Int. J. Nonlinear Anal. Appl. 7 (2016) No. 2, 279-301
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.1803.1472

Solutions and stability of a variant of Van Vleck's and d'Alembert's functional equations

Th.M. Rassias[a,*], E. Elqorachi[b], A. Redouani[b]

[a]Department of Mathematics, National Technical University of Athens, Zofrafou Campus, 15780 Athens, Greece
[b]Ibn Zohr University, Faculty of Sciences Department of Mathematic, Agadir, Morocco

(Communicated by M. Eshaghi)

Abstract

In this paper. (1) We determine the complex-valued solutions of the following variant of Van Vleck's functional equation

$$\int_S f(\sigma(y)xt)d\mu(t) - \int_S f(xyt)d\mu(t) = 2f(x)f(y),\ x, y \in S,$$

where S is a semigroup, σ is an involutive morphism of S, and μ is a complex measure that is linear combinations of Dirac measures $(\delta_{z_i})_{i\in I}$, such that for all $i \in I$, z_i is contained in the center of S. (2) We determine the complex-valued continuous solutions of the following variant of d'Alembert's functional equation

$$\int_S f(xty)dv(t) + \int_S f(\sigma(y)tx)dv(t) = 2f(x)f(y),\ x, y \in S,$$

where S is a topological semigroup, σ is a continuous involutive automorphism of S, and v is a complex measure with compact support and which is σ-invariant. (3) We prove the superstability theorems of the first functional equation.

Keywords: d'Alembert's equation; Van Vleck's equation; sine function; multiplicative function; superstability.
2010 MSC: Primary 39B32; Secondary 39B52.

*Corresponding author
Email addresses: trassias@math.ntua.gr (Th.M. Rassias), elqorachi@hotmail.com (E. Elqorachi), redouaniahmed@yahoo.fr (A. Redouani)

Received: December 2015 *Revised:* August 2016

1. Introduction

In his two papers [27, 28] Van Vleck studied the continuous solutions $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f \neq 0$ of the following functional equation

$$f(x - y + z_0) - f(x + y + z_0) = 2f(x)f(y),\ x, y \in \mathbb{R}, \tag{1.1}$$

where $z_0 > 0$ is fixed. He showed first that all solutions are periodic with period $4z_0$, and then he selected for his study any continuous solution with minimal period $4z_0$. He proved that any such solution has to be the sine function
$f(x) = \sin(\frac{\pi}{2z_0}x) = \cos(\frac{\pi}{2z_0}(x - z_0),\ x \in \mathbb{R}$.
Stetkær [9, Exercise 9.18] found the complex-valued solution of equation

$$f(xy^{-1}z_0) - f(xyz_0) = 2f(x)f(y),\ x, y \in G, \tag{1.2}$$

on non abelian groups G and where z_0 is a fixed element in the center of G.
Perkins and Sahoo [20] replaced the group inversion by an involution anti-automorphism τ: $G \longrightarrow G$ and they obtained the abelian, complex-valued solutions of equation

$$f(x\tau(y)z_0) - f(xyz_0) = 2f(x)f(y),\ x, y \in G. \tag{1.3}$$

Stetkær [22] extends the results of Perkins and Sahoo [20] about equation (1.3) to the more general case where G is a semigroup and the solutions are not assumed to be abelian.
Recently, Bouikhalene and Elqorachi [1] extends the results of Stetkær's [22] and obtain the solutions of the following extension of Van Vleck's functional equations

$$\chi(y)f(x\tau(y)z_0) - f(xyz_0) = 2f(x)f(y),\ x, y \in S$$

and

$$\chi(y)f(\sigma(y)xz_0) - f(xyz_0) = 2f(x)f(y),\ x, y \in M, \tag{1.4}$$

where S is a semigroup, χ is a multiplicative, M is a monoid, τ is an involution anti-automorphism of S and σ is an involutive automorphism of M.
There has been quite a development of the theory of d'Alembert's functional equation

$$f(xy) + f(x\tau(y)) = 2f(x)f(y),\ x, y \in G, \tag{1.5}$$

on non abelian groups, as shown in works by Davison [3, 4] for general groups, even monoids. The non-zero solutions of equation (1.5) for general groups, even monoids are the normalized traces of certain representations of the group G on $\mathbb{C}^2$ [3, 4].

Stetkær [24] obtained the complex valued solutions of the following variant of d'Alembert's functional equation

$$f(xy) + f(\sigma(y)x) = 2f(x)f(y),\ x, y \in S, \tag{1.6}$$

where σ is an involutive automorphism of the semigroup S. The solutions of equation (1.6) are of the form $f(x) = \frac{\chi(x)+\chi(\sigma(x))}{2}$, $x \in G$, where χ is multiplicative.

In [5] Ebanks and Stetkær obtained the solutions f, g: $G \longrightarrow \mathbb{C}$ of the following variant of Wilson's functional equation (see also [26])

$$f(xy) + f(y^{-1}x) = 2f(x)g(y),\ x, y \in G.$$

In 1979, a type of stability was observed by J. Baker, J. Lawrence and F. Zorzitto [5]. Indeed, they proved that if a function is approximately exponential, then it is either a true exponential function or bounded. Then the exponential functional equation is said to be superstable. This result was the first result concerning the superstability phenomenon of functional equations. Later, J. Baker [4] (see also [1, 3, 6, 13]) generalized this result as follows: Let $(S, .)$ be an arbitrary semigroup, and let $f : S \longrightarrow \mathbb{C}$. Assume that f is an approximately exponential function, i.e., there exists a nonnegative number δ such that $|f(xy) - f(x)f(y)| \leq \delta$ for all $x, y \in S$. Then f is either bounded or f is a multiplicative function. The result of Baker, Lawrence and Zorzitto [5] was generalized by L. Székelyhidi [29, 30] in another way. We refer also to [2], [8], [11], [12], [13], [14], [16], [18], [19] and [21] for other results concerning the stability and the superstability of functional equations

In the first part of this paper we extend the above results to the following generalization of Van Vleck's functional equation for the sine:

$$\int_S f(\sigma(y)xt)d\mu(t) - \int_S f(xyt)d\mu(t) = 2f(x)f(y),\ x, y \in S, \tag{1.7}$$

where S is a semigroup and σ is an involutive morphism : That is σ is an involutive automorphism: $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma(\sigma(x)) = x$ for all x, y or σ is an involutive anti-automorphism: $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$ for all x, y, and μ is assumed to be a complex measure that is linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$, with z_i contained in the center of S, for all $i \in I$.

The main idea is to relate the functional equation (1.7) to to the following variant of d'Alembert's functional equation

$$g(xy) + g(\sigma(y)x) = 2g(x)g(y),\ x, y \in S \tag{1.8}$$

and we apply the result obtained by Stetkær [24, 26].

In section 3, we obtain the the complex-valued continuous solutions of the following variant of d'Alembert's functional equation

$$\int_S f(xty)dv(t) + \int_S f(\sigma(y)tx)dv(t) = 2f(x)f(y),\ x, y \in S, \tag{1.9}$$

where S is a topological semigroup, σ is a continuous involutive automorphism of S, and v is a complex measure with compact support and which is σ-invariant. That is $\int_S h(\sigma(t))dv(t) = \int_S h(t)dv(t)$ for all continuous function h on S.

In the last section we obtain the superstability theorems of the functional equation (1.7).

In all proofs of the results of this paper we use without explicit mentioning the assumption that z_i is contained in the center of S for all $i \in I$ and its consequence $\sigma(z_i)$ is contained in the center of S.

2. The complex-valued solutions of equation (1.7) on semigroups

In this section we determine the solutions of the variant Van Vleck's functional equation (1.7) on semigroups. We first prove the following useful lemmas.

