx,$$

for all $u, v \in W^{1,\vec{p}(\cdot)}(\Omega)$.

Lemma 3.3. ([15]). Let $(H1)$-$(H4)$ holds. Then Ψ, Φ and J are sequentially weakly lower semi-continuous.

Lemma 3.4. Let $\frac{1}{p_i'} + \frac{1}{p_i} = 1$. Then $\Psi' : W^{1,\vec{p}(\cdot)}(\Omega) \longrightarrow W^{-1,\vec{p}(\cdot)'}(\Omega)$ is coercive, a homeomorphism and uniformly monotone, where $\vec{p}(\cdot)' = \left\{p_0', \dots p_N'\right\}$, (cf. [5] for the constant exponent case).

Proof . When $\|u\|_{1,\vec{p}(\cdot)} > 1$, we have

$$\begin{aligned}
\Psi(u) &= \sum_{i=1}^{N}\int_{\Omega}\frac{1}{p_i(x)}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)}\,dx + \int_{\Omega}\frac{\lambda(x)}{p_0(x)}\Big|u\Big|^{p_0(x)}\,dx \\
&\geq \sum_{i=1}^{N}\frac{1}{p_i^+}\Big(\Big\|\frac{\partial u}{\partial x_i}\Big\|_{p_i(\cdot)}^{\underline{p}}\Big) + \frac{\lambda^-}{p_0^+}\Big(\|u\|_{p_0(\cdot)}^{\underline{p}}\Big) \qquad \text{(by Proposition 2.1)} \\
&\geq \frac{\min\{1,\lambda^-\}}{(N+1)^{\underline{p}-1}\overline{p}}\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}},
\end{aligned}$$

which shows that Ψ is coercive. It is obvious that $(\Psi')^{-1} : W^{-1,\vec{p}(\cdot)'}(\Omega) \longrightarrow W^{1,\vec{p}(\cdot)}(\Omega)$ exists and continuous, because $\Psi' : W^{1,\vec{p}(\cdot)}(\Omega) \longrightarrow W^{-1,\vec{p}(\cdot)'}(\Omega)$ is a homeomorphism. Recalling the following well-known inequality

$$\Big(|\, a\,|^{\theta-2}\, a- |\, b\,|^{\theta-2}\, b\Big)(a-b) \geq \frac{1}{2^\theta}|a-b|^\theta, \ \ \forall a,b\in\mathbb{R}^N,\ \forall\theta\geq 2,$$

then we have for all $u, v \in W^{1,\vec{p}(\cdot)}(\Omega)$

$$\begin{aligned}
\langle\Psi'(u)-\Psi'(v),u-v\rangle &= \sum_{i=1}^{N}\int_{\Omega}\Big[\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)-2}\frac{\partial u}{\partial x_i} - \Big|\frac{\partial v}{\partial x_i}\Big|^{p_i(x)-2}\frac{\partial v}{\partial x_i}\Big]\Big(\frac{\partial u}{\partial x_i}-\frac{\partial v}{\partial x_i}\Big)\,dx \\
&\quad + \int_{\Omega}\lambda(x)\Big(|u|^{p_0(x)-2}u - |v|^{p_0(x)-2}v\Big)(u-v)\,dx \\
&\geq \frac{1}{2\overline{p}}\Big[\sum_{i=1}^{N}\int_{\Omega}\Big(\Big|\frac{\partial u}{\partial x_i}-\frac{\partial v}{\partial x_i}\Big|^{p_i(x)}\Big)\,dx + \lambda^-\int_{\Omega}\Big(|u-v|^{p_0(x)}\Big)\,dx\Big] \\
&\geq \frac{1}{2\overline{p}}\Big[\sum_{i=1}^{N}\Big(\Big\|\frac{\partial u}{\partial x_i}-\frac{\partial v}{\partial x_i}\Big\|_{p_i(\cdot)}^{\underline{p}}-1\Big) + \lambda^-\Big(\|u-v\|_{p_0(\cdot)}^{\underline{p}}-1\Big)\Big] \qquad \text{(by Proposition 2.1)} \\
&\geq \frac{\min\{1,\lambda^-\}}{2(N+1)^{\underline{p}-1}\overline{p}}\|u-v\|_{1,\vec{p}(\cdot)}^{\underline{p}}, \ \ \forall u,v\in W^{1,\vec{p}(\cdot)}(\Omega),
\end{aligned}$$

i.e. Ψ' is uniformly monotone. We deduce that $(\Psi')^{-1}$ exists and it is continuous. □

4 Main Results

Now, we formulate our main result.

