Int. J. Nonlinear Anal. Appl. 14 (2023) 10, 19–34
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.26602.3363

Semi-Fredholmness on a weighted geometric realization of 2-simplexes and 3-simplexes

Azeddine Baalal[a], Khalid Hatim[b,*]

[a] *Laboratoire de Mathématiques Fondamentales et Appliquées, Département de Mathématiques et Informatique, Faculté des Sciences Aïn Chock, Université Hassan II de Casablanca, Morocco*

[b] *Laboratoire de Mathématiques Fondamentales et Appliquées, Faculté des Sciences Aïn Chock, Université Hassan II de Casablanca, Morocco*

(Communicated by Mugur Alexandru Acu)

Abstract

In this present article, we introduce the notion of oriented 2-simplexes and the notion of oriented 3-simplexes and we use them to create a new framework that we call a weighted geometric realization of 2-simplexes and 3-simplexes. Next, we define the weighted geometric realization Gauss-Bonnet operator L. After that, we present and study the non-parabolicity at the infinity of L. Finally, we develop general conditions to ensure semi-Fredholmness of L based on its non-parabolicity at infinity.

Keywords: Weighted geometric realization of 2-simplexes and 3-simplexes, weighted geometric realization Gauss-Bonnet operator, non-parabolicity at infinity, semi-Fredholmness
2020 MSC: 47A53, 05C63, 39A12

1 Introduction

The concept of non-parabolicity at infinity was investigated in [1, 3]. The weighted geometric realization associated with the set of 2-simplexes and 3-simplexes is a notion of algebraic topology, see for instance [2, 6, 7, 8]. In this present work, we construct a weighted geometric realization of the set of 2-simplexes and 3-simplexes and its Gauss-Bonnet operator. Next, we study the non-parabolicity at infinity of the weighted geometric realization Gauss-Bonnet operator and we use it to ensure semi-Fredholmness of the weighted geometric realization Gauss-Bonnet operator. This current paper is structured as follows : In the second section, we introduce the notion of oriented 2-simplexes and the notion of oriented 3-simplexes, we refer to [2, 6, 7, 8] for surveys on the matter. After that, we create a new framework that's we call the weighted geometric realization of 2-simplexes and 3-simplexes. In the third section, we create the weighted simplexes cochains spaces and the weighted simplexes operators. Next, we construct the weighted geometric realization Gauss-Bonnet operator. In the fourth section, we introduce and study the non-parabolicity at infinity of the weighted geometric realization Gauss-Bonnet operator. In the last section, we develop general conditions to ensure semi-Fredholmness of the weighted geometric realization Gauss-Bonnet operator based on its non-parabolicity at infinity.

*Corresponding author
Email addresses: abaalal@gmail.com (Azeddine Baalal), hatimfriends@gmail.com (Khalid Hatim)

Received: March 2022 *Accepted:* May 2023

2 Weighted geometric realization of 2-simplexes and 3-simplexes

The aim of this section is to create a new framework that's we call the weighted geometric realization of 2 and 3-simplexes, see [2, 4, 5, 6, 7, 8, 11, 13].

Let V be the set of vertices at most countable, E be the set of oriented edges and (V,E) a graph. We take E symmetric, i.e., if $(x,y)\in E$, then $(y,x)\in E$. We take E irreflexive, i.e., if $x\in E$, then $(x,x)\notin E$. Let (E^{+},E^{-}) be a partition of E. If $(x,y)\in E$, then $(x,y)\in E^{+}$ or $(x,y)\in E^{-}$. We have $(x,y)\in E^{+}$ if and only if $(y,x)\in E^{-}$. Orient the graph (V,E) means define the partition (E^{+},E^{-}) of E. For $e=(x,y)$, we set $e^{-}=x$ and $e^{+}=y$. The path between x and y is a finite set of oriented edges $e_1,e_2,e_3,...,e_k$ such that $k\in\mathbb{N}^{*}, e_1^{-}=x, e_k^{+}=y$ and $\forall i\in\{1,2,3,...,k-1\}, e_i^{+}=e_{i+1}^{-}$. The simple path is a path where each edge appears only once time. The cycle is a path where the origin and the end are identical. The connected graph is a graph such that for all $x,y\in V$, there exists a path between x and y. The locally finite graph is a graph such that each vertice belongs to a finite number of edges. In our paper, we work with a graph that's oriented, connected, irreflexive, symmetric and locally finite. An oriented 2-simplex is a surface surrounded by a simple cycle of length equals 3 and it is an element of V^3. Let $S_2=\left\{(x,y,z)\in V^3 \mid (x,y,z) \text{ is an oriented 2-simplex}\right\}$ be the set of oriented 2-simplexes. An oriented 3-simplex is a volume surrounded by four oriented 2-simplexes and it is an element of V^4. Let $S_3=\{(x,y,z,t)\in V^4 \mid (x,y,z,t)$ is an oriented 3-simplex$\}$ be the set of oriented 3-simplexes. The odd permutation means we change the positions of two vertices an odd number of times. The even permutation means we change the positions of two vertices an even number of times. Let $(\alpha,\beta)\in S_2^2$ or $(\alpha,\beta)\in S_3^2$. We have $\alpha=\beta$ if we use the even permutation to pass from α to β. We have $\alpha=-\beta$ if we use the odd permutation to pass from α to β. The geometric realization of 2-simplexes and 3-simplexes, denoted by R, is the pair (S_2,S_3). We define a weight on S_3 by $w_3: S_3\to\mathbb{R}_{+}^{*}$ such that $\forall (a,b,c,d)\in S_3,\ w_3(-(a,b,c,d))=w_3(a,b,c,d)$. We define a weight on S_2 by $w_2: S_2\to\mathbb{R}_{+}^{*}$ such that $\forall (a,b,c)\in S_2,\ w_2(-(a,b,c))=w_2(a,b,c)$. The weighted geometric realization of 2-simplexes and 3-simplexes, denoted by R_w, is the quadruplet (S_2,S_3,w_2,w_3) that's equals to (R,w_2,w_3). The sub-weighted geometric realization R_w^M of $R_w=(S_2,S_3,w_2,w_3)$ is the quadruplet $R_w^M=\left(M,S_3^M,w_2,w_3\right)$ where $M\subset S_2$ and

$$S_3^M=\{(a,b,c,d)\in S_3 \mid (b,c,d),(d,c,a),(a,b,d),(c,b,a)\in M\}.$$

The 3-simplexes boundary, denoted by ∂S_3^M, is defined as

$$\begin{aligned}\partial S_3^M=\{(a,b,c,d)\in S_3 \mid &((b,c,d)\in M \text{ and } (d,c,a),(a,b,d),(c,b,a)\notin M) \text{ or}\\ &((d,c,a)\in M \text{ and } (b,c,d),(a,b,d),(c,b,a)\notin M) \text{ or } ((a,b,d)\in M \text{ and } (b,c,d),(d,c,a),\\ &(c,b,a)\notin M) \text{ or } ((c,b,a)\in M \text{ and } (b,c,d),(d,c,a),(a,b,d)\notin M)\}.\end{aligned}$$

The 2-simplex path from (x,y,z) to (x_0,y_0,z_0) is a finite sequence of 2-simplexes (a_1,b_1,c_1),..., (a_m,b_m,c_m) such that

$$(x,y,z)=(a_1,b_1,c_1),(x_0,y_0,z_0)=(a_m,b_m,c_m),$$

and

$$\forall j\in\{1,2,...,m-1\},(a_{j+1},b_{j+1},c_{j+1})\in S_2(a_j,b_j,c_j),$$

where

$$\begin{aligned}S_2(a_j,b_j,c_j)=\{(x,y,z)\in S_2 \mid &(x\in\{a_j,b_j,c_j\} \text{ and } y,z\notin\{a_j,b_j,c_j\}) \text{ or}\\ &(y\in\{a_j,b_j,c_j\} \text{ and } x,z\notin\{a_j,b_j,c_j\}) \text{ or } (z\in\{a_j,b_j,c_j\} \text{ and}\\ &x,y\notin\{a_j,b_j,c_j\})\}.\end{aligned}$$

The 2-simplex connected weighted geometric realization is a weighted geometric realization such that for all (x,y,z), $(x_0,y_0,z_0)\in S_2$, we have a 2-simplex path from (x,y,z) to (x_0,y_0,z_0). In the sequel of this work, we suppose that R_w is a 2-simplex connected weighted geometric realization.

