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STABILITY OF THE QUADRATIC FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN $\mathcal{L}$-FUZZY NORMED SPACES

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Dedicated to the 70th Anniversary of S.M.Ulam's Problem for Approximate Homomorphisms

Abstract. In this paper, we prove the generalized Hyers-Ulam stability of the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

in non-Archimedean $\mathcal{L}$-fuzzy normed spaces.

1. Introduction

The theory of fuzzy sets was introduced by Zadeh in 1965 [31]. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive development is made in the field of fuzzy topology [1, 8, 9, 12, 13, 14, 18, 22, 29]. One of problems in $\mathcal{L}$-fuzzy topology is to obtain an appropriate concept of $\mathcal{L}$-fuzzy metric spaces and $\mathcal{L}$-fuzzy normed spaces. J. Park [23], Saadati and J. Park [27], respectively, introduced and studied a notion of intuitionistic fuzzy metric spaces and fuzzy normed spaces, respectively.

On the other hand, the study of stability problems for functional equations is related to a question of Ulam [30] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [19]. Subsequently, the result of Hyers was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [24] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias has provided a lot of influence in the development of what we now call a *generalized Hyers-Ulam stability* of functional equations. We refer the interested readers for more information on such problems to the papers [4, 6, 11, 20, 21, 25].

In this paper, we prove the stability of the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

in non-Archimedean $\mathcal{L}$-fuzzy normed spaces.

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2. Preliminaries

In this section, we recall some definitions and results for our main results in this paper.

A *triangular norm* (shortly, *t-norm*) is a binary operation $T : [0,1] \times [0,1] \to [0,1]$ which is commutative, associative, monotone and has 1 as the unit element. Basic examples are the Lkasiewicz t-norm T_L, $T_L(a,b) = \max\{a+b-1, 0\}$ for all $a, b \in [0,1]$ and the t-norms T_P, T_M, T_D, where $T_P(a,b) := ab$, $T_M(a,b) := \min\{a,b\}$,

$$T_D(a,b) := \begin{cases} \min\{x,y\}, & \text{if } \max\{\text{x,y}\}=1; \\ 0, & \text{otherwise.} \end{cases}$$

A t-norm T is said to be *of Hadžić-type* (denoted by $T \in \mathcal{H}$) ([15]) if the family $(x_T^{(n)})_{n \in N}$ is equicontinuous at $x = 1$, where $x_T^{(n)}$ is defined by

$$x_T^{(1)} = x, \quad x_T^{(n)} = T(x_T^{(n-1)}, x), \quad \forall n \geq 2,\ x \in [0,1].$$

Other important triangular norms are as follows (see [16]):

(1) The *Sugeno-Weber family* $\{T_\lambda^{SW}\}_{\lambda \in [-1,\infty]}$ is defined by $T_{-1}^{SW} = T_D$, $T_\infty^{SW} = T_P$ and

$$T_\lambda^{SW} = \max\left\{0, \frac{x+y-1+\lambda xy}{1+\lambda}\right\}, \quad \forall \lambda \in (-1,\infty).$$

(2) The *Domby family* $\{T_\lambda^D\}_{\lambda \in [0,\infty]}$ is defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and

$$T_\lambda^D(x,y) = \frac{1}{1 + ((\frac{1-x}{x})^\lambda + (\frac{1-y}{y})^\lambda)^{1/\lambda}}, \quad \forall \lambda \in (0,\infty).$$

(3) The *Aczel-Alsina family* $\{T_\lambda^{AA}\}_{\lambda \in [0,\infty]}$ is defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and

$$T_\lambda^{AA}(x,y) = e^{-(|\log x|^\lambda + |\log y|^\lambda)^{1/\lambda}}, \quad \forall \lambda \in (0,\infty).$$

A t-norm T can be extended (by associativity) in a unique way to an n-ary operation taking, for all $(x_1, \cdots, x_n) \in [0,1]^n$, the value $T(x_1, \cdots, x_n)$ defined by

$$\mathrm{T}_{i=1}^0 x_i = 1, \quad \mathrm{T}_{i=1}^n x_i = T(\mathrm{T}_{i=1}^{n-1} x_i, x_n) = T(x_1, \cdots, x_n).$$

A t-norm T can also be extended to a countable operation taking, for any sequence $\{x_n\}_{n \in N}$ in $[0,1]$, the value

$$\mathrm{T}_{i=1}^\infty x_i = \lim_{n \to \infty} \mathrm{T}_{i=1}^n x_i.$$

Proposition 2.1. ([16]) (1) *For $T \geq T_L$ the following implication holds:*

$$\lim_{n \to \infty} \mathrm{T}_{i=1}^\infty x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty.$$

(2) *If T is of Hadžić-type, then*

$$\lim_{n \to \infty} \mathrm{T}_{i=1}^\infty x_{n+i} = 1$$

for every sequence $(x_n)_{n \in N}$ in $[0,1]$ such that $\lim_{n \to \infty} x_n = 1$.