Lemma 2.1. *Let S be semigroup, let $\sigma : S \longrightarrow S$ be an involutive morphism of S and μ be a complex measure that is linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$, with z_i contained in the center of S for all $i \in I$.*

Let $f\colon S \longrightarrow \mathbb{C}$ be a non-zero solution of equation (1.7). *Then for all $x, y \in S$ we have*

$$f(x) = -f(\sigma(x)), \tag{2.1}$$

$$\int_S f(t)d\mu(t) \neq 0, \tag{2.2}$$

$$f(\sigma(y)x) = -f(\sigma(x)y), \tag{2.3}$$

$$\int_S \int_S f(x\sigma(t)s)d\mu(t)d\mu(s) = f(x) \int_S f(t)d\mu(t), \tag{2.4}$$

$$\int_S \int_S f(xts)d\mu(t)d\mu(s) = -f(x) \int_S f(t)d\mu(t), \tag{2.5}$$

$$\int_S f(\sigma(x)t)d\mu(t) = \int_S f(xt)d\mu(t), \tag{2.6}$$

$$\int_S f(x\sigma(t))d\mu(t)) = \int_S f(\sigma(x)\sigma(t))d\mu(t), \tag{2.7}$$

$$\int_S \int_S f(ts)d\mu(t)d\mu(s) = \int_S \int_S f(t\sigma(s))d\mu(t)d\mu(s) = 0. \tag{2.8}$$

Proof . Equation (2.2): Let $f \neq 0$ be a non-zero solution of equation (1.7) and assume that $\int_S f(t)d\mu(t) = 0$. Taking $y = s$ in equation (1.7) and integrate the result obtained with respect to s we get

$$\int_S \int_S f(\sigma(s)xt)d\mu(s)d\mu(t) - \int_S \int_S f(xst)d\mu(s)d\mu(t) = 2f(x) \int_S f(s)d\mu(s) = 0 \tag{2.9}$$

Replacing y by ys in (1.7) and integrating the result with respect to s and using (2.9) and (1.7) we get

$$\int_S \int_S f(\sigma(y)xt\sigma(s))d\mu(t)d\mu(s) - \int_S \int_S f(xyst)d\mu(s)d\mu(t) = 2f(x) \int_S f(ys)d\mu(s)$$
$$= \int_S \int_S f(\sigma(y)xts)d\mu(t)d\mu(s) - \int_S \int_S f(xyst)d\mu(s)d\mu(t)$$
$$= 2f(y) \int_S f(xt)d\mu(t),$$

which implies that $f(y) \int_S f(xs)d\mu(s) = f(x) \int_S f(ys)d\mu(s)$ for all $x, y \in S$. Since $f \neq 0$, then there exists $\alpha \in \mathbb{C}\backslash\{0\}$ such that $\int_S f(xs)d\mu(s) = -\alpha f(x)$ for all $x \in S$. Substituting this into (1.7) we get

$$f(xy) - f(\sigma(y)x) = 2\frac{f(x)}{\alpha} f(y) \text{ for all } x, y \in S. \tag{2.10}$$

By interchanging x with y in (2.10) we get

$$f(yx) - f(\sigma(x)y) = \frac{2}{\alpha} f(x)f(y) \tag{2.11}$$

If we replace y by $\sigma(y)$ in (2.10) we have

$$f(x\sigma(y)) - f(yx) = \frac{2}{\alpha} f(x)f(\sigma(y)) \tag{2.12}$$

By adding (2.12) and (2.11) we obtain

$$f(x\sigma(y)) - f(\sigma(x)y) = \frac{2}{\alpha} f(x)[f(\sigma(y)) + f(\sigma(y))] \tag{2.13}$$

By replacing x by $\sigma(x)$ in (2.13) we get

$$-f(xy)+f(\sigma(x)\sigma(y))=\frac{2}{\alpha}f(\sigma(x))[f(\sigma(y))+f(\sigma(y))] \tag{2.14}$$

If we replace y by $\sigma(y)$ in (2.13) we get

$$-f(\sigma(x)\sigma(y))+f(xy)=\frac{2}{\alpha}f(x)[f(\sigma(y))+f(\sigma(y))]. \tag{2.15}$$

Now, by adding (2.14) and (2.15) we have

$$[f(x)+f(\sigma(x))][f(\sigma(y))+f(\sigma(y))]=0 \tag{2.16}$$

That is $f(\sigma(x))=-f(x)$ for all $x\in S$. Now, we will discuss two cases. Case 1: If σ is an involutive anti-automorphism. Then from $f(\sigma(x))=-f(x)$ for all $x\in S$ we have $f(\sigma(y)x)=-f(\sigma(x)y)$ for all $x,y\in S$ and equation (2.10) can be written as follows

$$f(xy)+f(\sigma(x)y)=2\frac{f(x)}{\alpha}f(y) \text{ for all } x,y\in S. \tag{2.17}$$

The left hand side of (2.18) is unchanged under interchange of x and $\sigma(x)$, so we get $f(x)=f(\sigma(x))$ for all $x\in S$. Now, $f(x)=-f(\sigma(x))=-f(x)$ implies that $f=0$. This contradicts the assumption that $f\neq 0$.
Case 2: If σ is an involutive automorphism. Then from $f(\sigma(x))=-f(x)$ for all $x\in S$ we have $f(\sigma(y)x)=-f(y\sigma(x))$ for all $x,y\in S$ and equation (2.10) can be written as follows

$$f(xy)+f(y\sigma(x))=2\frac{f(x)}{\alpha}f(y) \text{ for all } x,y\in S. \tag{2.18}$$

By replacing x by $\sigma(x)$ in (2.18) and using $f(\sigma(x))=-f(x)$ we get

$$l(yx)+l(\sigma(x)y)=2l(x)l(y) \text{ for all } x,y\in S.$$

where $l=-\frac{f}{\alpha}$. So, from [24] $f(\sigma(x))=f(x)$ for all $x\in S$. Consequently, $f(\sigma(x))=f(x)=-f(x)$ for all $x\in S$, which implies that $f=0$. This contradict the assumption that $f\neq 0$ and this proves the assertion (2.2).

Equation (2.3): Replacing y by yt in (1.7) and integrating the result obtained with respect to t we get

$$\int_S\int_S f(\sigma(y)xs\sigma(t))d\mu(t)d\mu(s)-\int_S\int_S f(xyst)d\mu(s)d\mu(t)=2f(x)\int_S f(yt)d\mu(t). \tag{2.19}$$

Replacing x by xs in (1.7) and integrating the result obtained with respect to s we get

$$\int_S\int_S f(\sigma(y)xst)d\mu(t)d\mu(s)-\int_S\int_S f(xyst)d\mu(s)d\mu(t)=2f(y)\int_S f(xs)d\mu(s). \tag{2.20}$$

Subtracting these equations results in

$$\begin{aligned}\int_S\int_S f(\sigma(y)xs\sigma(t))d\mu(t)d\mu(s)&-\int_S\int_S f(\sigma(y)xst)d\mu(t)d\mu(s)\\ &=2f(x)\int_S f(yt)d\mu(t)-2f(y)\int_S f(xs)d\mu(s).\end{aligned} \tag{2.21}$$

Since from (1.7) we have

$$\begin{aligned}&\int_S\int_S f(\sigma(y)xs\sigma(t))d\mu(t)d\mu(s)-\int_S\int_S f(\sigma(y)xst)d\mu(t)d\mu(s)\\&=\int_S[\int_S f(\sigma(t)\sigma(y)xs)d\mu(s)-\int_S f(\sigma(y)xts)d\mu(s)]d\mu(t)\\&=\int_S 2f(t)f(\sigma(y)x)d\mu(t)=2f(\sigma(y)x)\int_S f(t)d\mu(t).\end{aligned}$$

Then we get

$$f(x)\int_S f(yt)d\mu(t)-f(y)\int_S f(xs)d\mu(s)=f(\sigma(y)x)\int_S f(t)d\mu(t) \tag{2.22}$$

for all $x,y\in S$. Since $f(x)\int_S f(yt)d\mu(t)-f(y)\int_S f(xs)d\mu(s)=-[f(y)\int_S f(xt)d\mu(t)-f(x)\int_S f(ys)d\mu(s)]$, then we obtain $f(\sigma(y)x)\int_S f(t)d\mu(t)=-f(\sigma(x)y)\int_S f(t)d\mu(t)$. Now, by using (2.2) we deduce (2.3).

Equation (2.7): Replacing x by $x\sigma(s)$ in (1.7) and integrating the result obtained with respect to s we get

$$\begin{aligned}&\int_S\int_S f(\sigma(y)xs\sigma(s)t)d\mu(s)d\mu(t)-\int_S\int_S f(xy\sigma(s)t)d\mu(s)d\mu(t)\\&\qquad=2f(y)\int_S f(x\sigma(s))d\mu(s).\end{aligned} \tag{2.23}$$

By using (2.3) we have $f(x(y\sigma(s)t))=-f(\sigma(y\sigma(s)t)\sigma(x))=-f(\sigma(y)\sigma(x)\sigma(t)s)$ and then

$$\int_S\int_S f(xy\sigma(s)t)d\mu(s)d\mu(t)=-\int_S\int_S f(\sigma(y)\sigma(x)\sigma(s)t)d\mu(s)d\mu(t),$$

so equation (2.23) can be written as follows

$$\begin{aligned}&\int_S\int_S f(\sigma(y)xs\sigma(s)t)d\mu(s)d\mu(t)+\int_S\int_S f(\sigma(y)\sigma(x)\sigma(s)t)d\mu(s)d\mu(t)\\&\qquad=2f(y)\int_S f(x\sigma(s)d\mu(s).\end{aligned} \tag{2.24}$$

The left hand side of (2.24) is unchanged under interchange of x and $\sigma(x)$, so since $f\neq 0$ we get (2.7). By using (2.3) and (2.7) we get

$$\begin{aligned}&\int_S f(xs)d\mu(s)=-\int_S f(\sigma(s)\sigma(x))d\mu(s)\\&=-\int_S f(x\sigma(s))d\mu(s)=\int_S f(\sigma(x)s)d\mu(s).\end{aligned}$$

This proves (2.6).