3 Weighted geometric realization Gauss-Bonnet operator

The aim of this section is to construct the weighted geometric realization Gauss-Bonnet operator, see [9, 10, 12, 14].

We start by introducing the following simplexes functional spaces associated to the weighted geometric realization R_w :

- The 2-simplex cochains set, denoted by $C(S_2)$, is defined as

$$C(S_2) = \{f : S_2 \to \mathbb{R} \mid f(-(a,b,c)) = -f(a,b,c)\}.$$

We set

$$C_0(S_2) = \{f \in C(S_2) \mid f \text{ has a finite support}\}.$$

Let $(f,g) \in C_0(S_2) \times C_0(S_2)$. We define an inner product on $C_0(S_2)$ as

$$\langle f,g\rangle_{S_2} = \frac{1}{6}\sum_{(a,b,c)\in S_2} w_2(a,b,c)\, f(a,b,c)\, g(a,b,c).$$

Then

$$\|f\|_{S_2} = \sqrt{\langle f,f\rangle_{S_2}}.$$

The Hilbert space associated to S_2, denoted by $H(S_2)$, is given by

$$H(S_2) = \left\{f \in C_0(S_2) \mid \|f\|_{S_2} < \infty\right\}.$$

- The 3-simplex cochains set, denoted by $C(S_3)$, is defined as

$$C(S_3) = \{f : S_3 \to \mathbb{R} \mid f(-(a,b,c,d)) = -f(a,b,c,d)\}.$$

We set

$$C_0(S_3) = \{f \in C(S_3) \mid f \text{ has a finite support}\}.$$

Let $(f,g) \in C_0(S_3) \times C_0(S_3)$. We define a scalar product on $C_0(S_3)$ as

$$\langle f,g\rangle_{S_3} = \frac{1}{24}\sum_{(a,b,c,d)\in S_3} w_3(a,b,c,d)\, f(a,b,c,d) g(a,b,c,d).$$

Then

$$\|f\|_{S_3} = \sqrt{\langle f,f\rangle_{S_3}}.$$

The Hilbert space associated to S_3, denoted by $H(S_3)$, is given by

$$H(S_3) = \left\{f \in C_0(S_3) \mid \|f\|_{S_3} < \infty\right\}.$$

- We define the direct sum of $H(S_2)$ and $H(S_3)$ as

$$H(R_w) = H(S_2) \oplus H(S_3) = \{(f,g) \mid f \in H(S_2) \text{ and } g \in H(S_3)\},$$

where it's associated norm is given by

$$\|(f,g)\|^2_{R_w} = \|f\|^2_{S_2} + \|g\|^2_{S_3}.$$

In the next, we define the weighted simplexes operators.

- Let S be the operator defined as

$$S : C_0(S_2) \to C_0(S_3),$$

such that

$$S(f)(a,b,c,d) = f(b,c,d) + f(d,c,a) + f(a,b,d) + f(c,b,a),$$

for all $f \in C_0(S_2)$ and $(a,b,c,d) \in S_3$.

▶ Let δ be the adjoint operator of S defined as

$$\delta : C_0(S_3) \to C_0(S_2),$$

such that

$$\langle S(f), g\rangle_{S_3} = \langle f, \delta(g)\rangle_{S_2},$$

for all $f \in C_0(S_2)$ and $g \in C_0(S_3)$.

Theorem 3.1. Let R_w be a weighted geometric realization. Then, we have

$$\delta(f)(b,c,d) = \frac{1}{w_2(b,c,d)} \sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d) f(a,b,c,d),$$

for all $f \in C_0(S_3)$ and $(b,c,d) \in S_2$.

Proof . Let $(g,f) \in C_0(S_2) \times C_0(S_3)$. We have

$$\begin{aligned}
\langle S(g), f\rangle_{S_3} &= \frac{1}{24} \sum_{(a,b,c,d)\in S_3} w_3(a,b,c,d) S(g)(a,b,c,d) f(a,b,c,d) \\
&= \frac{1}{24} \sum_{(a,b,c,d)\in S_3} \left[g(b,c,d) + g(d,c,a) + g(a,b,d) + g(c,b,a)\right] w_3(a,b,c,d) f(a,b,c,d) \\
&= \frac{1}{24} \sum_{(a,b,c,d)\in S_3} w_3(a,b,c,d) f(a,b,c,d) g(b,c,d) + \frac{1}{24} \sum_{(a,b,c,d)\in S_3} w_3(a,b,c,d) f(a,b,c,d) g(d,c,a) \\
&+ \frac{1}{24} \sum_{(a,b,c,d)\in S_3} w_3(a,b,c,d) f(a,b,c,d) g(a,b,d) + \frac{1}{24} \sum_{(a,b,c,d)\in S_3} w_3(a,b,c,d) f(a,b,c,d) g(c,b,a).
\end{aligned}$$

Since we have four similar parts,

$$\begin{aligned}
\langle S(g), f\rangle_{S_3} &= \frac{1}{6} \sum_{(a,b,c,d)\in S_3} w_3(a,b,c,d) g(b,c,d) f(a,b,c,d) \\
&= \frac{1}{6} \sum_{(b,c,d)\in S_2} \sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d) g(b,c,d) f(a,b,c,d) \\
&= \frac{1}{6} \sum_{(b,c,d)\in S_2} \left[g(b,c,d) \sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d) f(a,b,c,d) \right].
\end{aligned}$$

Moreover, we have

$$\langle g, \delta(f)\rangle_{S_2} = \frac{1}{6} \sum_{(b,c,d)\in S_2} w_2(b,c,d) g(b,c,d) \delta(f)(b,c,d).$$

Since

$$\langle g, \delta(f)\rangle_{S_2} = \langle S(g), f\rangle_{S_3},$$

we get

$$\frac{1}{6} \sum_{(b,c,d)\in S_2} \left[g(b,c,d) \sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d) f(a,b,c,d) \right] = \frac{1}{6} \sum_{(b,c,d)\in S_2} w_2(b,c,d)\ g(b,c,d) \delta(f)(b,c,d).$$

Therefore, we obtain

$$\delta(f)(b,c,d) = \frac{1}{w_2(b,c,d)} \sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d) f(a,b,c,d).$$

□

Now, we present the weighted geometric realization Gauss-Bonnet operator.

Definition 3.2. The weighted geometric realization Gauss-Bonnet operator, denoted by L, is defined as

$$L = S + \delta : C_0(S_2) \oplus C_0(S_3) \to C_0(S_2) \oplus C_0(S_3),$$

such that

$$L(f,g) = S(f) + \delta(g),$$

for all $(f,g) \in C_0(S_2) \oplus C_0(S_3)$.

4 Non-parabolicity at infinity of the Gauss-Bonnet operator

This section is devoted to introduce and study the concept of non-parabolicity at infinity of the weighted geometric realization Gauss-Bonnet operator. The concept of non-parabolicity at infinity was investigated in [1, 3].

In the next, we give a useful theorem in the study of non-parabolicity at infinity.

Theorem 4.1. Let R_w be a weighted geometric realization and $(x, y, z), (x_0, y_0, z_0) \in S_2$. Then, $\exists \beta^x_{x_0} \in \mathbb{R}^+$ such that

$$|g(x,y,z)| \leq \beta^x_{x_0} \left(|g(x_0,y_0,z_0)| + \|Sg\|_{S_3}\right),$$

for all $g \in C_0(S_2)$.