(3) *If* $T \in \{T_\lambda^{AA}\}_{\lambda\in(0,\infty)} \cup \{T_\lambda^{D}\}_{\lambda\in(0,\infty)}$, *then*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n)^{\alpha} < \infty.$$

(4) *If* $T \in \{T_\lambda^{SW}\}_{\lambda\in[-1,\infty)}$, *then*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty.$$

3. $\mathcal{L}$-FUZZY NORMED SPACES

In the sequel, we shall adopt the usual terminology, notation and some definitions introduced by authors [5, 28].

Definition 3.1. [10] Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice and let U be a non-empty set called the universe. An $\mathcal{L}$-*fuzzy set* in U is defined as a mapping $\mathcal{A} : U \to L$. For each u in U, $\mathcal{A}(u)$ represents the *degree* (in L) to which u is an element of $\mathcal{A}$.

Consider the set L^* and operation $\leq_{L^*}$ defined by

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1,\ x_2 \geq y_2$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice (see [7]).

Definition 3.2. [3] An *intuitionistic fuzzy set* $\mathcal{A}_{\zeta,\eta}$ in the universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(u, \zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) : u \in U\}$, where $\zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ for all $u \in U$ are called the *membership degree* and the *non-membership degree*, respectively, of u in $\mathcal{A}_{\zeta,\eta}$ and, furthermore, satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

In the last section, t-norms on $([0,1], \leq)$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(1, x) = x$ for all $x \in [0,1]$. This definition can be straightforwardly extended to any lattice $\mathcal{L} = (L, \leq_L)$.

Definition 3.3. A *triangular norm* (t-norm) on $\mathcal{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:
(i) $(\forall x \in L)(\mathcal{T}(x, 1_{\mathcal{L}}) = x)$ (: boundary condition);
(ii) $(\forall (x,y) \in L^2)(\mathcal{T}(x,y) = \mathcal{T}(y,x))$ (: commutativity);
(iii) $(\forall (x,y,z) \in L^3)(\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y), z))$ (: associativity);
(iv) $(\forall (x,x',y,y') \in L^4)(x \leq_L x' \text{ and } y \leq_L y' \Rightarrow \mathcal{T}(x,y) \leq_L \mathcal{T}(x',y'))$ (: monotonicity).

A t-norm $\mathcal{T}$ on $\mathcal{L}$ is said to be *continuous* if, for any $x, y \in \mathcal{L}$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y, respectively,

$$\lim_{n\to\infty} \mathcal{T}(x_n, y_n) = \mathcal{T}(x,y).$$

For example, $\mathcal{T}(x,y) = \min(x,y)$ and $\mathcal{T}(x,y) = xy$ are two continuous t-norms on $[0,1]$. The t-norm $\wedge$ defined by

$$\wedge(x,y) = \begin{cases} x \text{ if } x \leq_L y \\ y \text{ if } y \leq_L x \end{cases}$$

is a continuous t-norm.

A t-norm $\mathcal{T}$ can also be defined recursively as an $(n+1)$-ary operation ($n \in \mathbf{N}$) by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x_1, \cdots, x_{n+1}) = \mathcal{T}(\mathcal{T}^{n-1}(x_1, \cdots, x_n), x_{n+1})$$

for all $n \geq 2$ and $x_i \in L$.

Definition 3.4. (1) A *negator* on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$.

(2) If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L$, then $\mathcal{N}$ is called an *involutive negator*.

(3) The negator N_s on $([0,1], \leq)$ defined as $N_s(x) = 1-x$ for all $x \in [0,1]$ is called the *standard negator* on $([0,1], \leq)$.

In this paper, the involutive negator $\mathcal{N}$ is fixed.

Definition 3.5. (1) The triple $(X, \mathcal{M}, \mathcal{T})$ is said to be an *$\mathcal{L}$-fuzzy metric space* if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}$-fuzzy set on $X^2 \times]0, +\infty[$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s \in]0, +\infty[$,
(a) $\mathcal{M}(x, y, t) >_L 0_{\mathcal{L}}$;
(b) $\mathcal{M}(x, y, t) = 1_{\mathcal{L}}$ for all $t > 0$ if and only if $x = y$;
(c) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$;
(d) $\mathcal{T}(\mathcal{M}(x, y, t), \mathcal{M}(y, z, s)) \leq_L \mathcal{M}(x, z, t+s)$;
(e) $\mathcal{M}(x, y, \cdot) :]0, +\infty[\to L$ is continuous.
In this case, $\mathcal{M}$ is called an *$\mathcal{L}$-fuzzy metric*.