Equation (2.1): By replacing x by $\sigma(x)$ in (1.7) we obtain

$$\int_S f(\sigma(y)\sigma(x)t)d\mu(t)-\int_S f(\sigma(x)yt)d\mu(t)=2f(\sigma(x))f(y). \tag{2.25}$$

If σ is an involutive automorphism then from (2.6) we have

$$\int_S f(\sigma(y)\sigma(x)t)d\mu(t)=\int_S f(\sigma(yx)t)d\mu(t)=\int_S f(yxt)d\mu(t)$$

and it follows from (2.25) that

$$\int_S f(yxt)d\mu(t) - \int_S f(\sigma(x)yt)d\mu(t) = 2f(\sigma(x))f(y).$$

Since

$$\int_S f(yxt)d\mu(t) - \int_S f(\sigma(x)yt)d\mu(t) = -\left[\int_S f(\sigma(x)yt)d\mu(t) - \int_S f(yxt)d\mu(t)\right] = -2f(y)f(x),$$

then we conclude that

$$-2f(x)f(y) = 2f(\sigma(x))f(y)$$

for all $x, y \in S$. Since $f \neq 0$ then we get (2.1).

If σ is an involutive anti-automorphism then from (2.6) we have

$$\int_S f(\sigma(y)\sigma(x)t)d\mu(t) = \int_S f(\sigma(xy)t)d\mu(t) = \int_S f(xyt)d\mu(t)$$

and $\int_S f(\sigma(x)yt)d\mu(t) = \int_S f(\sigma(y)xt)d\mu(t)$. Equation (2.25) implies that

$$\int_S f(xyt)d\mu(t) - \int_S f(\sigma(x)yt)d\mu(t) = 2f(\sigma(x))f(y) = -\left[\int_S f(\sigma(x)yt)d\mu(t) - \int_S f(xyt)d\mu(t)\right]$$

$$= -\left[\int_S f(\sigma(y)xt)d\mu(t) - \int_S f(xyt)d\mu(t)\right] = -2f(x)f(y).$$

Since $f \neq 0$ then we get again (2.1).

Equation (2.4): Putting $x = \sigma(s)$ in (1.7), using (2.1) and integrating the result obtained with respect to s we get by a computation that

$$\int_S\int_S f(\sigma(y)\sigma(s)t)d\mu(s)d\mu(t) - \int_S\int_S f(\sigma(s)yt)d\mu(s)d\mu(t) = 2f(y)\int_S f(\sigma(s))d\mu(s)$$

$$= -2f(y)\int_S f(s)d\mu(s).$$

Since

$$\int_S\int_S f(\sigma(y)\sigma(s)t)d\mu(s)d\mu(t) = -\int_S\int_S f(ys\sigma(t))d\mu(s)d\mu(t),$$

then we get

$$\int_S\int_S f(\sigma(s)yt)d\mu(s)d\mu(t) = f(y)\int_S f(s)d\mu(s)$$

for all $y \in S$, which proves (2.4).

Equation (2.5): By using (2.4), replacing y by s in (1.7) and integrating the result obtained with respect to s we get

$$\int_S\int_S f(\sigma(s)xt)d\mu(s)d\mu(t) - \int_S\int_S f(xst)d\mu(s)d\mu(t)$$

$$= 2f(x)\int_S f(s)d\mu(s) = f(x)\int_S f(s)d\mu(s) - \int_S\int_S f(xst)d\mu(s)d\mu(t).$$

Equation (2.8): By replacing x by s in (2.6) and integrating the result obtained with respect to s we get $\int_S\int_S f(\sigma(s)t)d\mu(s)d\mu(t)=\int_S\int_S f(st)d\mu(s)d\mu(t)$. From (2.3) we have $f(\sigma(s)t)=-f(\sigma(t)s)$ for all $s,t\in S$, then

$$\int_S\int_S f(\sigma(s)t)d\mu(s)d\mu(t)=-\int_S\int_S f(\sigma(s)t)d\mu(s)d\mu(t)$$

which implies (2.8) and this completes the proof. □

Lemma 2.2. *Let $f\colon S\longrightarrow\mathbb{C}$ be a non-zero solution of equation* (1.7). *Then*
(1) The function defined by

$$g(x)\ :=\frac{\int_S f(xt)d\mu(t)}{\int_S f(t)d\mu(t)}\ \textit{for } x\in S$$

is a non-zero solution of the variant of d'Alembert's functional equation (1.6). *Furthermore,* $\int_S g(s)d\mu(s)=0$. *(2) The function g from (1) has the form $g=\frac{\chi+\chi\circ\sigma}{2}$, where $\chi:S\longrightarrow\mathbb{C}$, $\chi\neq 0$, is a multiplicative function.*

Proof . (1) From (2.4), (2.5), (1.7) and the definition of g we have

$$\left(\int_S f(t)d\mu(t)\right)^2[g(xy)+g(\sigma(y)x)]=\int_S f(t)d\mu(t)\int_S f(\sigma(y)xs)d\mu(s)+\int_S f(t)d\mu(t)\int_S f(xys)d\mu(s)$$

$$=\int_S\int_S\int_S f(\sigma(y)xs\sigma(t)k)d\mu(s)d\mu(t)d\mu(k)-\int_S\int_S\int_S f(xystk)d\mu(s)d\mu(t)d\mu(k)$$
$$=\int_S\int_S\int_S f(\sigma(yt)(xs)k)d\mu(s)d\mu(t)d\mu(k)-\int_S\int_S\int_S f((xs)(yt)k)d\mu(s)d\mu(t)d\mu(k)$$
$$=2\int_S f(xs)d\mu(s)\int_S f(yt)d\mu(t).$$

Dividing by $(\int_S f(t)d\mu(t))^2$ we get g satisfies the variant of d'Alembert's functional equation (1.6).

From (2.5) and the definition of g we get

$$\int_S\int_S g(ts)d\mu(t)d\mu(s)=\frac{\int_S\int_S\int_S f(s'ts)d\mu(t)d\mu(s)d\mu(s')}{\int_S f(s)d\mu(s)}$$

$$=\frac{-\int_S f(s')d\mu(s')\int_S f(s)d\mu(s)}{\int_S f(s)d\mu(s)}=-\int_S f(s)d\mu(s)\neq 0.$$

From (2.8) and the definition of g we get

$$\int_S g(s)d\mu(s)=\frac{\int_S f(st)d\mu(s)d\mu(t)}{\int_S f(s)d\mu(s)}=\frac{0}{\int_S f(s)d\mu(s)}=0.$$

Furthermore, $\int_S\int_S g(st)d\mu(t)d\mu(s)\neq 0$, so g is non-zero solution of equation (1.6).
(2) Let f be a non-zero solution of (1.7). Replacing x by xs in (1.7) and integrating the result obtained with respect to s we get

$$\int_S\int_S f(\sigma(y)xst)d\mu(s)d\mu(t)-\int_S\int_S f(xyst)d\mu(s)d\mu(t)=2f(y)\int_S f(xs)d\mu(s). \tag{2.26}$$

By using (2.4), (2.5) equation (2.26) can be written as follows

$$-f(\sigma(y)x)+f(xy)=2f(y)g(x),\ x,y\in S, \tag{2.27}$$

where g is the function defined above. If we replace y by ys in (1.7) and integrating the result obtained with respect to s we get

$$\int_S\int_S f(\sigma(y)x\sigma(s)t)d\mu(s)d\mu(t)-\int_S\int_S f(xyst)d\mu(s)d\mu(t)=2f(x)\int_S f(ys)d\mu(s). \tag{2.28}$$

By using (2.4), (2.5) we obtain

$$f(\sigma(y)x)+f(xy)=2f(x)g(y),\ x,y\in S. \tag{2.29}$$

By adding (2.29) and (2.27) we get that the pairf,g satisfies the sine addition law

$$f(xy)=f(x)g(y)+f(y)g(x) \text{ for all } x,y\in S.$$

Now, in view of [[6], Lemma 3.4.] g is abelian. Since g is a non-zero solution of d'Alembert's functional equation (1.6) then from [[23], Theorem 9.21] there exists a non-zero multiplicative function χ: $S\longrightarrow\mathbb{C}$ such that $g=\frac{\chi+\chi\circ\sigma}{2}$. This completes the proof. □

Now we are ready to prove the main result of the present section.