Proof . Let (x, y, z), $(x_0, y_0, z_0) \in S_2$ and $g \in C_0(S_2)$. Since R_w is a 2-simplex connected weighted geometric realization, then we have a 2-simplex path from (x, y, z) to (x_0, y_0, z_0), i.e., there exists a finite sequence of 2-simplexes (a_1, b_1, c_1),..., (a_m, b_m, c_m) such that $(x, y, z) = (a_1, b_1, c_1)$ and $(x_0, y_0, z_0) = (a_m, b_m, c_m)$ and $\forall j \in \{1, 2, ..., m-1\}, (a_{j+1}, b_{j+1}, c_{j+1}) \in S_2(a_j, b_j, c_j)$, where

$$\begin{aligned} S_2(a_j,b_j,c_j) = \{(x,y,z) \in S_2 \mid &(x \in \{a_j,b_j,c_j\} \text{ and } y,z \notin \{a_j,b_j,c_j\}) \\ &\text{or } (y \in \{a_j,b_j,c_j\} \text{ and } x,z \notin \{a_j,b_j,c_j\}) \\ &\text{or } (z \in \{a_j,b_j,c_j\} \text{ and } x,y \notin \{a_j,b_j,c_j\})\}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} |g(x,y,z) - g(x_0,y_0,z_0)| \leq\ &|Sg(a_1,b_1,c_1,d_1)| + |Sg(a_2,b_2,c_2,d_2)| \\ &+ |Sg(a_3,b_3,c_3,d_3)| + ... + |Sf(a_m,b_m,c_m,d_m)|, \end{aligned}$$

where

$$\forall i \in \{1,2,...,m-1\}, d_i \in \{a_{i+1}, b_{i+1}, c_{i+1}\} \setminus \{a_i, b_i, c_i\}$$

and

$$d_m \in \{a_{m-1}, b_{m-1}, c_{m-1}\} \setminus \{a_m, b_m, c_m\}.$$

We set

$$\pi^x_{x_0} = \{(a_1,b_1,c_1,d_1), (a_2,b_2,c_2,d_2), (a_3,b_3,c_3,d_3), ..., (a_m,b_m,c_m,d_m)\}.$$

So, we get

$$|g(x,y,z) - g(x_0,y_0,z_0)| \leq \sum_{(a,b,c,d)\in\pi^x_{x_0}} \frac{1}{(w_3(a,b,c,d))^{\frac{1}{2}}} (w_3(a,b,c,d))^{\frac{1}{2}} |Sg(a,b,c,d)|.$$

We use the Cauchy-Schwarz inequality, we find

$$\begin{aligned} |g(x,y,z) - g(x_0,y_0,z_0)| &\leq \left(\sum_{(a,b,c,d)\in\pi^x_{x_0}} \frac{1}{w_3(a,b,c,d)}\right)^{\frac{1}{2}} \left(\sum_{(a,b,c,d)\in\pi^x_{x_0}} w_3(a,b,c,d)(Sg(a,b,c,d))^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{(a,b,c,d)\in\pi^x_{x_0}} \frac{1}{w_3(a,b,c,d)}\right)^{\frac{1}{2}} \left(\sum_{(a,b,c,d)\in S_3} w_3(a,b,c,d)(Sg(a,b,c,d))^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{(a,b,c,d)\in\pi^x_{x_0}} \frac{1}{w_3(a,b,c,d)}\right)^{\frac{1}{2}} \|Sf\|_{S_3}. \end{aligned}$$

Then, we get

$$\begin{aligned}|g(x,y,z)| &\le |g(x,y,z)-g(x_0,y_0,z_0)|+|g(x_0,y_0,z_0)|\\ &\le \left(\sum_{(a,b,c,d)\in\pi^x_{x_0}}\frac{1}{w_3(a,b,c,d)}\right)^{\frac12}\|Sg\|_{S_3}+|g(x_0,y_0,z_0)|\\ &\le \max\left(\left(\sum_{(a,b,c,d)\in\pi^x_{x_0}}\frac{1}{w_3(a,b,c,d)}\right)^{\frac12},1\right)\left(\|Sg\|_{S_3}+|g(x_0,y_0,z_0)|\right).\end{aligned}$$

We set

$$\beta^x_{x_0}=\max\left(\left(\sum_{(a,b,c,d)\in\pi^x_{x_0}}\frac{1}{w_3(a,b,c,d)}\right)^{\frac12},1\right).$$

Therefore, we obtain

$$|g(x,y,z)|\le\beta^x_{x_0}\left(|g(x_0,y_0,z_0)|+\|Sg\|_{S_3}\right).$$

□

We want now define the non-parabolic at infinity for L and study it.

Definition 4.2. The couple $N=(S_2^N,S_3^N)$ is a finite subset of $R_w=(S_2,S_3)$ if S_2^N is a finite subset of S_2 and S_3^N is a finite subset of S_3.

For all $(f,\varphi)\in N=(S_2^N,S_3^N)$, we have

$$\|(f,\varphi)\|^2_N=\|f\|^2_{S_2^N}+\|\varphi\|^2_{S_3^N}.$$

Definition 4.3. The weighted geometric realization Gauss-Bonnet operator L is said non-parabolic at infinity if there is a finite sub-weighted geometric realization $R_w^M=\left(M,S_3^M\right)$ of $R_w=(S_2,S_3)$ such that for all finite subset N of $R_w\setminus R_w^M$, $\exists\beta=\beta_N\in\mathbb{R}^+$ such that

$$\beta\|(g,h)\|_N\le\|L(g,h)\|_{R_w\setminus R_w^M},\forall(g,h)\in C_0(S_2\setminus M)\times C_0\left(S_3\setminus S_3^M\right).$$

Definition 4.4. The combinatorial simplexes neighborhood of $R_w^M=\left(M,S_3^M\right)$, denoted by $R_w^{M^*}=(M^*,S_3^{M^*})$, is a finite sub-weighted geometric realization of R_w satisfies the following :

1. $M\subset M^*$ finite.
2. $S_3^M\cup\partial S_3^M\subset S_3^{M^*}$.
3. $(x,y,z,t)\in S_3^{M^*}\implies(y,z,t),(z,x,t),(x,y,t),(z,y,x)\in M^*$.

Definition 4.5. The smallest combinatorial simplexes neighborhood of R_w^M, denoted by $R_w^{M_s^*}$, is a finite sub-weighted geometric realization of R_w contains R_w^M and its 3-simplexes boundary.

- The mean value of $g\in C_0(S_2)$, denoted by $\widetilde{g}$, is defined as

$$\begin{aligned}\widetilde{g}(a,b,c,d)&=\frac{g(b,c,d)+g(d,c,a)+g(a,b,d)+g(c,b,a)}{4}\\ &=\frac14 S(g)(a,b,c,d),\end{aligned}$$

for all $(a,b,c,d)\in S_3$.