(2) If $\mathcal{M} = \mathcal{M}_{M,N}$ is an intuitionistic fuzzy set (see Definition 3.2), then the 3-tuple $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is said to be an *intuitionistic fuzzy metric space*.

Example 3.6. Let (X, d) be a metric space. Denote $\mathcal{T}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and let $\mathcal{M}_{M,N}$ be the intuitionistic fuzzy set on $X \times]0, \infty[$ defined as follows:

$$\mathcal{M}_{M,N}(x, t) = \left(\frac{ht^n}{ht^n + md(x,y)}, \frac{md(x,y)}{ht^n + md(x,y)} \right)$$

for all $t, h, m, n \in \mathbb{R}^+$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Example 3.7. Let $X = \mathbb{N}$. Define $\mathcal{T}(a, b) = (\max(0, a_1 + b_1 - 1), a_2 + b_2 - a_2 b_2)$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and let $\mathcal{M}_{M,N}$ be the intuitionistic fuzzy set on $X \times]0, \infty[$ defined as follows:

$$\mathcal{M}_{M,N}(x, t) = \begin{cases} (\frac{x}{y}, \frac{y-x}{y}) & \text{if } x \leq y \\ (\frac{y}{x}, \frac{x-y}{x}) & \text{if } y \leq x \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Definition 3.8. (1) The triple $(V, \mathcal{P}, \mathcal{T})$ is said to be an *$\mathcal{L}$-fuzzy normed space* if V is a vector space, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $V \times]0, +\infty[$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in]0, +\infty[$,
(a) $\mathcal{P}(x, t) >_L 0_{\mathcal{L}}$;
(b) $\mathcal{P}(x, t) = 1_{\mathcal{L}}$ if and only if $x = 0$;

(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}\big(x, \frac{t}{|\alpha|}\big)$ for each $\alpha \neq 0$;
(d) $\mathcal{T}(\mathcal{P}(x,t), \mathcal{P}(y,s)) \leq_L \mathcal{P}(x+y, t+s)$;
(e) $\mathcal{P}(x, \cdot) :]0, \infty[\to L$ is continuous;
(f) $\lim_{t\to 0} \mathcal{P}(x,t) = 0_{\mathcal{L}}$ and $\lim_{t\to\infty} \mathcal{P}(x,t) = 1_{\mathcal{L}}$.
In this case, $\mathcal{P}$ is called an *$\mathcal{L}$-fuzzy norm.*

(2) If $\mathcal{P} = \mathcal{P}_{\mu,\nu}$ is an intuitionistic fuzzy set (see Definition 3.2), then the 3-tuple $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an *intuitionistic fuzzy normed space.*

Example 3.9. Let $(V, \|\cdot\|)$ be a normed space. Denote $\mathcal{T}(a,b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and let $\mathcal{P}_{\mu,\nu}$ be the intuitionistic fuzzy set on $V\times]0, +\infty[$ defined as follows:

$$\mathcal{P}_{\mu,\nu}(x,t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|} \right)$$

for all $t \in \mathbb{R}^+$. Then $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is an intuitionistic fuzzy normed space.

Definition 3.10. (1) A sequence $(x_n)_{n\in\mathbb{N}}$ in an $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, \mathcal{T})$ is called a *Cauchy sequence* if, for each $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$,

$$\mathcal{P}(x_n - x_m, t) >_L \mathcal{N}(\varepsilon),$$

where $\mathcal{N}$ is a negator on $\mathcal{L}$.

(2) A sequence $(x_n)_{n\in\mathbb{N}}$ is said to be *convergent* to $x \in V$ in the $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, \mathcal{T})$, which is denoted by $x_n \xrightarrow{\mathcal{P}} x$ if $\mathcal{P}(x_n - x, t) \to 1_{\mathcal{L}}$, whenever $n \to +\infty$ for all $t > 0$.

(3) An $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, \mathcal{T})$ is said to be *complete* if and only if every Cauchy sequence in V is convergent.

Note that, if $\mathcal{P}$ is an $\mathcal{L}$-fuzzy norm on V, then the following are satisfied:
(1) $\mathcal{P}(x,t)$ is nondecreasing with respect to t for all $x \in V$.
(2) $\mathcal{P}(x - y, t) = \mathcal{P}(y - x, t)$ for all $x, y \in V$ and $t \in]0, +\infty[$.

Let $(V, \mathcal{P}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy normed space. If we define

$$\mathcal{M}(x, y, t) = \mathcal{P}(x - y, t)$$

for all $x, y \in V$ and $t \in]0, +\infty[$, then $\mathcal{M}$ is an $\mathcal{L}$-fuzzy metric on V, which is called the *$\mathcal{L}$-fuzzy metric* induced by the $\mathcal{L}$-fuzzy norm $\mathcal{P}$.