Theorem 4.1. Let $(H1)$-$(H4)$ holds and $\underline{p} > N$. Then there exist an open interval $\Lambda \subset]0,+\infty[$ and a constant $\rho > 0$ such that for any $\alpha\in\Lambda$ and every function $g : \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ which satisfying $(H3)$, there exists $\sigma > 0$ such that for each $\beta\in[0,\sigma]$ problem (1.1) has at least three solutions in $W^{1,\vec{p}(\cdot)}(\Omega)$ whose norms are less than ρ.

Proof . In order to prove this result, we apply Theorem 1.1. Below we denote by d_i a generic positive constant. Since $1 < h(x) \leq h^+ = \max_{x\in\overline{\Omega}} h(x) < \underline{p}$, we obtain $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W^{1,\underline{p}}(\Omega) \hookrightarrow L^{\underline{p}}(\Omega) \hookrightarrow L^{h^+}(\Omega)$, so we can find two positive constants d_1, d_2 such that

$$\|u\|_{L^{h^+}(\Omega)} \leq d_1\|u\|_{1,\vec{p}(\cdot)}, \tag{4.1}$$

$$\|u\|_{L^{h^-}(\Omega)} \leq d_2\|u\|_{1,\vec{p}(\cdot)}, \tag{4.2}$$

and by using $(H3)$, we have

$$\begin{aligned}
\Psi(u)+\alpha\Phi(u) &= \sum_{i=1}^{N}\int_{\Omega}\frac{1}{p_i(x)}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)}\,dx + \int_{\Omega}\frac{\lambda(x)}{p_0(x)}|u|^{p_0(x)}\,dx - \alpha\int_{\Omega}F(x,u)\,dx \\
&\geq \frac{\min\{1,\lambda^-\}}{(N+1)^{\underline{p}-1}\overline{p}}\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - \alpha c_1\int_{\Omega}|u|\,dx - \alpha c_2\int_{\Omega}\frac{1}{h(x)}|u|^{h(x)}\,dx \\
&\geq \frac{\min\{1,\lambda^-\}}{(N+1)^{\underline{p}-1}\overline{p}}\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - \alpha c_1\|u\|_{L^h(\Omega)} - \alpha d_3\Big(\|u\|_{L^h(\Omega)}^{h^+} + \|u\|_{L^h(\Omega)}^{h^-}\Big) \\
&\geq \frac{\min\{1,\lambda^-\}}{(N+1)^{\underline{p}-1}\overline{p}}\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - d_4\|u\|_{1,\vec{p}(\cdot)} - d_5\|u\|_{1,\vec{p}(\cdot)}^{h^+},
\end{aligned}$$

for $\|u\|_{1,\vec{p}(\cdot)} > 1$ and any $\alpha > 0$. Since $h^+ < \underline{p}$, then $\lim\limits_{\|u\|_{1,\vec{p}(\cdot)} \longrightarrow +\infty} (\Psi(u) + \alpha\Phi(u)) = +\infty$ and (a) is verified.

In the following, we will verify the conditions (b) and (c) in Theorem 1.1. By $F'_t(x,t) = f(x,t)$ and assumption (H4), it follows that $F(x,t)$ is increasing for $t \in [t_0, +\infty[$ and decreasing for $t \in [0, t_0]$. Obviously

$$F(x,0) = 0 \quad \text{and} \quad \lim_{t \longrightarrow +\infty} F(x,t) = +\infty.$$

Then there exists a real number $\delta > t_0$ such that

$$F(x,t) \geq 0 = F(x,0) \geq F(x,\tau), \quad \forall x \in \Omega,\ \forall t > \delta, \text{ and } \tau \in [0, t_0]. \tag{4.3}$$

Let c, m be two real numbers such that $0 < c < \min\{1, C_0\}$ with C_0 given in remark 2.3 and $m > \delta$ satisfies

$$m^{\underline{p}} \|\lambda\|_{L^1(\Omega)} > 1, \tag{4.4}$$

and

$$m^{\overline{p}} \|\lambda\|_{L^1(\Omega)} > 1. \tag{4.5}$$

By (4.3) we obtain

$$\int_\Omega \sup_{0 \leq t \leq c} F(x,t)\, dx \leq 0 < \frac{1}{m^{\underline{p}}} \Big(\frac{c}{C_0}\Big)^{\overline{p}} \int_\Omega F(x,m)\, dx.$$