Theorem 4.6. Let R_w be a weighted geometric realization and L be non-parabolic at infinity. Then $\forall N\subset R_w$ and N is finite, $\exists\beta'=\beta'_N\in\mathbb{R}^+$ such that

$$\beta'\|(g,h)\|_N\le\|L(g,h)\|_{R_w}+\|(g,\varphi)\|_{R_w^{M^*}},\forall(g,h)\in C_0(S_2)\oplus C_0(S_3).$$

Proof . We have N is a finite subset of R_w, then we can reduce it to a 2-simplex or a 3-simplex. Let $(x,y,z)\in S_2$, $(x_0,y_0,z_0)\in M^*$ and $R_w^{M^*}$ be a finite sub-weighted geometric realization of R_w. We show that

$$\beta' \left|g\left(x,y,z\right)\right| \leq \left\|Sg\right\|_{S_3} + \left\|g\right\|_{M^*}, \forall g\in C_0\left(S_2\right).$$

We use Theorem 4.1, $\exists \beta_1\in\mathbb{R}^+$ such that

$$g^2\left(x,y,z\right)\leq \beta_1\left(\left\|g\right\|_{M^*}^2+\left\|Sg\right\|_{S_3}^2\right)$$

and

$$g^2\left(x,y,z\right)\leq \beta_{x_0}^{x}\left(g^2\left(x_0,y_0,z_0\right)+\left\|Sg\right\|_{S_3}^2\right).$$

Moreover, we have

$$\begin{aligned} w_2(x_0,y_0,z_0)g^2\left(x,y,z\right) &\leq \beta_{x_0}^{x}\left(w_2(x_0,y_0,z_0)g^2\left(x_0,y_0,z_0\right)+w_2(x_0,y_0,z_0)\left\|Sg\right\|_{S_3}^2\right)\\ &\leq \beta_{x_0}^{x}\left(\left\|g\right\|_{M^*}^2+w_2(x_0,y_0,z_0)\left\|Sg\right\|_{S_3}^2\right).\end{aligned}$$

We take

$$\beta_{x_0}^{\prime x}=\max\left(\beta_{x_0}^{x}, w_2(x_0,y_0,z_0)\beta_{x_0}^{x}\right).$$

We obtain

$$w_2(x_0,y_0,z_0)g^2\left(x,y,z\right)\leq \beta_{x_0}^{\prime x}\left(\left\|g\right\|_{M^*}^2+\left\|Sg\right\|_{S_3}^2\right).$$

Then, we find

$$g^2\left(x,y,z\right)\leq \frac{\beta_{x_0}^{\prime x}}{w_2(x_0,y_0,z_0)}\left(\left\|g\right\|_{M^*}^2+\left\|Sg\right\|_{S_3}^2\right).$$

We take

$$\beta_1=\frac{\beta_{x_0}^{\prime x}}{w_2(x_0,y_0,z_0)}.$$

Therefore, we obtain

$$f^2\left(x,y,z\right)\leq \beta_1\left(\left\|g\right\|_{M^*}^2+\left\|Sg\right\|_{S_3}^2\right).$$

We show that

$$\beta''\left|h\left(a,b,c,d\right)\right|\leq \left\|h\right\|_{S_3^{M^*}}+\left\|\delta h\right\|_{S_2}, \forall\varphi\in C_0\left(S_3\right), \forall\left(a,b,c,d\right)\in S_3.$$

Let $(a,b,c,d)\in S_3^M\subset S_3^{M^*}$ finite. We have

$$h^2\left(a,b,c,d\right)\leq \left\|h\right\|_{S_3^{M^*}}^2\leq \left\|h\right\|_{S_3^{M^*}}^2+\left\|\delta\varphi\right\|_{S_2}^2.$$

If $(a,b,c,d)\in S_3\setminus S_3^M$, the indicator function of M^c, denoted by χ, is defined as

$$\chi\left(x,y,z\right)=\begin{cases} 0 \text{ if } (x,y,z)\in M\\ 1 \text{ otherwise .}\end{cases}$$

So, we find

$$S\chi\left(a,b,c,d\right)=\begin{cases} 0 \text{ if } (a,b,c,d)\in S_3^M\\ \pm 1 \text{ if } (a,b,c,d)\in \partial S_3^M\\ 0 \text{ otherwise}\end{cases}$$

and

$$\widetilde{\chi}\left(a,b,c,d\right)=\begin{cases} 0 \text{ if } (a,b,c,d)\in S_3^M\\ \frac{1}{4} \text{ if } (a,b,c,d)\in \partial S_3^M\\ 1 \text{ otherwise.}\end{cases}$$

If $h \in C_0(S_3)$, then $\widetilde{\chi}h$ is with a finite support in $S_3 \setminus S_3^M$. Then, we apply the definition of the non-parabolicity at infinity of L to the function $(0, \widetilde{\chi}h)$, we get

$$\|\widetilde{\chi}h\|_N^2 \leq \beta \|\delta(\widetilde{\chi}h)\|_{S_2}^2,$$

where $\beta = \frac{1}{\beta(N)}$. We have $(a,b,c,d) \in S_3 \setminus S_3^M$, then

$$h^2(a,b,c,d) \leq \beta \|\delta(\widetilde{\chi}h)\|_{S_2}^2.$$

Moreover, we have

$$\begin{aligned}
\delta(\widetilde{\chi}h)(b,c,d) &= \frac{1}{w_2(b,c,d)} \sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d)(\widetilde{\chi}h)(a,b,c,d) \\
&= \frac{1}{w_2(b,c,d)} \sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d)\,\widetilde{\chi}(a,b,c,d)\,h(a,b,c,d) \\
&= \frac{1}{4\times w_2(b,c,d)} \sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d)\,S(\chi)(a,b,c,d)\,h(a,b,c,d).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\|\delta(\widetilde{\chi}h)\|_{S_2}^2 &= \frac{1}{6} \sum_{(b,c,d)\in S_2} w_2(b,c,d)\left(\delta(\widetilde{\chi}h)(b,c,d)\right)^2 \\
&\leq 16 \sum_{(b,c,d)\in S_2} w_2(b,c,d)\left(\delta(\widetilde{\chi}h)(b,c,d)\right)^2 \\
&= 16 \sum_{(b,c,d)\in S_2} w_2(b,c,d)\left[\frac{1}{4\times w_2(b,c,d)} \sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d)\,S(\chi)(a,b,c,d)\,h(a,b,c,d)\right]^2 \\
&= \sum_{(b,c,d)\in S_2} \frac{1}{w_2(b,c,d)}\left[\sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d)\,S(\chi)(a,b,c,d)\,h(a,b,c,d)\right]^2 \\
&\leq \sum_{(b,c,d)\in M} \frac{1}{w_2(b,c,d)}\left[\sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d)\,S(\chi)(a,b,c,d)\,h(a,b,c,d)\right]^2 \\
&+ \sum_{(b,c,d)\in S_2\setminus M} \frac{1}{w_2(b,c,d)}\left[\sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d)\,S(\chi)(a,b,c,d)\,h(a,b,c,d)\right]^2.
\end{aligned}$$

We use $\text{supp}(d\chi) = \partial S_3^M \subset S_3^{M^*}$, we obtain

$$\begin{aligned}
&\sum_{(b,c,d)\in M} \frac{1}{w_2(b,c,d)}\left[\sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d)\,S(\chi)(a,b,c,d)\,\varphi(a,b,c,d)\right]^2 \\
&= \sum_{(b,c,d)\in M} \frac{1}{w_2(b,c,d)}\left[\sum_{a;(a,b,c,d)\in S_3 \text{ and } (a,b,c,d)\in\text{supp}(d\chi)} w_3(a,b,c,d)\,\varphi(a,b,c,d)\right]^2 \\
&= \max_{(b,c,d)\in M} \frac{1}{w_2(b,c,d)}\left[\sum_{(a,b,c,d)\in\text{supp}(S\chi)} w_3(a,b,c,d)\,\varphi(a,b,c,d)\right]^2 \\
&\leq \max_{(b,c,d)\in M} \frac{1}{w_2(b,c,d)}\left[\sum_{(a,b,c,d)\in\text{supp}(S\chi)} w_3(a,b,c,d)\right]\left[\sum_{(a,b,c,d)\in\text{supp}(S\chi)} w_3(a,b,c,d)\,h^2(a,b,c,d)\right] \\
&\leq \max_{(b,c,d)\in M} \frac{1}{w_2(b,c,d)} \times \#S_3^{M^*} \max_{(a,b,c,d)\in S_3^{M^*}} w_3(a,b,c,d) \sum_{(a,b,c,d)\in S_3^{M^*}} w_3(a,b,c,d)\,h^2(a,b,c,d).
\end{aligned}$$

We set

$$\beta_2 = \max_{(b,c,d)\in M} \frac{1}{w_2(b,c,d)} \times \#S_3^{M^*} \max_{(a,b,c,d)\in S_3^{M^*}} w_3(a,b,c,d).$$