Theorem 2.3. *The non-zero solutions $f:S\longrightarrow\mathbb{C}$ of the functional equation* (1.7) *are the functions of the form*

$$f=\frac{\chi\circ\sigma-\chi}{2}\int_S\chi(t)d\mu(t), \tag{2.30}$$

where $\chi:S\longrightarrow\mathbb{C}$ is a multiplicative function such that $\int_S\chi(t)d\mu(t)\neq 0$ and $\int_S\chi(\sigma(t))d\mu(t)=-\int_S\chi(t)d\mu(t)$.
If S is a topological semigroup and that $\sigma:S\longrightarrow S$ is continuous then the non-zero solution f of equation (1.7) *is continuous, if and only if χ is continuous.*

Proof . Simple computations show that f defined by (2.30) is a solution of (1.7). Conversely, let $f:S\longrightarrow\mathbb{C}$ be a non-zero solution of the functional equation (1.7). Putting $y=s$ in (1.7) and integrating the result obtained with respect to s we get

$$\begin{aligned} f(x)&=\frac{\int_S\int_S f(\sigma(s)xt)d\mu(s)d\mu(t)-\int_S\int_S\int_S f(xst)d\mu(s)d\mu(t)}{2\int_S f(s)d\mu(s)}\\ &=\frac{1}{2}(\int_S g(\sigma(s)x)d\mu(s)-\int_S g(xs)d\mu(s)), \end{aligned} \tag{2.31}$$

where g is the function defined by $g=\frac{\chi+\chi\circ\sigma}{2}$, and where $\chi:S\longrightarrow\mathbb{C}$, $\chi\neq 0$ is a multiplicative function. Substituting this into (2.31) we find that f has the form

$$f=\frac{\int_S\chi(s)d\mu(s)-\int_S\chi(\sigma(s))d\mu(s)}{2}\frac{\chi\circ\sigma-\chi}{2}. \tag{2.32}$$

Furthermore, from (2.6) f satisfies $\int_S f(\sigma(x)s)d\mu(s)=\int_S f(xs)d\mu(s)$ for all $x\in S$. By applying the last expression of f in (2.6) we get after computations that

$$\left[\int_S\chi(\sigma(t))d\mu(t)+\int_S\chi(t)d\mu(t)\right][\chi-\chi\circ\sigma]=0.$$

Since $\chi \neq \chi \circ \sigma$, we obtain $\int_S \chi(\sigma(t))d\mu(t) + \int_S \chi(t)d\mu(t) = 0$ and then from (2.32) we have

$$f = \frac{\chi \circ \sigma - \chi}{2} \int_S \chi(t)d\mu(t).$$

For the topological statement we use [[23], Theorem 3.18(d)]. This completes the proof. □

If $\mu = \delta_{z_0}$ where z_0 is a fixed element of the center of S we get the following particular result obtained by Bouikhalene and Elqorachi on monoids [1].

Corollary 2.4. [1] Let M be a monoid, let σ: $M \longrightarrow M$ be an involutive automorphism. The non-zero solutions $f : S \longrightarrow \mathbb{C}$ of the functional equation

$$f(\sigma(y)xz_0) - f(xyz_0) = 2f(x)f(y),\ x, y \in M$$

are the functions of the form

$$f = \chi(z_0)\frac{\chi \circ \sigma - \chi}{2}, \tag{2.33}$$

where $\chi : M \longrightarrow \mathbb{C}$ is a multiplicative function such that $\chi(z_0) \neq 0$ and $\chi(\sigma(z_0)) = -\chi(z_0)$.

3. The complex-valued continuous solutions of equation (1.9) on a topological semigroup.

The following lemma will be used to construct some particular solutions of equation (1.9).

Lemma 3.1. *Let S be a locally compact semigroup, let υ be a complex measure with compact support. The continuous solutions of the functional equation*

$$\int_S f(xyt)d\upsilon(t) = f(x)f(y),\ x, y \in S \tag{3.1}$$

are the functions

$$f = \chi \int_S \chi(t)d\upsilon(t), \tag{3.2}$$

where χ: $S \longrightarrow \mathbb{C}$ is a continuous multiplicative function on S.

Proof . We use in the proof similar Stetkær's computations [[25], Proposition 16] used for the special case of $\upsilon = \delta_{s_0}$, where $s_0 \in S$. Let f be a continuous solution of (3.1). Replacing y by s in (3.1) and integrating the result obtained with respect to s we get

$$\int_S \int_S f(xst)d\upsilon(s)d\upsilon(t) = f(x) \int_S f(s)d\upsilon(s),\ x \in S. \tag{3.3}$$

Assume that $f \neq 0$, we will show that $\int_S f(s)d\upsilon(s) \neq 0$. By replacing x by xs, y by yk in (3.1) and integrating the result obtained with respect to s and k we get $\int_S \int_S \int_S f(xsykt)d\upsilon(s)d\upsilon(k)d\upsilon(t) = \int_S f(xs)d\upsilon(s) \int_S f(yk)d\upsilon(k)$. On the other hand, from (3.3) we have

$$\int_S \int_S \int_S f(xsykt)d\upsilon(s)d\upsilon(k)d\upsilon(t) = \int_S f(xsy)d\upsilon(s) \int_S f(t)d\upsilon(t).$$

So, if $\int_S f(s)d\upsilon(s) = 0$ then we get $\int_S f(xs)d\upsilon(s)\int_S f(yk)d\upsilon(k) = 0$ for all $x, y \in S$ and it follows that $\int_S f(xys)d\upsilon(s) = 0 = f(x)f(y)$ for all $x, y \in S$. Which contradicts the assumption that $f \neq 0$.

From (3.1) and (3.3) we have

$$\int_S f(s)d\upsilon(s)\int_S f(xty)d\upsilon(t) = \int_S\int_S f(xtysk)d\upsilon(t)d\upsilon(s)d\upsilon(k)$$

$$= f(x)\int_S\int_S f(tys)d\upsilon(t)d\upsilon(s) = f(x)f(y)\int_S f(t)d\upsilon(t).$$

This implies that

$$\int_S f(xty)d\upsilon(t) = f(x)f(y) \tag{3.4}$$

for all $x, y \in S$. Now, let

$$\chi(x) = \frac{\int_S f(xt)d\upsilon(t)}{\int_S f(t)d\upsilon(t)},\ x \in S.$$

In view of (3.1), (3.3) and (3.4) we have

$$\left(\int_S f(t)d\upsilon(t)\right)^2\chi(x)\chi(y) = \int_S f(xt)d\upsilon(t)\int_S f(ys)d\upsilon(s)$$

$$= \int_S\int_S [f(xt)f(ys)]d\upsilon(t)d\upsilon(s) = \int_S\int_S [\int_S f(xtysk)d\upsilon(k)]d\upsilon(t)d\upsilon(s)$$

$$= \int_S f(k)d\upsilon(k)\int_S f(xty)d\upsilon(t) = \int_S f(k)d\upsilon(k)\int_S f(xyt)d\upsilon(t)$$

$$= \left(\int_S f(k)d\upsilon(k)\right)^2\frac{\int_S f(xyt)d\upsilon(t)}{\int_S f(t)d\upsilon(t)} = \left(\int_S f(k)d\upsilon(k)\right)^2\chi(xy).$$

Which proves that χ is a multiplicative function. Finally, from (3.3) we have

$$f(x) = \frac{\int_S f(xst)d\upsilon(s)d\upsilon(t)}{\int_S f(t)d\upsilon(t)} = \int_S \chi(xs)d\upsilon(s) = \chi(x)\int_S \chi(s)d\upsilon(s).$$

This completes the proof. □

The continuous solutions of (1.9) are described in the following theorem.

Theorem 3.2. *Let S be a locally compact semigroup, let $\sigma : S \longrightarrow S$ be a continuous involutive automorphism of S, and let υ be a Borel complex measure with compact support and which is σ-invariant. The continuous solutions of the functional equation* (1.9) *are the functions*

$$f = \frac{\psi + \psi\circ\sigma}{2}, \tag{3.5}$$

where ψ: $S \longrightarrow \mathbb{C}$ is a continuous υ-spherical function. That is $\int_S \psi(xsy)d\upsilon(s) = \psi(x)\psi(y)$ for all $x, y \in S$.