Consider u_0, $u_1 \in W^{1,\vec{p}(\cdot)}(\Omega)$, with $u_0(x) = 0$ and $u_1(x) = m$ for any $x \in \Omega$. We define $r = \frac{\min\{1,\lambda^-\}}{(N+1)^{\underline{p}-1}\overline{p}} \Big(\frac{c}{C_0}\Big)^{\overline{p}}$. Clearly, $r \in]0,1[$. A simple computation implies

$$\Psi(u_0) = \Phi(u_0) = 0.$$

Let $m > 1$. Then, if we consider formula (4.4) we get

$$\begin{aligned}
\Psi(u_1) &= \int_\Omega \frac{1}{p_0(x)} \lambda(x) m^{p_0(x)}\, dx \\
&\geq \frac{1}{\overline{p}} m^{\underline{p}} \int_\Omega \lambda(x)\, dx \\
&= \frac{1}{\overline{p}} m^{\underline{p}} \|\lambda\|_{L^1(\Omega)} \\
&> \frac{1}{\overline{p}} \\
&> \frac{\min\{1,\lambda^-\}}{(N+1)^{\underline{p}-1}\overline{p}} \Big(\frac{c}{C_0}\Big)^{\overline{p}} \\
&= r.
\end{aligned} \tag{4.6}$$

Similarly for $m < 1$, by help of (4.5), we get the desired result. Thus, we deduce that

$$\Psi(u_0) < r < \Psi(u_1),$$

and (b) in Theorem 1.1 is verified. Finally, we will verify that condition (c) of Theorem 1.1 is fulfilled. Moreover, we have

$$\begin{aligned}
\Phi(u_1) &= -\int_\Omega F(x,m)\, dx \\
&= -F(x,m)|\Omega|,
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
\frac{\Big(\Psi(u_1) - r\Big)\Phi(u_0) + \Big(r - \Psi(u_0)\Big)\Phi(u_1)}{\Psi(u_1) - \Psi(u_0)} &= r\frac{\Phi(u_1)}{\Psi(u_1)} \\
&= -r\frac{F(x,m)|\Omega|}{\displaystyle\int_\Omega \frac{\alpha(x)}{p_0(x)} m^{p_0(x)}\, dx} \\
&< 0.
\end{aligned} \tag{4.8}$$

Next, we consider the case $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ such that $\|u\|_{1,\vec{p}(\cdot)} \le 1$ with $\Psi(u) \le r < 1$. Since

$$\begin{aligned} \frac{\min\{1,\alpha^-\}}{(N+1)^{\underline{p}-1}\overline{p}}\Big\|u\Big\|^{\overline{p}}_{1,\vec{p}(\cdot)} &\le \frac{1}{\overline{p}}\Big(\sum_{i=1}^{N}\int_\Omega\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)}\,dx + \int_\Omega |u|^{p_0(x)}\,dx\Big) \le \Psi(u) \le r \\ &< 1. \end{aligned}$$

Thus, using remark 2.3, we have

$$\|u\|_{L^\infty(\Omega)} \le C_0\Big(\overline{p}r\Big)^{\frac{1}{\overline{p}}} = c.$$

The above inequality shows that

$$\begin{aligned} -\inf_{u\in\Psi^{-1}(]-\infty,r])}\phi(u) &= \sup_{u\in\Psi^{-1}(]-\infty,r])} -\phi(u) \\ &\le \int_\Omega \sup_{0\le t\le c} F(x,t)\,dx \\ &\le 0. \end{aligned}$$

It follows that

$$-\inf_{u\in\Psi^{-1}(]-\infty,r])}\phi(u) < r\frac{\displaystyle\int_\Omega F(x,b)\,dx}{\displaystyle\int_\Omega \frac{1}{p_0(x)}\lambda(x)m^{p_0(x)}\,dx}.$$

That is

$$\inf_{u\in\Psi^{-1}(]-\infty,r])}\phi(u) > \frac{\Big(\Psi(u_1)-r\Big)\phi(u_0)+\Big(r-\Psi(u_0)\Big)\phi(u_1)}{\Psi(u_1)-\Psi(u_0)}.$$

which means that condition (c) in Theorem 1.1 is verified. So, all the assumptions of Theorem 1.1 are satisfied and the conclusion follows. □

Acknowledgements

The authors are grateful to the reviewer(s) for useful comments.

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