Then

$$\sum_{(b,c,d)\in M}\frac{1}{w_2(b,c,d)}\left[\sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d)\,S(\chi)(a,b,c,d)\,h(a,b,c,d)\right]^2=\beta_2\,\|\varphi\|^2_{S_3^{M^*}}.$$

In addition, we have

$$\begin{aligned}
&\sum_{(b,c,d)\in S_2\setminus M}\frac{1}{w_2(b,c,d)}\left[\sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d)\,S(\chi)(a,b,c,d)\,h(a,b,c,d)\right]^2\\
&=\sum_{(b,c,d)\in\partial M}\frac{1}{w_2(b,c,d)}\left[\sum_{a;(a,b,c,d)\in S_3 \text{ and } (a,b,c,d)\in\operatorname{supp}(S\chi)} w_3(a,b,c,d)\,h(a,b,c,d)\right]^2\\
&=\max_{(b,c,d)\in\partial M}\frac{1}{w_2(b,c,d)}\times\left[\sum_{(a,b,c,d)\in\operatorname{supp}(S\chi)} w_3(a,b,c,d)\,h(a,b,c,d)\right]^2\\
&\le\max_{(b,c,d)\in\partial M}\frac{1}{w_2(b,c,d)}\left[\sum_{(a,b,c,d)\in\operatorname{supp}(S\chi)} w_3(a,b,c,d)\right]
\end{aligned}$$

$$\left[\sum_{(a,b,c,d)\in supp(S\chi)} w_3(a,b,c,d)\,h^2(a,b,c,d)\right]\le\max_{(b,c,d)\in\partial M}\frac{1}{w_2(b,c,d)}\#S_3^{M^*}\max_{(a,b,c,d)\in S_3^{M^*}} w_3(a,b,c,d)\sum_{(a,b,c,d)\in S_3^{M^*}} w_3(a,b,c,d)\,h^2(a,b,c,d).$$

We set

$$\beta_2'=\max_{(b,c,d)\in\partial M}\frac{1}{w_2(b,c,d)}\times\#S_3^{M^*}\max_{(a,b,c,d)\in S_3^{M^*}} w_3(a,b,c,d).$$

Hence

$$\begin{aligned}
&\sum_{(b,c,d)\in S_2\setminus M}\frac{1}{w_2(b,c,d)}\left[\sum_{a;(a,b,c,d)\in S_3} w_3(a,b,c,d)\,S(\chi)(a,b,c,d)\,h(a,b,c,d)\right]^2\\
&=\beta_2'\,\|h\|^2_{S_3^{M^*}}.
\end{aligned}$$

We take

$$\hat{\beta}_2=\max(\beta_2,\beta_2').$$

We find

$$\|\delta(\widetilde{\chi}h)\|^2_{S_2}\le\hat{\beta}_2\,\|h\|^2_{S_3^{M^*}}\le\max\left(1,\hat{\beta}_2\right)\left(\|\delta h\|^2_{S_2}+\|h\|^2_{S_3^{M^*}}\right).$$

Therefore, we get

$$h^2(a,b,c,d)\le\beta\,\|\delta(\widetilde{\chi}h)\|^2_{S_2}$$

and

$$\|\delta(\widetilde{\chi}h)\|^2_{S_2}\le\max\left(1,\hat{\beta}_2\right)\left(\|\delta h\|^2_{S_2}+\|h\|^2_{S_3^{M^*}}\right).$$

We put

$$\beta^*=\frac{\max\left(1,\hat{\beta}_2\right)}{\beta}.$$

Thus, we find

$$h^2(a,b,c,d)\le\beta^*\left(\|\delta h\|^2_{S_2}+\|h\|^2_{S_3^{M^*}}\right).$$

□

Theorem 4.7. Let R_w be a weighted geometric realization and L be non-parabolic at infinity. Then, we can construct a Hilbert space P satisfies the following :

1. $C_0(S_2) \oplus C_0(S_3)$ is dense in P.
2. The injection of $C_0(S_2) \oplus C_0(S_3)$ to $C_0(S_2) \oplus C_0(S_3)$ extends by continuity to P.
3. $L : P \to H(R_w)$ is a bounded operator.

Proof . Let $R_w^{M^*}$ be a combinatorial simplexes neighborhood of R_w. We take P the closure of $C_0(S_2) \oplus C_0(S_3)$ under the norm

$$N_{M^*}(g,h) = \left(\|(g,h)\|^2_{R_w^{M^*}} + \|L(g,h)\|^2_{R_w} \right)^{\frac{1}{2}}.$$

Aim 1 We have N_{M^*} is a norm on P. Then, we look only at the nullity. we have

$$N_{M^*}(g,h) = 0 \iff \|(g,h)\|_{R_w^{M^*}} = 0, \|L(g,h)\|_{R_w} = 0 \iff \|g\|_{M^*} = 0,$$
$$\|h\|_{S_3^{M^*}} = 0, \|Sg\|_{S_3} = 0 \text{ and } \|\delta h\|_{S_2} = 0.$$

We have $\#M^* < \infty$, we use Theorem 4.6, we find

$$g^2(x,y,z) \leq \beta_1 \left(\|g\|^2_{M^*} + \|Sg\|^2_{S_3} \right), \forall (x,y,z) \in S_2.$$

Moreover, we have $\|g\|_{M^*} = 0$ and $\|Sg\|_{S_3} = 0$. So, we get $f = 0$ on S_2. We show that if $\|h\|_{S_3^{M^*}} = 0$ and $\|\delta h\|_{S_2} = 0$, then $h = 0$. Let $h \neq 0$. We have h is a finite support function in $S_3 \setminus S_3^{M^*}$. We apply Theorem 3 with N equals to the support of h, then $\exists \beta \in \mathbb{R}^+$ such that

$$\beta \|h\|_{S_3^N} \leq \|h\|_{S_3^{M^*}} + \|\delta h\|_{S_2}.$$

We have $\|h\|_{S_3^{M^*}} + \|\delta h\|_{S_2} = 0$, then we obtain $h = 0$ on S_3^N, which is impossible.

Aim 2 We prove that the space P is independent of the choice of $R_w^{M^*}$. We take $R_w^{M_1^*}$ another combinatorial simplexes neighborhood of R_w such that $M \subset M_0^* \subset M_1^*$. We have

$$N_{M_0^*}(g,h) \leq N_{M_1^*}(g,h).$$

To prove that $\exists \beta \in \mathbb{R}_+^*$ such that

$$N_{M_1^*}(g,h) \leq \beta N_{M_0^*}(g,h),$$

we need to prove that $\exists \beta \in \mathbb{R}_+^*$ such that

$$\|(g,h)\|^2_{M_1^* \setminus M_0^*} \leq \beta N^2_{M_0^*}(g,h).$$

We have

$$\begin{aligned} N^2_{M_1^*}(g,h) &= \|(g,h)\|^2_{R_w^{M_1^*}} + \|L(g,h)\|^2_{R_w} \\ &= \|(g,h)\|^2_{M_1^* \setminus M_0^*} + \|(g,h)\|^2_{R_w^{M_0^*}} + \|L(g,h)\|^2_{R_w} \\ &= \|(g,h)\|^2_{M_1^* \setminus M_0^*} + N^2_{M_0^*}(g,h). \end{aligned}$$

Since $\#M_1^* \setminus M_0^* < \infty$, using Theorem 4.6 we obtain

$$\|g\|^2_{M_1^* \setminus M_0^*} \leq \beta \left(\|g\|^2_{M_0^*} + \|Sg\|^2_{S_3} \right)$$

and

$$\|h\|^2_{S_3^{M_1^*} \setminus S_3^{M_0^*}} \leq \beta \left(\|h\|^2_{S_3^{M_0^*}} + \|\delta h\|^2_{S_2} \right),$$

where $\beta = \beta(M_1^* \setminus M_0^*, M_0^*)$.