Proof . Same computations that are needed in our discussion are due to Stetkær [24]. Let f be a solution of (1.9) and let $x, y, z \in S$. If we replace x by xsy and y by z in (1.9) and integrating the result obtained with respect to s we get

$$\int_S\int_S f(xsytz)d\upsilon(s)d\upsilon(t) + \int_S\int_S f(\sigma(z)txsy)d\upsilon(t)d\upsilon(s) = 2f(z)\int_S f(xsy)d\upsilon(s). \tag{3.6}$$

On the other hand if we replace x by $\sigma(z)sx$ in (1.9) and integrate the resul obtained with respect to s we obtain

$$\int_S\int_S f(\sigma(z)sxty)d\upsilon(s)d\upsilon(t) + \int_S\int_S f(\sigma(y)t\sigma(z)sx)d\upsilon(t)d\upsilon(s) = 2f(y)\int_S f(\sigma(z)sx)d\upsilon(s)$$
$$= 2f(y)\Big[2f(x)f(z) - \int_S f(xsz)d\upsilon(s)\Big].$$

Since, υ is σ-invariant, so we have

$$\int_S\int_S f(\sigma(y)t\sigma(z)sx)d\upsilon(t)d\upsilon(s) = \int_S\int_S f(\sigma(ytz)sx)d\upsilon(t)d\upsilon(s)$$
$$= 2f(x)\int_S f(ytz)d\upsilon(t) - \int_S\int_S f(xsytz)d\upsilon(s)d\upsilon(t).$$

Thus, we get

$$\int_S\int_S f(\sigma(z)sxty)d\upsilon(s)d\upsilon(t) + 2f(x)\int_S f(ysz)d\upsilon(s) - \int_S\int_S f(xsytz)d\upsilon(s)d\upsilon(t) \tag{3.7}$$
$$= 2f(y)\Big[2f(x)f(z) - \int_S f(xsz)d\upsilon(s)\Big].$$

Subtracting this from (3.6) we get

$$\int_S\int_S f(xsytz)d\upsilon(s)d\upsilon(t) \tag{3.8}$$
$$= f(x)\int_S f(ysz)d\upsilon(s) + f(z)\int_S f(xsy)d\upsilon(s) + f(y)\int_S f(xsz)d\upsilon(s) - 2f(y)f(x)f(z).$$

With the notation

$$f_x(y) = \int_S f(xsy)d\upsilon(s) - f(x)f(y) \tag{3.9}$$

equation (3.8) can be written as follows

$$\int_S f_a(xsy)d\upsilon(s) = f_a(x)f(y) + f_a(y)f(x),\ x, y \in S, \tag{3.10}$$

equation which was solved on groups in [9].

If $f_a = 0$ for all $a \in S$, then f is a υ-spherical function. Substituting f into (1.9) we obtain that $f = f \circ \sigma$. So, $f = \frac{\psi + \psi\circ\sigma}{2}$, with $\psi = f$ a υ-spherical function.

If there exists $a \in S$ such that $f_a \neq 0$ then from [9], there exist two υ-spherical functions ψ_1, ψ_2: $S \longrightarrow \mathbb{C}$ such that $f = \frac{\psi_1+\psi_2}{2}$. If $\psi_1 \neq \psi_2$ by substituting $f = \frac{\psi_1+\psi_2}{2}$ into (1.9) we get after a computation that

$$\psi_1(x)[\psi_2(y) - \psi_1(\sigma(y))] + \psi_2(x)[\psi_1(y) - \psi_2(\sigma(y))] = 0 \tag{3.11}$$

for all $x, y \in$. $\psi_1 \neq \psi_2$, then (ψ_1, ψ_2) are linearly independent [9], so from (3.11) we get $\psi_2(y) = \psi_1(\sigma(y))$ for all $y \in S$. This completes the proof. $\square$

By using Theorem 3.2 and Lemma 3.1 we get the following result.

Corollary 3.3. Let S be a locally compact semigroup, let $\sigma : S \longrightarrow S$ be a continuous involutive automorphism of S, and let υ be a complex Borel σ-invariant measure with compact support contained in the center of S. The continuous solutions of the functional equation

$$\int_S f(xyt)d\upsilon(t) + \int_S f(\sigma(y)xt)d\upsilon(t) = 2f(x)f(y),\ x, y \in S \tag{3.12}$$

are the functions

$$f = \frac{\chi + \chi \circ \sigma}{2} \int_S \chi(t)d\upsilon(t), \tag{3.13}$$

where χ: $S \longrightarrow \mathbb{C}$ is a continuous multiplicative function.

Remark 3.4. Let S be a locally compact semigroup, σ be a continuous involutive anti-automorphism of S, and υ be a complex measure with compact support and which is σ-invariant. If f is a continuous solution of the functional equation (1.9) then by adapting the computations used in [26] with the following mappings $R(y)h(x) = \int_S h(xty)d\upsilon(t)$; $L(y) = \int_S f(\sigma(y)tx)d\upsilon(t)$ we get that $\int_S f(xty)d\upsilon(t) = \int_S f(ytx)d\upsilon(t)$ for all $x, y \in S$. So, equation (1.9) can be written as follow

$$\int_S f(xty)d\upsilon(t) + \int_S f(xt\sigma(y))d\upsilon(t) = 2f(x)f(y)\ x, y \in S.$$

The last functional equation has been studied in [9].

4. The superstability of the functional equation (1.7)

In this section we obtain the superstability of the variant Van Vleck's functional equation (1.7) on semigroups.

Lemma 4.1. *Let S be a semigroup, let σ be an involutive morphism of S. Let μ be a complex measure that is a linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$, such that for all $i \in I$, z_i is contained in the center of S. Let $\delta > 0$ be fixed. If $f : S \longrightarrow \mathbb{C}$ is an unbounded function which satisfies the inequality*

$$\left| \int_S f(\sigma(y)xt)d\mu(t) - \int_S f(xyt)d\mu(t) - 2f(x)f(y) \right| \leq \delta \tag{4.1}$$

for all $x, y \in S$. Then, for all $x, y \in S$

$$f(\sigma(x)) = -f(x), \tag{4.2}$$

$$|f(\sigma(x)y) + f(\sigma(y)x)| \leq \frac{3\delta\|\mu\|}{|\int_S f(s)d\mu(s)|}, \tag{4.3}$$

$$\left| \int_S \int_S f(x\sigma(s)t)d\mu(s)d\mu(t) - f(x)\int_S f(s)d\mu(s) \right| \leq \frac{\delta\|\mu\|}{2}, \tag{4.4}$$

$$\left| \int_S \int_S f(xst)d\mu(s)d\mu(t) + f(x)\int_S f(t)d\mu(t) \right| \leq \frac{3\delta\|\mu\|}{2}, \tag{4.5}$$

$$\int_S f(t)d\mu(t) \neq 0, \tag{4.6}$$

$$\int_S f(x\sigma(s)d\mu(s) = \int_S f(\sigma(x)\sigma(s))d\mu(s), \tag{4.7}$$

$$\left| \int_S f(xs)d\mu(s) - \int_S f(\sigma(x)s)d\mu(s) \right| \leq \frac{6\delta\|\mu\|^2}{|\int_S f(s)d\mu(s)|}. \tag{4.8}$$

The function g defined by

$$g(x) = \frac{\int_S f(xt)d\mu(t)}{\int_S f(t)d\mu(t)} \; for\; x \in S \tag{4.9}$$

is unbounded on S and satisfies the following inequality

$$|g(xy) + g(\sigma(y)x) - 2g(x)g(y)| \leq \frac{3\delta\|\mu\|^2}{(|\int_S f(s)d\mu(s))^2|}. \tag{4.10}$$

for all $x, y \in S$. Furthermore, g satisfies (1.8)*, $\int_S \int_S g(st)d\mu(s)d\mu(t) \neq 0$ and $\int_S g(s)d\mu(s) = 0$.*

Proof . Equation (4.6): Let $f\colon S \longrightarrow \mathbb{C}$ be an unbounded function which satisfies (4.58). Equation (4.6): First, We prove that $\int_S f(s)d\mu(s) \neq 0$. Assume that $\int_S f(s)d\mu(s) = 0$. By replacing y by s and x by $\sigma(y)x$ in (4.58) and integrating the result obtained with respect to s we get

$$\left| \int_S \int_S f(\sigma(y)x\sigma(s)t)d\mu(s)d\mu(t) - \int_S \int_S f(\sigma(y)xst)d\mu(s)d\mu(t) \right| \leq \delta\|\mu\|. \tag{4.11}$$

Replacing y by ys in (4.58) and integrating the result obtained with respect to s we have

$$\left| \int_S \int_S f(\sigma(y)x\sigma(s)t)d\mu(s)d\mu(t) - \int_S \int_S f(xyst)d\mu(s)d\mu(t) - 2f(x) \int_S f(ys)d\mu(s) \right| \leq \delta\|\mu\|. \tag{4.12}$$

On the other hand by replacing x by xs in (4.58) and integrating the result obtained with respect to s we obtain

$$\left| \int_S \int_S f(\sigma(y)xst)d\mu(s)d\mu(t) - \int_S \int_S f(xyst)d\mu(s)d\mu(t) - 2f(y) \int_S f(xs)d\mu(s) \right| \leq \delta\|\mu\|. \tag{4.13}$$