Definition 3.11. Let $(V, \mathcal{P}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy normed space and $\mathcal{N}$ a negator on $\mathcal{L}$.

(1) For all $t \in]0, +\infty[$, we define the *open ball* $B(x, r, t)$ with center $x \in V$ and radius $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ as follows:

$$B(x, r, t) = \{y \in V : \mathcal{P}(x - y, t) >_L \mathcal{N}(r)\}$$

and define the *unit ball* of V by

$$B(0, r, 1) = \{x : \mathcal{P}(x, 1) >_L \mathcal{N}(r)\}.$$

(2) A subset $A \subseteq V$ is said to be *open* if, for each $x \in A$, there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $B(x, r, t) \subseteq A$.

(3) Let $\tau_{\mathcal{P}}$ denote the family of all open subsets of V. Then $\tau_{\mathcal{P}}$ is called the *topology induced by the $\mathcal{L}$-fuzzy norm $\mathcal{P}$.*

Note that, in the case of an intuitionistic fuzzy normed space, this topology is the same as the topology induced by intuitionistic fuzzy metric which is Hausdorff (see Remark 3.3 and Theorem 3.5 of [23]).

Definition 3.12. Let $(V, \mathcal{P}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy normed space and let $\mathcal{N}$ be a negator on $\mathcal{L}$. A subset A of V is said to be *$\mathcal{L}F$-bounded* if there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{P}(x,t) >_L \mathcal{N}(r)$ for all $x \in A$.

Note that, in an $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, \mathcal{T})$, every compact set is closed and $\mathcal{L}F$-bounded (see Remark 3.10 of [23]).

4. Non-Archimedean $\mathcal{L}$-fuzzy normed spaces

In 1897, Hensel [17] introduced a field with a valuation in which does not have the Archimedean property.

Definition 4.1. Let $\mathcal{K}$ be a field. A *non-Archimedean absolute value* on $\mathcal{K}$ is a function $|\cdot| : \mathcal{K} \to [0, +\infty[$ such that, for any $a, b \in \mathcal{K}$,
(i) $|a| \geq 0$ and equality holds if and only if $a = 0$,
(ii) $|ab| = |a||b|$,
(iii) $|a + b| \leq \max\{|a|, |b|\}$ (the strict triangle inequality).

Note that $|n| \leq 1$ for each integer n. We always assume, in addition, that $|\cdot|$ is non-trivial, i.e., there exists an $a_0 \in \mathcal{K}$ such that $|a_0| \neq 0, 1$.

Definition 4.2. A *non-Archimedean $\mathcal{L}$-fuzzy normed space* is a triple $(V, \mathcal{P}, \mathcal{T})$, where V is a vector space, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $V \times]0, +\infty[$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in]0, +\infty[$,
(a) $0_{\mathcal{L}} <_L \mathcal{P}(x,t)$;
(b) $\mathcal{P}(x,t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
(d) $\mathcal{T}(\mathcal{P}(x,t), \mathcal{P}(y,s)) \leq_L \mathcal{P}(x+y, \max\{t,s\})$;
(e) $\mathcal{P}(x, \cdot) :]0, \infty[\to L$ is continuous;
(f) $\lim_{t\to 0} \mathcal{P}(x,t) = 0_{\mathcal{L}}$ and $\lim_{t\to\infty} \mathcal{P}(x,t) = 1_{\mathcal{L}}$.

Example 4.3. Let $(X, \|.\|)$ be a non-Archimedean normed linear space. Then the triple $(X, \mathcal{P}, \min)$, where

$$\mathcal{P}(x,t) = \begin{cases} 0, & \text{if } t \leq \|x\|; \\ 1, & \text{if } t > \|x\|, \end{cases}$$

is a non-Archimedean $\mathcal{L}$-fuzzy normed space in which $L = [0,1]$.

Example 4.4. Let $(X, \|\cdot\|)$ be a non-Archimedean normed linear space. Denote $\mathcal{T}_M(a,b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and let $\mathcal{P}_{\mu,\nu}$ be the intuitionistic fuzzy set on $X \times]0, +\infty[$ defined as follows:

$$\mathcal{P}_{\mu,\nu}(x,t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|} \right)$$

for all $t \in \mathbb{R}^+$. Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T}_M)$ is a non-Archimedean intuitionistic fuzzy normed space.

5. Generalized $\mathcal{L}$-fuzzy Hyers-Ulam stability

Let $\mathcal{K}$ be a non-Archimedean field, X a vector space over $\mathcal{K}$ and $(Y, \mathcal{P}, \mathcal{T})$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathcal{K}$.