Then, we get
$$\|(g,h)\|^2_{R_w^{M_1^*}\setminus R_w^{M_0^*}} \leq \beta N^2_{M_0^*}(g,h).$$

Therefore, we have proved that the construction of a norm on P is independent of the choice of the combinatorial simplexes neighborhood associated to the sub-weighted geometric realization R_w^M. We put

$$\|(g,h)\|_P = \left(\|(g,h)\|^2_{R_w^{M^*}} + \|L(g,h)\|^2_{R_w}\right)^{\frac{1}{2}}, \forall (g,h) \in C_0(S_2) \oplus C_0(S_3).$$

Aim 3 We use Theorem 4.6, the injection of $C_0(S_2) \oplus C_0(S_3)$ to $C_0(S_2) \oplus C_0(S_3)$ extends by continuity to P.

Aim 4 Since

$$\begin{aligned}\|L(g,h)\|^2_{R_w} &\leq \|(g,h)\|^2_{R_w^{M^*}} + \|L(g,h)\|^2_{R_w}\\ &= \|(g,h)\|^2_P.\end{aligned}$$

We obtain that $L : P \to H(R_w)$ is a bounded operator.

□

5 Semi-Fredholmness of the Gauss-Bonnet operator

The purpose of this section is to develop necessary and sufficient conditions for semi-Fredholmness of the weighted geometric realization Gauss-Bonnet operator by using its non-parabolicity at infinity.

Definition 5.1. An operator is semi-Fredholm if its range is closed and its kernel is finite dimensional .

Theorem 5.2. Let R_w be a weighted geometric realization and P be a Hilbert space satisfies the following :

1. $C_0(S_2) \oplus C_0(S_3)$ is dense in P.
2. The injection of $C_0(S_2) \oplus C_0(S_3)$ to $C_0(S_2) \oplus C_0(S_3)$ extends by continuity to P.
3. The operator $L : P \to H(R_w)$ is bounded.

Then, the following two conditions are equivalent :

i) The operator $L : P \to H(R_w)$ is semi-Fredholm.

ii) There exists a finite sub-weighted geometric realization R_w^M of R_w and $\beta = \beta_M \in \mathbb{R}_+^*$ such that

$$\beta\|(g,h)\|_P \leq \|L(g,h)\|_{R_w}, \forall g \in C_0(S_2 \setminus M), \forall h \in C_0\left(S_3 \setminus S_3^M\right).$$

Proof . We show the direct implication, we suppose that the conclusion is false. So, we find an increasing sequence of finite sub-weighted geometric realization $\left\{R_w^{M_n}\right\}_n$ such that $R_w = \bigcup\limits_n R_w^{M_n}$ a sequence $\{\phi_n\}_n$ with finite support in $S_2 \setminus M_n$ satisfies for all $n \in \mathbb{N}^*$ the following :

$$\begin{cases} \phi_n = (g_n,h_n) \in C_0(S_2 \setminus M_n) \times C_0\left(S_3 \setminus S_3^{M_n}\right)\\ \|\phi_n\|_P = 1\\ \|L\phi_n\|_{R_w} \leq \frac{1}{n}.\end{cases}$$

We suppose that the operator $L : P \to H(R_w)$ is semi-Fredholm. We use [15], there exists a bounded operator $\Delta : H(R_w) \to P$ satisfies
$$\Delta \circ L = Id_P - T,$$

where T is the orthogonal projection onto the $KerL$, T is an operator with finite rank. Therefore, we find

$$\begin{aligned}\|\phi_n\|_P &\leq \|(\Delta \circ L)\,\phi_n\|_P + \|T\phi_n\|_P \\ &\leq \|\Delta\|\,\|L\phi_n\|_{R_w} + \|T\phi_n\|_P \\ &\leq \left(\frac{\|\Delta\|}{n} + \|T\phi_n\|_P\right).\end{aligned}$$

If $\lim\limits_{n\to\infty} \|T\phi_n\|_P = 0$, then $\lim\limits_{n\to\infty} \|\phi_n\|_P = 0$, which contradicts the assumption $\|\phi_n\|_P = 1$. The aim now is to show that $\{T\phi_n\}_n$ converges to 0 in P. We take $\phi_n^1 = T\phi_n \in KerL$, $\phi_n^2 \in (KerL)^\perp$ and

$$\phi_n = \phi_n^1 + \phi_n^2,$$

such as

$$\begin{cases} (\Delta \circ L)\,\phi_n = \phi_n^2 \\ \|\Delta \circ L\phi_n\|_P \leq \|\Delta\|\,\|L\phi_n\|_{R_w} \underset{n\to\infty}{\rightarrow} 0. \end{cases}$$

For the norm of P, we have $\lim\limits_{n\to\infty} \phi_n^2 = 0$. The sequence $\left\{\phi_n^1\right\}_n$ is bounded of $kerL$ which is of finite dimension. Then, we can extract a subsequence converging to ϕ in P, denoted by $\left\{\phi_{h(n)}^1\right\}_n$. Since $\phi_n = \phi_n^1 + \phi_n^2$ and $\lim\limits_{n\to\infty} \phi_n^2 = 0$, the sequence $\left\{\phi_{h(n)}\right\}_n$ converges in P to ϕ and we get that $\|\phi\|_P = 1$. We prove that

$$\phi = \lim_{n\to\infty} \phi_{h(n)} = \lim_{n\to\infty} \phi_{h(n)}^1 = 0.$$

We assume that $\phi \neq 0$. Since P is injected continuously in $C_0(S_2) \oplus C_0(S_3)$, $\exists (x,y,z) \in S_2$ such that $\left\{\phi_{h(n)}(x,y,z)\right\}_n$ converges to $\phi(x,y,z) \neq 0$. We have $\left\{\phi_{h(n)}\right\}_n$ converges ponctually to 0 by construction. Then, we find that $\phi(x,y,z) = 0$ which is absurd. We remain to show $ii) \implies i)$.

First step To prove that $L : P \to H(R_w)$ has a finite kernel and a closed range, we need to build a bounded operator $U : H(R_w) \to P$ such that $U \circ L - Id_P$ is a compact operator. We have

$$P\left(R_w \setminus R_w^M\right) = \left\{\phi = (f,g) \in P \mid \phi = 0 \text{ on } R_w^M\right\}.$$

Let $L_1 = L_{|R_w \setminus R_w^M} : P\left(R_w \setminus R_w^M\right) \to H(R_w)$ be the restriction of the operator L on $R_w \setminus R_w^M$. Using the assumption we have

$$\beta\,\|(g,h)\|_P \leq \|L(g,h)\|_{R_w}, \forall (g,h) \in C_0(S_2 \setminus M) \times C_0\left(S_3 \setminus S_3^M\right).$$

Thus, we get that the restriction operator L_1 is injective with closed range. So, there exists a left inverse Δ_1 satisfies

$$\Delta_1 \circ L_1 = Id.$$

Let M_0^* be the smallest combinatorial simplexes neighborhood of M and M_1^* be a combinatorial simplexes neighborhood of M_0^*. We denote

$$L_2 : H(M_1^*) \to H(R_w).$$

We have L_2 is continuous with closed range, as $H(M_1^*)$ is a vector space of finite dimension. We take a continuous operator Δ_2 satisfies

$$\Delta_2 \circ L_2 = Id - U_2,$$

where U_2 is the orthogonal projection onto $kerL_2$. We define the indicator function χ on ${M_0^*}^c$ as

$$\chi(x,y,z) = \begin{cases} 0 \text{ if } (x,y,z) \in M_0^* \\ 1 \text{ otherwise.} \end{cases}$$