By subtracting the result of equation (4.13) from the result of (4.12) and using the triangle inequality, we get after computation that

$$\begin{aligned} \Bigg| \int_S \int_S f(\sigma(y)x\sigma(s)t)d\mu(s)d\mu(t) - \int_S \int_S f(\sigma(y)xst)d\mu(s)d\mu(t) \\ - 2[f(x) \int_S f(ys)d\mu(s) - f(y) \int_S f(xs)d\mu(s)] \Bigg| \leq 2\delta\|\mu\|. \end{aligned} \tag{4.14}$$

From (4.11), (4.14) and the triangle inequality we get

$$\left| f(x) \int_S f(ys)d\mu(s) - f(y) \int_S f(xs)d\mu(s) \right| \leq \frac{3\delta}{2}\|\mu\|. \tag{4.15}$$

f is assumed to be unbounded function on S then $f \neq 0$. Let $y_0 \in S$ such that $f(y_0) \neq 0$. Equation (4.15) can be written as follows

$$\Big|\int_S f(xs)d\mu(s) - \alpha f(x)| \leq \frac{3\delta}{2|f(y_0)\Big|}\|\mu\|, \tag{4.16}$$

where $\alpha = \frac{\int_S f(y_0 s)d\mu(s)}{f(y_0)}$. Of course $\alpha \neq 0$ because if $\alpha = 0$ then by using (4.16) we deduce that the function $x \longmapsto \int_S f(xs)d\mu(s)$ is bounded and from (4.58) and the triangle inequality we get f bounded. Which contradict the assumption that f is an unbounded function on S.

Now, from (4.16) and the triangle inequality equation (4.58) can be written as follows

$$|f(\sigma(y)x) - f(xy) - \frac{2}{\alpha}f(x)f(y)| \leq \frac{\frac{3\delta\|\mu\|}{|f(y_0)|} + \delta}{|\alpha|} = M \tag{4.17}$$

for all $x, y \in S$. Since $\int_S f(s)d\mu(s) = 0$. So, if we replace y by s in (4.17) and integrating the result obtained with respect to s we get

$$\Big|\int_S f(\sigma(s)x)d\mu(s) - \int_S f(xs)d\mu(s)\Big| \leq M\|\mu\| \text{ for all } x \in S. \tag{4.18}$$

Replacing y by x and x by s in (4.17) and integrating the result obtained with respect to s we get

$$\Big|\int_S f(\sigma(x)s)d\mu(s) - \int_S f(sx)d\mu(s)\Big| \leq M\|\mu\| \text{ for all } x \in S. \tag{4.19}$$

Subtracting the result of (4.18) from the result of (4.19) and using the triangle inequality we get

$$\Big|\int_S f(\sigma(x)s)d\mu(s) - \int_S f(\sigma(s)x)d\mu(s)\Big| \leq 2M\|\mu\| \text{ for all } x \in S. \tag{4.20}$$

By interchanging x with y in (4.17) we get

$$|f(\sigma(x)y) - f(yx) - \frac{2}{\alpha}f(x)f(y)| \leq M. \tag{4.21}$$

If we replace y by $\sigma(y)$ in (4.17) we have

$$|f(yx) - f(x\sigma(y)) - \frac{2}{\alpha}f(x)f(\sigma(y))| \leq M. \tag{4.22}$$

By adding the results of (4.22) and (4.21) and using the triangle inequality we obtain

$$|f(\sigma(x)y) - f(x\sigma(y)) - \frac{2}{\alpha}f(x)[f(\sigma(y)) + f(\sigma(y))]| \leq 2M. \tag{4.23}$$

By replacing x by $\sigma(x)$ in (4.23) we get

$$|f(xy) - f(\sigma(x)\sigma(y)) - \frac{2}{\alpha}f(\sigma(x))[f(\sigma(y)) + f(\sigma(y))]| \leq 2M. \tag{4.24}$$

If we replace y by $\sigma(y)$ in (4.23) we get

$$|f(\sigma(x)\sigma(y)) - f(xy) - \frac{2}{\alpha}f(x)[f(\sigma(y)) + f(\sigma(y))]| \leq 2M. \tag{4.25}$$

Now, by adding the results of (4.24) and (4.25) and using the triangle inequality we have

$$|[f(x) + f(\sigma(x))][f(\sigma(y)) + f(\sigma(y))]| \leq 2M|\alpha|. \tag{4.26}$$

That is $x \longmapsto f(x) + f(\sigma(x))$ is a bounded function on S. So, the function $x \longmapsto \int_S f(\sigma(x)s)d\mu(s) + \int_S f(x\sigma(s))d\mu(s)$ is also a bounded function on S. Since, from (4.20) we have $x \longmapsto \int_S f(\sigma(x)s)d\mu(s) - \int_S f(x\sigma(s))d\mu(s)$ is a bounded function on S. Consequently, the function $x \longmapsto \int_S f(xs)d\mu(s)$ is a bounded function on S and from (4.58) and the triangle inequality we get that f is a bounded function on S. Which contradict the assumption that f is an unbounded function on S and this proves (4.6).

Equation (4.3): By replacing y by ys in (4.58) and integrating the result obtained with respect to s we get

$$\left|\int_S\int_S f(\sigma(y)x\sigma(s)t)d\mu(s)d\mu(t) - \int_S\int_S f(xyst)d\mu(s)d\mu(t) - 2f(x)\int_S f(ys)d\mu(s)\right| \leq \delta\|\mu\|. \tag{4.27}$$

If we replace x by xs in (4.58) and integrating the result obtained with respect to s we get

$$\left|\int_S\int_S f(\sigma(y)xst)d\mu(s)d\mu(t) - \int_S\int_S f(xyst)d\mu(s)d\mu(t) - 2f(y)\int_S f(xs)d\mu(s)\right| \leq \delta\|\mu\|. \tag{4.28}$$

By subtracting the result of (4.28) from the result of (4.27) and using the triangle inequality we obtain

$$\begin{aligned}\Bigg|\int_S\int_S f(\sigma(y)x\sigma(s)t)d\mu(s)d\mu(t) &- \int_S\int_S f(\sigma(y)xst)d\mu(s)d\mu(t) \\ &- 2[f(x)\int_S f(ys)d\mu(s) - f(y)\int_S f(xs)d\mu(s)]\Bigg| \leq 2\delta\|\mu\|.\end{aligned} \tag{4.29}$$

Replacing y by s and x by $\sigma(y)x$ in (4.58) and integrating the result obtained with respect to s we get

$$\left|\int_S\int_S f(\sigma(y)x\sigma(s)t)d\mu(s)d\mu(t) - \int_S\int_S f(\sigma(y)xst)d\mu(s)d\mu(t) - 2f(\sigma(y)x)\int_S f(s)d\mu(s)\right| \leq \delta\|\mu\|. \tag{4.30}$$

By subtracting the result of (4.29) from the result of (4.30) and using the triangle inequality we obtain

$$|f(\sigma(y)x)\int_S f(s)d\mu(s) - [f(x)\int_S f(ys)d\mu(s) - f(y)\int_S f(xs)d\mu(s)]| \leq \frac{3\delta\|\mu\|}{2}. \tag{4.31}$$

By interchanging x and y in (4.31) we have

$$|f(\sigma(x)y)\int_S f(s)d\mu(s) - [f(y)\int_S f(xs)d\mu(s) - f(x)\int_S f(ys)d\mu(s)]| \leq \frac{3\delta\|\mu\|}{2}. \tag{4.32}$$

By adding the result of (4.31) and the result of (4.32) and using the triangle inequality we get

$$|f(\sigma(x)y)+f(\sigma(y)x)| \leq \frac{3\delta\|\mu\|}{|\int_S f(s)d\mu(s)|}. \tag{4.33}$$

for all $x, y \in S$. This proves (4.3).
Equation (4.7): Replacing x by $x\sigma(s)$ in (4.58) and integrating the result obtained with respect to s we get

$$\left|\int_S\int_S f(\sigma(y)x\sigma(s)t)d\mu(s)d\mu(t) - \int_S\int_S f(xy\sigma(s)t)d\mu(s)d\mu(t) - 2f(y)\int_S f(x\sigma(s))d\mu(s)\right| \leq \delta\|\mu\|. \tag{4.34}$$

If we replace y by ys and x by xt in (4.33) and integrating the result obtained with respect to s and t we obtain

$$\left|\int_S\int_S f(\sigma(x)y\sigma(t)s)d\mu(s)d\mu(t) + \int_S\int_S f(\sigma(y)x\sigma(s)t)d\mu(s)d\mu(t)\right| \leq \frac{3\delta\|\mu\|^3}{|\int_S f(s)d\mu(s)|}. \tag{4.35}$$