In this section, we investigate the generalized Hyers-Ulam stability of the quadratic functional equation (1.1).

We define an $\mathcal{L}$-fuzzy approximately quadratic mapping. Let Ψ be an $\mathcal{L}$-fuzzy set on $X \times X \times [0, \infty)$ such that $\Psi(x, y, \cdot)$ is nondecreasing,

$$\Psi(cx, cx, t) \geq_L \Psi\left(x, x, \frac{t}{|c|}\right), \quad \forall x \in X,\ c \neq 0$$

and

$$\lim_{t\to\infty} \Psi(x, y, t) = 1_{\mathcal{L}}, \quad \forall x, y \in X,\ t > 0.$$

Definition 5.1. A mapping $f : X \to Y$ is said to be Ψ-*approximately quadratic* if

$$\begin{aligned} &\mathcal{P}(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\quad \geq_L \Psi(x, y, t), \quad \forall x, y \in X,\ t > 0. \end{aligned} \tag{5.1}$$

The following is one of our main results in this section.

Theorem 5.2. *Let $\mathcal{K}$ be a non-Archimedean field, X a vector space over $\mathcal{K}$ and $(Y, \mathcal{P}, \mathcal{T})$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathcal{K}$. Let $f : X \to Y$ be a Ψ-approximately quadratic mapping. If there exist an $\alpha \in \mathbb{R}$ $(\alpha > 0)$ and an integer $k, k \geq 2$ with $|2^k| < \alpha$ and $|2| \neq 0$ such that*

$$\Psi(2^{-k}x, 2^{-k}y, t) \geq_L \Psi(x, y, \alpha t), \quad \forall x \in X,\ t > 0, \tag{5.2}$$

and

$$\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty} \mathcal{M}\left(x, \frac{\alpha^j t}{|2|^{kj}}\right) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0,$$

then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mathcal{P}(f(x) - Q(x), t) \geq \mathcal{T}_{i=1}^{\infty} \mathcal{M}\left(x, \frac{\alpha^{i+1} t}{|2|^{ki}}\right), \quad \forall x \in X,\ t > 0, \tag{5.3}$$

where

$$\mathcal{M}(x, t) := \mathcal{T}(\Psi(x, x, t), \Psi(2x, 2x, t), \cdots, \Psi(2^{k-1}x, 2^{k-1}x, t)), \quad \forall x \in X,\ t > 0.$$

Proof. First, we show, by induction on j, that, for all $x \in X$, $t > 0$ and $j \geq 1$,

$$\begin{aligned} \mathcal{P}(f(2^j x) - 4^j f(x), t) &\geq_L \mathcal{M}_j(x, t) \\ &:= \mathcal{T}(\Psi(x, x, t), \cdots, \Psi(2^{j-1}x, 2^{j-1}x, t)). \end{aligned} \tag{5.4}$$

Putting $y = x$ in (5.1), we obtain

$$\mathcal{P}(f(2x) - 4f(x), t) \geq_L \Psi(x, x, t), \quad \forall x \in X,\ t > 0.$$

This proves (5.4) for $j = 1$. Let (5.4) hold for some $j > 1$. Replacing y and x by $2^j x$ in (5.1), we get

$$\mathcal{P}(f(2^{j+1}x) - 4f(2^j x), t) \geq_L \Psi(2^j x, 2^j x, t), \quad \forall x \in X,\ t > 0.$$

Since $|2| \leq 1$, it follows that

$$\begin{aligned}
&\mathcal{P}(f(2^{j+1}x) - 4^{j+1}f(x), t) \\
&\quad \geq_L \mathcal{T}\Big(\mathcal{P}(f(2^{j+1}x) - 4f(2^j x), t), \mathcal{P}(4f(2^j x) - 4^{j+1}f(x), t)\Big) \\
&\quad = \mathcal{T}\left(\mathcal{P}(f(2^{j+1}x) - 4f(2^j x), t), \mathcal{P}\left(f(2^j x) - 4^j f(x), \frac{t}{|4|}\right)\right) \\
&\quad \geq_L \mathcal{T}\Big(\mathcal{P}(f(2^{j+1}x) - 4f(2^j x), t), \mathcal{P}\left(f(2^j x) - 4^j f(x), t\right)\Big) \\
&\quad \geq_L \mathcal{T}(\Psi(2^j x, 2^j x, t), \mathcal{M}_j(x, t)) \\
&\quad = \mathcal{M}_{j+1}(x, t), \quad \forall x \in X,\ t > 0.
\end{aligned}$$

Thus (5.4) holds for all $j \geq 1$. In particular, we have

$$\mathcal{P}(f(2^k x) - 4^k f(x), t) \geq_L \mathcal{M}(x, t), \quad \forall x \in X,\ t > 0. \tag{5.5}$$