So, we get

$$S\chi(a,b,c,d) = \begin{cases} 0 \text{ if } (a,b,c,d) \in S_3^{M_0^*} \\ \pm 1 \text{ if } (a,b,c,d) \in \partial S_3^{M_0^*} \\ 0 \text{ otherwise} \end{cases}, \widetilde{\chi}(a,b,c,d) = \begin{cases} 0 \text{ if } (a,b,c,d) \in S_3^{M_0^*} \\ \frac{1}{4} \text{ if } (a,b,c,d) \in \partial S_3^{M_0^*} \\ 1 \text{ otherwise} \end{cases}$$

and

$$(1-\chi)(x,y,z) = \begin{cases} 1 \text{ if } (x,y,z) \in M_0^* \\ 0 \text{ otherwise} \end{cases}, (1-\widetilde{\chi})(a,b,c,d) = \begin{cases} 1 \text{ if } (a,b,c,d) \in S_3^{M_0^*} \\ \frac{1}{4} \text{ if } (a,b,c,d) \in \partial S_3^{M_0^*} \\ 0 \text{ otherwise.} \end{cases}$$

We consider the operator χ^* depending on the domain as:
If $\chi^* : C_0(S_2) \to C_0(S_2)$, then

$$\chi^* g = \chi g, \forall g \in C_0(S_2).$$

If $\chi^* : C_0(S_3) \to C_0(S_3)$, then

$$\chi^* h = \widetilde{\chi} h, \forall h \in C_0(S_3).$$

If $\chi^* : C_0(S_2) \oplus C_0(S_3) \to C_0(S_2) \oplus C_0(S_3)$, then

$$\chi^*(g,h) = (\chi g, \widetilde{\chi} h), \forall (g,h) \in C_0(S_2) \oplus C_0(S_3).$$

We put

$$U\phi = \Delta_2(1-\chi)\phi + \Delta_1 \chi \phi,$$

where $\phi = (g,h)$.

Second step We prove that the operator $U \circ L - Id$ is compact. We set

$$[E,F] = EF - FE,$$

for any two operators E and F. We have

$$\begin{aligned} U \circ L &= \Delta_2(1-\chi)L + \Delta_1 \chi L \\ &= \Delta_2 L(1-\chi) + \Delta_2[1-\chi, L] + \Delta_1 L \chi + \Delta_1[\chi, L] \\ &= \Delta_2 L_2(1-\chi) + \Delta_2[1-\chi, L] + \Delta_1 L_1 \chi + \Delta_1[\chi, L] \\ &= (Id - T_2)(1-\chi) + \Delta_2[1-\chi, L] + Id(\chi) + \Delta_1[\chi, L] \\ &= Id - T_2(1-\chi) + \Delta_2[1-\chi, L] + \Delta_1[\chi, L]. \end{aligned}$$

We calculate $\Delta_2[1-\chi, L]$ and $\Delta_1[\chi, L]$, we find

$$[1-\chi, L] = [1-\chi, S] + [1-\chi, \delta].$$

We have

$$\begin{aligned} \left[(1-\chi)^*, S\right] g(a,b,c,d) &= (1-\widetilde{\chi})(a,b,c,d)\, S(g)(a,b,c,d) - S((1-\chi)g)(a,b,c,d) \\ &= \frac{1}{4} S(1-\chi)(a,b,c,d)\, S(g)(a,b,c,d) - (1-\chi)(b,c,d)g(b,c,d) \\ &- (1-\chi)(d,c,a)g(d,c,a) - (1-\chi)(a,b,d)g(a,b,d) - (1-\chi)(c,b,a)g(c,b,a). \end{aligned}$$

We have

$$\begin{aligned} \left[(1-\chi)^*, \delta\right] h(x,y,z) &= (1-\chi)(x,y,z)\, \delta(h)(x,y,z) + \delta((1-\widetilde{\chi})h)(x,y,z) \\ &= (1-\chi)(x,y,z)\, \delta(h)(x,y,z) + \\ &\frac{1}{w_2(x,y,z)} \sum_{t;(t,x,y,z)\in S_3} w_3(x,y,z,t)(1-\widetilde{\chi})(x,y,z,t)\, h(x,y,z,t) \\ &= (1-\chi)(x,y,z)\, \delta(h)(x,y,z) + \\ &\frac{1}{4 \times w_2(x,y,z)} \sum_{t;(t,x,y,z)\in S_3} w_3(x,y,z,t)\, S(1-\chi)(x,y,z,t) \times h(x,y,z,t). \end{aligned}$$

The support of $S(1-\chi)$ is included in $\partial S_3^{M_0^*} \subset M_1^*$ which is finite. So, Δ_2 has a finite range then it is a compact operator. We use the same method, we prove that Δ_1 has a finite range so it is a compact operator. Therefore, we get that $U \circ L = Id + T$ where T is a compact operator.

□

Theorem 5.3. Let R_w be a weighted geometric realization and P be a Hilbert space satisfies the following :

1. $C_0(S_2) \oplus C_0(S_3)$ is dense in P.
2. The injection of $C_0(S_2) \oplus C_0(S_3)$ to $C_0(S_2) \oplus C_0(S_3)$ extends by continuity to P.
3. The operator $L : P \to H(R_w)$ is a bounded.

So, if there exists a finite sub-weighted geometric realization R_w^M of R_w and $\beta = \beta_M \in \mathbb{R}_+^*$ such that

$$\beta \left\|(g,h)\right\|_P \leq \left\|L(g,h)\right\|_{R_w}, \forall (g,h) \in C_0(S_2 \setminus M) \times C_0\left(S_3 \setminus S_3^M\right),$$

Then, the operator $L : P \to H(R_w)$ is semi-Fredholm.

Proof . First result : We show that if $\phi_n = (g_n, h_n) \in C_0(S_2) \times C_0(S_3)$ is P-bounded and $(L\phi_n)_n$ is convergent in $H(R_w)$, then $(\phi)_n$ has a P-convergent subsequence. We take a combinatorial simplexes neighborhood $R_w^{M^*}$ of the sub-weighted geometric realization R_w. The sequence $(\phi_n\ |_{M^*})_n$ is bounded in a vector space with finite dimension. Therefore, the sequence $(\phi_n\ |_{M^*})_n$ has a convergent subsequence. We define the indicator function χ on ${M^*}^c$ as

$$\chi(x,y,z) = \begin{cases} 0 \text{ if } (x,y,z) \in M^* \\ 1 \text{ otherwise.} \end{cases}$$

So, we have

$$S\chi(a,b,c,d) = \begin{cases} 0 \text{ if } (a,b,c,d) \in S_3^{M^*} \\ \pm 1 \text{ if } (a,b,c,d) \in \partial S_3^{M^*} \\ 0 \text{ otherwise} \end{cases}, \widetilde{\chi}(a,b,c,d) = \begin{cases} 0 \text{ if } (a,b,c,d) \in S_3^{M^*} \\ \frac{1}{4} \text{ if } (a,b,c,d) \in \partial S_3^{M^*} \\ 1 \text{ otherwise.} \end{cases}$$

Thus, we get a function $\chi\phi_n$ with finite support in $R_w \setminus R_w^M$. We apply the inequality $\beta \left\|(g,h)\right\|_P \leq \left\|L(g,h)\right\|_{R_w}$ to $\chi\phi_n$, exactly to $(\chi g_n, 0)$ and $(0, \widetilde{\chi}\phi_n)$, we find

$$\left\|\chi g_n\right\|_P \leq \beta \left\|S(\chi g_n)\right\|_{S_3}.$$

Since the sequence $(S(g_n))_n$ is convergent and $supp(S\chi) \subset S_3^{M^*}$ is finite, $g_n(x,y,z)\ |_{M^*}$ has a convergent subsequence. Therefore, we obtain that χg_n has a P-convergent subsequence, i.e., $(g_n\ |_{S_2 \setminus M^*})_n$ hass a P-convergent subsequence. Moreover, we have

$$\left\|\widetilde{\chi}h_n\right\|_P \leq \beta \left\|\delta(\widetilde{\chi}h_n)\right\|_{S_2}.$$

Using the assumptions, we have $(\delta(h_n))_n$ is a convergent sequence and $supp(S\chi) \subset S_3^{M^*}$ is finite, thus $(h_n\ |_{S_3^{M^*}})_n$ has a convergent subsequence. So, we deduce that the sequence $(\widetilde{\chi}h_n)_n$ has a p-convergent subsequence. As a result, the sequence $(h_n\ |_{S_3 \setminus S_3^{M^*}})_n$ has a P-convergent subsequence.