By subtracting the result of (4.34) from the result of (4.35) and using the triangle inequality we get

$$\begin{aligned}\left|\int_S\int_S f(\sigma(x)y\sigma(t)s)d\mu(s)d\mu(t) + \int_S f(xy\sigma(s)t)d\mu(s)d\mu(t) + 2f(y)\int_S f(x\sigma(s)d\mu(s)\right| \\ \leq \frac{3\delta\|\mu\|^3}{|\int_S f(s)d\mu(s)|} + \delta\|\mu\|.\end{aligned} \tag{4.36}$$

Replacing x by $\sigma(x)$ in (4.36) we get

$$\left|\int_S\int_S f(xy\sigma(t)s)d\mu(s)d\mu(t) + \int_S f(\sigma(x)y\sigma(s)t)d\mu(s)d\mu(t) + 2f(y)\int_S f(\sigma(x)\sigma(s)d\mu(s)\right| \tag{4.37}$$

$$\leq \frac{3\delta\|\mu\|^3}{|\int_S f(s)d\mu(s)|} + \delta\|\mu\|.$$

Subtracting the result of (4.37) from the result of (4.36) and using the triangle inequality we get

$$\left|2f(y)[\int_S f(x\sigma(s)d\mu(s) - \int_S f(\sigma(x)\sigma(s)d\mu(s)]\right| \leq \frac{6\delta\|\mu\|^3}{|\int_S f(s)d\mu(s)|} + 2\delta\|\mu\|. \tag{4.38}$$

Since f is assumed to be unbounded then we deduce (4.7).
Equation (4.8): If we replace y by $\sigma(s)$ in (4.33) and integrating the result obtained with respect to s we obtain

$$\left|\int_S f(xs)d\mu(s) + \int_S f(\sigma(x)\sigma(s))d\mu(s)\right| \leq \frac{3\delta\|\mu\|^2}{|\int_S f(s)d\mu(s)|}. \tag{4.39}$$

In view of (4.7) the inequality (4.39) can be written as follows

$$\left|\int_S f(xs)d\mu(s) + \int_S f(x\sigma(s))d\mu(s)\right| \leq \frac{3\delta\|\mu\|^2}{|\int_S f(s)d\mu(s)|}. \tag{4.40}$$

Replacing y by s in (4.33) and integrating the result obtained with respect to s we get

$$\left|\int_S f(x\sigma(s))d\mu(s) + \int_S f(\sigma(x)s)d\mu(s)\right| \leq \frac{3\delta\|\mu\|^2}{|\int_S f(s)d\mu(s)|}. \tag{4.41}$$

By subtracting the result of (4.40) from the result of (4.41) we obtain (4.8).
Equation (4.2): Replacing x by $\sigma(x)$ in (4.58) we get

$$\left|\int_S f(\sigma(y)\sigma(x)t)d\mu(t) - \int_S f(\sigma(x)yt)d\mu(s)d\mu(t) - 2f(\sigma(x))f(y)\right| \leq \delta. \tag{4.42}$$

Now, we will discuss two cases.
Case 1. If σ is an involutive automorphism of S. Replacing x by yx in (4.8) we obtain

$$\left|\int_S f(yxs)d\mu(s) - \int_S f(\sigma(y)\sigma(x)s)d\mu(s)\right| \leq \frac{6\delta\|\mu\|^2}{|\int_S f(s)d\mu(s)|}. \tag{4.43}$$

Adding the result of (4.42) to the result of (4.43) and using the triangle inequality we get

$$\left|\int_S f(yxs)d\mu(s) - \int_S f(\sigma(x)ys)d\mu(s) - 2f(\sigma(x))f(y)\right| \leq \frac{6\delta\|\mu\|^2}{|\int_S f(s)d\mu(s)|} + \delta. \tag{4.44}$$

By interchanging x and y in (4.58) we get

$$\left|\int_S f(\sigma(x)ys)d\mu(s) - \int_S f(yxs)d\mu(s) - 2f(x)f(y)\right| \leq \delta. \tag{4.45}$$

By adding the result of (4.45) and the result of (4.44) and using the triangle inequality we obtain

$$|2f(y)[f(\sigma(x)) + f(x)]| \leq \frac{6\delta\|\mu\|^2}{|\int_S f(s)d\mu(s)|} + 2\delta. \tag{4.46}$$

Since f is unbounded then we get (4.2).
Case 2. If σ is an involutive anti-automorphism of S. By replacing x by yx in (4.8) we have

$$\left|\int_S f(yxs)d\mu(s) - \int_S f(\sigma(x)\sigma(y)s)d\mu(s)\right| \leq \frac{6\delta\|\mu\|^2}{|\int_S f(s)d\mu(s)|}. \tag{4.47}$$

If we replace y by x and y by $\sigma(y)$ in (4.58) we get

$$|\int_S f(\sigma(x)\sigma(y)t)d\mu(t) - \int_S f(\sigma(y)xt)d\mu(t) - 2f(\sigma(y))f(x)| \leq \delta. \tag{4.48}$$

By adding the results of (4.47) and (4.48) and using the triangle inequality we get

$$\left|\int_S f(yxt)d\mu(t) - \int_S f(\sigma(y)xt)d\mu(t) - 2f(\sigma(y))f(x)\right| \leq \delta + \frac{6\delta\|\mu\|^2}{|\int_S f(s)d\mu(s)}|. \tag{4.49}$$

From (4.58) we have

$$\left|\int_S f(\sigma(x)yt)d\mu(t) - \int_S f(yxt)d\mu(t) - 2f(y)f(x)\right| \leq \delta. \tag{4.50}$$

and from (4.8) we have

$$\left|\int_S f(\sigma(x)ys)d\mu(s) - \int_S f(\sigma(y)xs)d\mu(s)\right| \leq \frac{6\delta\|\mu\|^2}{|\int_S f(s)d\mu(s)|}. \tag{4.51}$$

By subtracting the results of (4.50) and (4.51) and using the triangle inequality we get

$$\left|\int_S f(\sigma(y)xt)d\mu(t)-\int_S f(yxt)d\mu(t)-2f(y)f(x)\right|\le\delta+\frac{6\delta\|\mu\|^2}{|\int_S f(s)d\mu(s)|}. \tag{4.52}$$

By adding the results of (4.52) and (4.49)

$$|2f(x)(f(y)+f(\sigma(y)))|\le\frac{6\delta\|\mu\|^2}{|\int_S f(s)d\mu(s)|}+\delta \tag{4.53}$$

for all $x,y\in S$. Since f is unbounded then we we get (4.2).

Equation (4.4): If we replace x by $\sigma(s)$ in (4.58) and using (4.2) and integrating the result obtained with respect to s we get

$$\left|-\int_S\int_S f(ys\sigma(t))d\mu(s)d\mu(t)-\int_S\int_S f(y\sigma(s)t)d\mu(s)d\mu(t)+2f(y)\int_S f(s)d\mu(s)\right|\le\delta\|\mu\|, \tag{4.54}$$

which proves (4.4).

Equation (4.5): By replacing y by s in (4.58) and integrating the result obtained with respect to s we get

$$\left|\int_S\int_S f(x\sigma(s)t)d\mu(s)d\mu(t)-\int_S\int_S f(xst)d\mu(s)d\mu(t)-2f(x)\int_S f(s)d\mu(s)\right|\le\delta\|\mu\|. \tag{4.55}$$

From (4.4) and the triangle inequality we obtain

$$\left|\int_S\int_S f(xst)d\mu(s)d\mu(t)+f(x)\int_S f(s)d\mu(s)\right|\le\delta\|\mu\|+\frac{\delta\|\mu\|}{2}.$$

for all $x\in S$. This proves (4.5).

Equation (4.10): Let g be the function defined by $g(x)=\frac{\int_S f(xt)d\mu(t)}{\int_S f(t)d\mu(t)}$ for $x\in S$. Then we have

$$\int_S f(s)d\mu(s)\int_S f(k)d\mu(k)[g(xy)+g(\sigma(y)x)-2g(x)g(y)] \tag{4.56}$$

$$\begin{aligned}
&=\int_S f(k)d\mu(k)\int_S f(xyt)d\mu(t)+\int_S f(s)d\mu(s)\int_S f(\sigma(y)xt)d\mu(t)-2\int_S f(xs)d\mu(s)\int_S f(ys)d\mu(s)\\
&=\int_S[f(xyt)\int_S f(k)d\mu(k)+\int_S\int_S f(xytks)d\mu(k)d\mu(s)]d\mu(t)\\
&+\int_S[f(\sigma(y)xt)\int_S f(s)d\mu(s)-\int_S\int_S f(\sigma(y)xt\sigma(k)s)d\mu(k)d\mu(s)]d\mu(t)\\
&+\int_S\int_S[\int_S f(\sigma(yk)xst)d\mu(t)-\int_S f(xsykst)d\mu(t)\\
&-2f(xs)f(yk)]d\mu(k)d\mu(s).
\end{aligned}$$

So, from (4.4), (4.5) and (4.58) we get

$$\int_S f(s)d\mu(s)\int_S f(k)d\mu(k)[g(xy)+g(\sigma(y)x)-2g(x)g(y)]$$

$$\leq \frac{3\delta\|\mu\|^2}{2} + \frac{\delta\|\mu\|^2}{2} + \delta\|\mu\|^2 = 3\delta\|\mu\|^2.$$

Which proves (4.10). Now, since f is unbounded then g is unbounded and satisfies (4.10). So, by using same computations used in [10] g satisfies the variant of d'Alembert's functional equation $g(xy) + g(\sigma(y)x) = 2g(x)g(y)$ for all $x, y \in S$.