Replacing x by $2^{-(kn+k)}x$ in (5.5) and using the inequality (5.2), we obtain

$$\begin{aligned}
\mathcal{P}\left(f\left(\frac{x}{2^{kn}}\right) - 4^k f\left(\frac{x}{2^{kn+k}}\right), t\right) \ &\geq_L\ \mathcal{M}\left(\frac{x}{2^{kn+k}}, t\right) \\
&\geq_L\ \mathcal{M}(x, \alpha^{n+1}t) \quad \forall x \in X,\ t > 0,\ n \geq 0
\end{aligned}$$

and so

$$\begin{aligned}
&\mathcal{P}\left((2^{2k})^n f\left(\frac{x}{(2^k)^n}\right) - (2^{2k})^{n+1} f\left(\frac{x}{(2^k)^{n+1}}\right), t\right) \\
&\quad \geq_L \mathcal{M}\left(x, \frac{\alpha^{n+1}}{|(2^{2k})^n|}t\right) \\
&\quad \geq_L \mathcal{M}\left(x, \frac{\alpha^{n+1}}{|(2^k)^n|}t\right), \quad \forall x \in X,\ t > 0,\ n \geq 0.
\end{aligned}$$

Hence it follow that

$$\begin{aligned}
&\mathcal{P}\left((2^{2k})^n f\left(\frac{x}{(2^k)^n}\right) - (2^{2k})^{n+p} f\left(\frac{x}{(2^k)^{n+p}}\right), t\right) \\
&\quad \geq_L \mathcal{T}_{j=n}^{n+p}\left(\mathcal{P}((2^{2k})^j f\left(\frac{x}{(2^k)^j}\right) - (2^{2k})^{j+p} f\left(\frac{x}{(2^k)^{j+p}}\right), t)\right) \\
&\quad \geq_L \mathcal{T}_{j=n}^{n+p}\mathcal{M}\left(x, \frac{\alpha^{j+1}}{|(2^k)^j|}t\right), \quad \forall x \in X,\ t > 0,\ n \geq 0.
\end{aligned}$$

Since $\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty}\mathcal{M}\left(x, \frac{\alpha^{j+1}}{|(2^k)^j|}t\right) = 1_{\mathcal{L}}$ for all $x \in X$ and $t > 0$, $\left\{(2^{2k})^n f\left(\frac{x}{(2^k)^n}\right)\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the non-Archimedean $\mathcal{L}$-fuzzy Banach space $(Y, \mathcal{P}, \mathcal{T})$. Hence we can define a mapping $Q : X \to Y$ such that

$$\lim_{n\to\infty} \mathcal{P}\left((2^{2k})^n f\left(\frac{x}{(2^k)^n}\right) - Q(x), t\right) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0. \tag{5.6}$$

Next, for all $n \geq 1$, $x \in X$ and $t > 0$, we have

$$\begin{aligned}
&\mathcal{P}\left(f(x) - (2^{2k})^n f\left(\frac{x}{(2^k)^n}\right), t\right) \\
&= \mathcal{P}\left(\sum_{i=0}^{n-1}(2^{2k})^i f\left(\frac{x}{(2^k)^i}\right) - (2^{2k})^{i+1} f\left(\frac{x}{(2^k)^{i+1}}\right), t\right) \\
&\geq_L \mathcal{T}_{i=0}^{n-1}\left(\mathcal{P}((2^{2k})^i f\left(\frac{x}{(2^k)^i}\right) - (2^{2k})^{i+1} f\left(\frac{x}{(2^k)^{i+1}}\right), t)\right) \\
&\geq_L \mathcal{T}_{i=0}^{n-1}\mathcal{M}\left(x, \frac{\alpha^{i+1}t}{|2^k|^i}\right)
\end{aligned}$$

and so

$$\begin{aligned}
&\mathcal{P}(f(x) - Q(x), t) \\
&\geq_L \mathcal{T}\left(\mathcal{P}(f(x) - (2^{2k})^n f\left(\frac{x}{(2^k)^n}\right), t), \mathcal{P}((2^{2k})^n f\left(\frac{x}{(2^k)^n}\right) - Q(x), t)\right) \\
&\geq_L \mathcal{P}\left(\mathcal{T}_{i=0}^{n-1}\mathcal{M}\left(x, \frac{\alpha^{i+1}t}{|2^k|^i}\right), \mathcal{P}((2^{2k})^n f\left(\frac{x}{(2^k)^n}\right) - Q(x), t)\right).
\end{aligned} \tag{5.7}$$

Taking the limit as $n \to \infty$ in (5.7), we obtain

$$\mathcal{P}(f(x) - Q(x), t) \geq_L \mathcal{T}_{i=1}^{\infty}\mathcal{M}\left(x, \frac{\alpha^{i+1}t}{|3^k|^i}\right),$$

which proves (5.3). As $\mathcal{T}$ is continuous, from a well known result in $\mathcal{L}$-fuzzy (probabilistic) normed space (see [26], Chapter 12), it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \mathcal{P}((4^k)^n f(2^{-kn}(x+y)) + (4^k)^n f(2^{-kn}(x-y)) - 2(4^k)^n f(2^{-kn}(x)) \\
&\quad -2(4^k)^n f(2^{-kn}(y)), t) \\
&\quad = \mathcal{P}(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y), t)
\end{aligned}$$

for almost all $t > 0$.