Now, we prove that the weighted geometric realization Gauss-Bonnet operator L is semi-Fredholm.

1. We prove that $kerL$ is finite dimensional, which is equivalent to prove that $\{\phi \in kerL \mid \left\|\phi\right\|_P = 1\}$ is compact. We take $(\phi_n)_n \subset kerL$ such that
$$\left\|\phi_n\right\|_P = 1 \text{ and } L\phi_n = 0.$$
We use the first result, we get that the sequence $(\phi_n)_n$ admits a convergent subsequence. So, the result occurs.
2. We prove that ImL is closed.
We take the sequence $(\varphi_n)_n$ of ImL such that
$$\lim_{n\to\infty} \varphi_n = \varphi \in H(R_w).$$
We have $(\varphi_n)_n \subset ImL$, then $\exists(\phi_n)_n \subset kerL^\perp$ and $\phi_n \neq 0\ \forall n$, such that $\varphi_n = L\phi_n$. The sequence $(\phi_n)_n$ must be bounded. If not, we construct $f_n = \frac{\phi_n}{\left\|\phi_n\right\|_P}$ such that
$$\begin{cases} (f_n)_n \subset kerL^\perp \\ \left\|f_n\right\|_P = 1 \\ Lf_n \to 0. \end{cases}$$

We use the first result, we get that the sequence$(f_n)_n$ has a convergent subsequence with limit denoted by ϕ such that

$$\begin{cases} f \in kerL^{\perp} \\ \|f_n\|_P = 1 \\ L_f = 0. \end{cases}$$

Therefore, we obtain

$$f \in kerL \cap kerL^{\perp} = \{0\}.$$

Thus, we find $f = 0$, which is absurd. So, the sequence $(\phi_n)_n$ is bounded and since

$$\lim_{n\to\infty} L\phi_n = \varphi.$$

We use the first result, the sequence $(\phi_n)_n$ has a convergent subsequence, we denote this limit by ϕ. We have the operator L is bounded. Then

$$\lim_{n\to\infty} L\phi_n = L\phi.$$

Using the uniqueness of the limit, we get $\varphi = L\phi$.

□

Corollary 5.4. Let R_w be a weighted geometric realization and P be a Hilbert space. The weighted geometric realization Gauss-Bonnet operator L is non-parabolic at infinity if and only if there exists a finite sub-weighted geometric realization R_w^M of R_w such that if we complete $C_0(S_2) \times C_0(S_3)$ by the following norm

$$\|(g,h)\|_P = \left(\|(g,h)\|^2_{R_w^{M*}} + \|L(g,h)\|^2_{R_w}\right)^{\frac{1}{2}},$$

to get a Hilbert space P satisfies the following :

1. The set $C_0(S_2) \oplus C_0(S_3)$ is dense in P.
2. The injection of $C_0(S_2) \oplus C_0(S_3)$ to $C_0(S_2) \oplus C_0(S_3)$ extends by continuity to P.
3. The operator $L : P \to H(R_w)$ is semi-Fredholm.

Proof . Let L be non-parabolic at infinity. We use Theorem 4.7, we find that P is well defined. We remain to prove that the operator $L : P \to H(R_w)$ is semi-Fredholm. The definition of the non-parabolicity at infinity gives the existence of a finite sub-weighted geometric realization R_w^M such that $\forall N \in R_w \setminus R_w^M, \exists \beta = \beta_N \in \mathbb{R}_+^*$,

$$\beta \|(g,h)\|_N \leq \|L(g,h)\|_{R_w}, \forall (g,h) \in C_0(S_2 \setminus M) \oplus C_0\left(S_3 \setminus S_3^M\right).$$

Let $N = R_w^{M^*}$ and $(g,h) \in C_0(S_2 \setminus M) \oplus C_0\left(S_3 \setminus S_3^M\right)$. Then, we obtain

$$\beta \|(g,h)\|_{R_w^{M*}} \leq \|B(g,h)\|_{R_w},$$

$$\beta^2 \|(g,h)\|^2_{R_w^{M*}} + \|L(g,h)\|^2_{R_w} \leq 2\|L(g,h)\|^2_{R_w}$$

and

$$\beta' \|(g,h)\|_P \leq \|L(g,h)\|_{R_w}.$$

We apply Theorem 5.2, we have the operator $L : P \to H(R_w)$ is semi-Fredholm.

Inversely, if the operator $L : P \to H(R_w)$ is semi-Fredholm. By Theorem 5.2, there exists a finite sub-weighted geometric realization R_w^M such that $\exists \beta = \beta_M \in \mathbb{R}_+^*$,

$$\beta \|(g,h)\|_P \leq \|L(g,h)\|_{R_w}, \forall (g,h) \in C_0(S_2 \setminus M) \oplus C_0\left(S_3 \setminus S_3^M\right).$$

The injection of $C_0(S_2) \oplus C_0(S_3)$ to $C_0(S_2) \oplus C_0(S_3)$ extends by continuity to P, implies $\forall N \in R_w \setminus R_w^M$,

$$\begin{aligned} \beta \|(g,h)\|_N &\leq \beta \|(g,h)\|_P \\ &\leq \|L(g,h)\|_{R_w}, \forall (g,h) \in C_0(S_2 \setminus M) \oplus C_0\left(S_3 \setminus S_3^M\right). \end{aligned}$$

Then, we get that the operator L is non-parabolic at infinity. □

References

[1] N. Anghel. *An abstract index theorem on noncompact Riemannian manifolds*, Houston J. Math. **19** (1993), no. 2, 223—237.

[2] A. Baalal and K. Hatim, *Weighted spectra on a weighted geometric realization of* 2- *simplexes and 3-simplexes*, Discrete Math. Algorithms Appl. **15** (2023), no. 5, 2250130.

[3] C. Carron, *Un théorème de l'indice relatif*, Pacific J. Math. **198** (2001), 81–107.

[4] F.R.K. Chung, *Spectral graph theory*, Reg. Conf. Ser. Math., Vol. 92, Amer., Math. Soc., Providence, RI, 1997.

[5] Y. Colin de Verdière, *Spectre de Graphes, Cours Spécialisés*, vol. 4, Soc. Math. France, 1998.

[6] H. Danijela and J. Jost, *Spectra of combinatorial Laplace operators on simplicial complexes*, Adv. Math. **244** (2013), 303–336.

[7] A.M. Duval, C.J. Klivans and J.L. Martin, *Critical groups of simplicial complexes*, Ann. Comb. **17** (2013), no. 1, 53–70.

[8] D.L. Ferrario and R.A. Piccinini, *Simplicial Structures in Topology*, CMS Books in Mathematics, 2011.

[9] M. Gaffney, *Hilbert space methods in the theory of harmonic integrals*, Ann. Math. **78** (1955), 426–444.

[10] J. Gilbert and M.A. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge Studies in Advanced Mathematics 26, 1991.

[11] A. Grigoryan, *Analysis on Graphs*, Lecture Notes. University Bielefeld, 2009.

[12] X. Huang, M. Keller, J. Masamune and R.K. Wojciechowski, *A note on self-adjoint extensions of the Laplacian on weighted graphs*, J. Funct. Anal. **265** (2013), 1556–1578.

[13] B. Mohar and W. Woess, *A survey on spectra of infinite graphs*, J. Bull. Lond. Math. Soc. 21, No.3, 209-234, 1989.

[14] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Academic Press, 1978.

[15] M.A. Shubin, *Pseudodifferential Operators and Spectral Theory*, Springer-Verlag Berlin Heidelberg New York, 2001.