Finally, from (4.4), (4.5) and the triangle inequality we have

$$\left| \int_S \int_S f(x\sigma(s)t)d\mu(s)d\mu(t) + \int_S f(xst)d\mu(s)d\mu(t) \right| \leq 2\delta\|\mu\|, \tag{4.57}$$

for all $x, y \in S$. By using the definition of g the inequality (4.57) can be written as follows

$$\left| \int_S f(k)d\mu(k)[\int_S g(x\sigma(k))d\mu(k) + \int_S g(xk)d\mu(k)] \right| \leq 2\delta\|\mu\|.$$

On the other hand g is a solution of d'Alembert's functional equation (1.8) then g is central and we get $|2g(x)\int_S g(k)d\mu(k)| \leq \frac{2\delta\|\mu\|}{|\int_S f(k)d\mu(k)|}$ for all $x \in S$. Since g is unbounded then we deduce that $\int_S g(k)d\mu(k) = 0$. That is $\int_S \int_S f(st)d\mu(s)d\mu(t) = 0$. □

Theorem 4.2. *Let S be a semigroup, let σ be an involutive morphism of S. Let μ be a complex measure that is a linear combination of Dirac measures $(\delta_{z_i})_{i\in I}$, such that for all $i \in I$, z_i is contained in the center of S. Let $\delta > 0$ be fixed. If $f : S \longrightarrow \mathbb{C}$ is a function which satisfies the inequality*

$$\left| \int_S f(\sigma(y)xt)d\mu(t) - \int_S f(xyt)d\mu(t) - 2f(x)f(y) \right| \leq \delta \tag{4.58}$$

for all $x, y \in S$. Then, either f is bounded on S and $|f(x)| \leq \frac{\|\mu\|+\sqrt{\|\mu\|^2+2\delta}}{2}$ for all $x \in S$ or f is a solution of the variant Van Vleck's functional equation (1.7)*.*

Proof . Assume that f is an unbounded solution of (4.58). Replacing y by ys in (4.58) and integrating the result obtained with respect to s we get

$$\left| \int_S \int_S f(\sigma(y)x\sigma(s)t)d\mu(s)d\mu(t) - \int_S \int_S f(xyst)d\mu(s)d\mu(t) - 2f(x)\int_S f(ys)d\mu(s) \right| \leq \delta\|\mu\| \tag{4.59}$$

for all $x, y \in S$. By using (4.58), (4.4) and (4.5) and the triangle inequality we get

$$\left| \int_S f(s)d\mu(s)f(xy) + \int_S f(s)d\mu(s)f(\sigma(y)x) - 2f(x)\int_S f(ys)d\mu(s) \right| \leq 3\delta\|\mu\| \tag{4.60}$$

for all $x, y \in S$ which can be written as follows

$$|f(xy) + f(\sigma(y)x) - 2f(x)g(y)| \leq \frac{3\delta\|\mu\|}{|\int_S f(s)d\mu(s)|} \tag{4.61}$$

for all $x, y \in S$, where g is the function defined above. Replacing x by xs in (4.58) and integrating the result obtained with respect to s we get

$$\left| \int_S \int_S f(\sigma(y)xst)d\mu(s)d\mu(t) - \int_S \int_S f(xyst)d\mu(s)d\mu(t) - 2f(y)\int_S f(xs)d\mu(s) \right| \leq \delta\|\mu\| \tag{4.62}$$

for all $x, y \in S$. By using (4.58), (4.5) and the triangle inequality we get

$$|f(xy) - f(\sigma(y)x) - 2f(y)g(x)| \leq \frac{4\delta\|\mu\|}{|\int_S f(s)d\mu(s)|} \tag{4.63}$$

for all $x, y \in S$. By adding the result of (4.63) and (4.61) we get

$$|f(xy) - f(x)g(y) - f(x)g(y)| \leq \frac{2\delta\|\mu\|}{|\int_S f(s)d\mu(s)|} \tag{4.64}$$

for all $x, y \in S$. Now, we will show that if $\alpha f + \beta g$ is a bounded function on S then $\alpha = \beta = 0$. Assume that there exits M such that

$$|\alpha f(x) + \beta \int_S f(xt)d\mu(t)| \leq M \tag{4.65}$$

for all $x \in S$. Then $|\alpha f(\sigma(x)) + \beta \int_S f(\sigma(x)t)d\mu(t)| \leq M$. Since $f(\sigma(x)) = -f(x)$. So, by using (4.8) and the triangle inequality we get

$$| - \alpha f(x) + \beta \int_S f(xt)d\mu(t)| \leq M + |\beta| \frac{6\delta\|\mu\|^2}{|\int_S f(t)d\mu(t)|}. \tag{4.66}$$

By adding the result of (4.65) and (4.66) we get $2\beta \int_S f(xt)d\mu(t)$ is a bounded function. Since g is unbounded then $\beta = 0$ and consequently $\alpha = 0$. Now, from [[31], Lemma 2.1] we conclude that f, g are solutions of the sine addition law

$$f(xy) = f(x)g(y) + f(y)g(x) \; x, y \in S. \tag{4.67}$$

Since $f(\sigma(x)) = -f(x)$ and $g(\sigma(x) = g(x)$ for all $x \in S$ then the pair f, g satisfies the variant Wilson's functional equation

$$f(xy) + f(\sigma(y)x) = 2f(x)g(y) \; x, y \in S. \tag{4.68}$$

Taking $y = s$ in (4.68) and integrating the result obtained with respect to s we get

$$\int_S f(xs)d\mu(s) + \int_S f(\sigma(s)x)d\mu(s) = 0, \tag{4.69}$$

because $\int_S g(s)d\mu(s) = 0$. By replacing y by $s\sigma(k)$ in in (4.67) and integrating the result obtained with respect to s and k we obtain

$$\int_S \int_S f(xs\sigma(k))d\mu(s)d\mu(k) + \int_S \int_S f(xs\sigma(k))d\mu(s)d\mu(k) = 2f(x) \int_S \int_S g(s\sigma(k))d\mu(s)d\mu(k).$$

That is

$$\int_S f(xs\sigma(k))d\mu(s)d\mu(k) = f(x) \int_S \int_S g(s\sigma(k))d\mu(s)d\mu(k). \tag{4.70}$$

Now from (4.4) and (4.70) we get

$$|f(x)(\int_S \int_S g(s\sigma(k))d\mu(s)d\mu(k) - \int_S f(t)d\mu(t))| \leq \frac{\delta\|\mu\|}{2}$$

for all $x \in S$. Since f is assumed to be unbounded then we get

$$\int_S \int_S g(s\sigma(k))d\mu(s)d\mu(k) = \int_S f(t)d\mu(t). \tag{4.71}$$

g satisfies (1.8) and $\int_S g(s)d\mu(s) = 0$ implies that $\int_S g(yk)d\mu(s) = -\int_S g(y\sigma(k))d\mu(s)$. So, by using the definition of g, equations (4.70) and (4.71) we have

$$\begin{aligned}\int_S g(yk)d\mu(k) = -\int_S g(y\sigma(k))d\mu(k) &= \frac{-\int_S\int_S f(y\sigma(k)t)d\mu(k)d\mu(t)}{\int_S f(s)d\mu(s)} \\ &= \frac{-f(y)\int_S\int_S g(\sigma(k)t)d\mu(k)d\mu(t)}{\int_S f(s)d\mu(s)} \\ &= \frac{-f(y)\int_S f(t)d\mu(t)}{\int_S f(s)d\mu(s)} = -f(y).\end{aligned} \tag{4.72}$$

Finally, From (4.67), (4.63), (4.70) and (4.72) for all $x, y \in S$ we have

$$\begin{aligned}\int_S f(\sigma(y)xt)d\mu(t) - \int_S f(xyt)d\mu(t) &= -\int_S f(\sigma(y)x\sigma(t))d\mu(t) - \int_S f(xyt)d\mu(t) \\ &= -[\int_S f(\sigma(yt)x)d\mu(t) + \int_S f(xyt)d\mu(t)] \\ &= -2f(x)\int_S g(yt)d\mu(t) = 2f(x)f(y).\end{aligned}$$

That is f is a solution of Van Vleck's functional equation (1.7). This completes the proof. □

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