On the other hand, replacing x, y by $2^{-kn}x, 2^{-kn}y$ in (5.1) and (5.2), we get

$$\begin{aligned}
&\mathcal{P}((4^k)^n f(2^{-kn}(x+y)) + (4^k)^n f(2^{-kn}(x-y)) - 2(4^k)^n f(2^{-kn}(x)) \\
&\quad -2(4^k)^n f(2^{-kn}(y)), t) \\
&\quad \geq_L \Psi\left(2^{-kn}x, 2^{-kn}y, \frac{t}{|2^{2k}|^n}\right) \\
&\quad \geq_L \Psi\left(x, y, \frac{\alpha^n t}{|2^k|^n}\right), \quad \forall x, y \in X,\ t > 0.
\end{aligned}$$

Since $\lim_{n\to\infty} \Psi\left(x, y, \frac{\alpha^n t}{|2^k|^n}\right) = 1_{\mathcal{L}}$, we infer that Q is a quadratic mapping.

For the uniqueness of Q, let $Q' : X \to Y$ be another quadratic mapping such that

$$\mathcal{P}(Q'(x) - f(x), t) \geq_L \mathcal{M}(x, t), \quad \forall x \in X,\ t > 0.$$

Then we have, for all $x, y \in X$ and $t > 0$,

$$\begin{aligned}&\mathcal{P}(Q(x) - Q'(x), t)\\ &\geq_L \mathcal{T}\left(\mathcal{P}(Q(x) - (2^{2k})^n f\left(\frac{x}{(2^k)^n}\right), t), \mathcal{P}((2^{2k})^n f\left(\frac{x}{(2^k)^n}\right) - Q'(x), t), t)\right).\end{aligned}$$

Therefore, from (5.6), we conclude that $Q = Q'$. This completes the proof. □

Corollary 5.3. *Let $\mathcal{K}$ be a non-Archimedean field, X a vector space over $\mathcal{K}$ and $(Y, \mathcal{P}, \mathcal{T})$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathcal{K}$ under a t-norm $\mathcal{T} \in \mathcal{H}$. Let $f : X \to Y$ be a Ψ-approximately quadratic mapping. If there exist an $\alpha \in \mathbb{R}$ $(\alpha > 0)$, $|2| \neq 0$ and an integer $k, k \geq 2$ with $|2^k| < \alpha$ such that*

$$\Psi(2^{-k}x, 2^{-k}y, t) \geq_L \Psi(x, y, \alpha t), \quad \forall x \in X,\ t > 0,$$

then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mathcal{P}(f(x) - Q(x), t) \geq_L \mathcal{T}_{i=1}^{\infty}\mathcal{M}\left(x, \frac{\alpha^{i+1}t}{|2|^{ki}}\right), \quad \forall x \in X,\ t > 0,$$

where

$$\mathcal{M}(x, t) := \mathcal{T}(\Psi(x, x, t), \Psi(2x, 2x, t), \cdots, \Psi(2^{k-1}x, 2^{k-1}x, t)), \quad \forall x \in X,\ t > 0.$$

Proof. Since

$$\lim_{n\to\infty} \mathcal{M}\left(x, \frac{\alpha^j t}{|2|^{kj}}\right) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0,$$

and $\mathcal{T}$ is of Hadžić type, it follows from Proposition 2.1 that

$$\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty}\mathcal{M}\left(x, \frac{\alpha^j t}{|2|^{kj}}\right) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0.$$

Now, if we apply Theorem 5.2, we get the conclusion. □

Now, we give an example to validate the main result as follows:

Example 5.4. Let $(X, \|.\|)$ be a non-Archimedean Banach space, $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T}_M)$ non-Archimedean $\mathcal{L}$-fuzzy normed space (intuitionistic fuzzy normed space) in which

$$\mathcal{P}_{\mu,\nu}(x, t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\right), \quad \forall x \in X,\ t > 0,$$

and $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T}_M)$ a complete non-Archimedean $\mathcal{L}$-fuzzy normed space (intuitionistic fuzzy normed space) (see Example 4.4). Define

$$\Psi(x, y, t) = \left(\frac{t}{1+t}, \frac{1}{1+t}\right).$$

It is easy to see that (5.2) holds for $\alpha = 1$. Also, since

$$\mathcal{M}(x, t) = \left(\frac{t}{1+t}, \frac{1}{1+t}\right),$$

we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{T}_{M,j=n}^{\infty}\mathcal{M}\left(x, \frac{\alpha^j t}{|2|^{kj}}\right) &= \lim_{n\to\infty}\left(\lim_{m\to\infty} \mathcal{T}_{M,j=n}^{m}\mathcal{M}\left(x, \frac{t}{|2|^{kj}}\right)\right)\\ &= \lim_{n\to\infty}\lim_{m\to\infty}\left(\frac{t}{t+|2^k|^n}, \frac{|2^k|^n}{t+|2^k|^n}\right)\\ &= (1,0) = 1_{L^*}, \quad \forall x\in X,\ t>0.\end{aligned}$$

Let $f : X \to Y$ be a Ψ-approximately quadratic mapping. Thus all the conditions of Theorem 5.2 hold and so there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - Q(x), t) \geq_{L^*} \left(\frac{t}{t+|2^k|}, \frac{|2^k|}{t+|2^k|}\right).$$

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References

1. M. Amini and R. Saadati, Topics in fuzzy metric space, J. Fuzzy Math., 4 (2003), 765–768.
2. T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, 2 (1950), 64–66.
3. K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1986), 87–96.
4. C. Baak and M.S. Moslehian, On the stability of J^*-homomorphisms, Nonlinear Anal.–TMA 63 (2005), 42–48.
5. Y.J. Cho and R. Saadati, $\mathcal{L}$-fuzzy normed spaces (preprint).
6. S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, 2002.
7. G. Deschrijver and E.E. Kerre, On the relationship between some extensions of fuzzy set theory, Fuzzy Sets and Systems, 133 (2003), 227–235.
8. A. George and P. Veeramani, On some result in fuzzy metric space, Fuzzy Sets and System, 64 (1994), 395–399.
9. A. George and P. Veeramani, On some result of analysis for fuzzy metric spaces, Fuzzy Sets and Systems, 90 (1997), 365–368.
10. J. Goguen, $\mathcal{L}$-fuzzy sets, J. Math. Anal. Appl., 18 (1967), 145–174.
11. M.E. Gordji, M.B. Ghaemi and H. Hajani, Generalized Hyers-Ulam-Rassias theorem in Menger probabilistic normed spaces, Discrete Dynamics in Nature and Society 2010 (2010), Article ID 162371, 11 pages.
12. V. Gregori and S. Romaguera, Some properties of fuzzy metric spaces, Fuzzy Sets and Systems, 115 (2000), 485–489.
13. V. Gregori and S. Romaguera, On completion of fuzzy metric spaces, Fuzzy Sets and Systems, 130 (2002), 399–404.
14. V. Gregori and S. Romaguera, Characterizing completable fuzzy metric spaces, Fuzzy Sets and Systems, 144 (2004), 411–420.

15. O. Hadžić and E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer Academic, Dordrecht, 2001.
16. O. Hadžić, E. Pap and M. Budincević, Countable extension of triangular norms and their applications to the fixed point theory in probabilistic metric spaces, Kybernetica 38 (2002), 363–381.
17. K. Hensel, Uber eine neue Begrundung der Theorie der algebraischen Zahlen. Jahres, Deutsch. Math. Verein, 6 (1897), 83–88.
18. C. Hu, $\mathcal{C}$-structure of FTS. V: Fuzzy metric spaces, J. Fuzzy Math. 3 (1995), 711–721.
19. D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA 27 (1941), 222–224.
20. D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
21. S. Jung, Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, 2001.
22. R. Lowen, *Fuzzy Set Theory*, Kluwer Academic Publishers, Dordrecht, 1996.
23. J. Park, Intuitionistic fuzzy metric spaces, Chaos, Solitons and Fractals, 22 (2004), 1039–1046.
24. Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., 72 (1978), 297–300.
25. Th.M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2003.
26. B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, Elsevier, North Holand, New York, 1983.
27. R. Saadati and J. Park, On the intuitionistic fuzzy topological spaces, Chaos, Solitons and Fractals, 27 (2006), 331–344.
28. R. Saadati, A. Razani and H. Adibi, A common fixed point theorem in $\mathcal{L}$-fuzzy metric spaces Chaos, Solitons and Fractals, 33 (2007), 358–363.
29. B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960), 314–334.
30. S.M. Ulam, *Problems in Modern Mathematics*, Chapter VI, Science Editions, Wiley, New York, 1964.
31. L.A. Zadeh, Fuzzy sets, Inform. Control, 8 (1965), 338–353.

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